Exercises for Mathematical Logic (24 Oct 2023)

14. Prove that if a term $t(x_0, \ldots, x_{n-1}, y)$ is free for y in a formula $\varphi(x_0, \ldots, x_{n-1}, y)$, then for all terms $s_0, \ldots, s_{n-1}, r$, the formula $\big(\varphi(t/y)\big)(s_0/x_0, \ldots, s_{n-1}/x_{n-1}, r/y)$ is syntactically identical to the formula $\varphi\big(s_0/x_0, \ldots, s_{n-1}/x_{n-1}, t(s_0/x_0, \ldots, s_{n-1}/x_{n-1}, r/y)/y\big)$.

15. Consider a modification of the first-order proof system given in the lecture such that the axioms of equality are replaced with the axiom $x = x$ and the axiom schema $t = s \wedge \varphi(t/s) \to \varphi(s/x)$ for all formulas φ and terms t, s free for x in φ. Show that this is equivalent to the original proof system.

16. For any formula $\varphi(x)$ and variable y free for x in φ, show that the formula $\exists y\,(\exists x\,\varphi(x) \to \varphi(y))$ is provable.