Exercises for Mathematical Logic (10 Oct 2023)

7. Prove the propositional soundness theorem: for all $\Gamma \subseteq \mathrm{Prop}(A)$ and $\varphi \in \mathrm{Prop}(A)$, if $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.

8. Let $\Gamma, \Delta \subseteq \mathrm{Prop}(A)$ and $\varphi, \psi \in \mathrm{Prop}(A)$. Show that if $\Gamma \vdash \varphi$ and $\Delta, \varphi \vdash \psi$, then $\Gamma, \Delta \vdash \psi$.

9. For every $\varphi \in \mathrm{Prop}(A)$, we define its *De Morgan dual* $\varphi^{\mathrm{d}} \in \mathrm{Prop}(A)$ by induction on the complexity of φ:

$$\begin{aligned} a^{\mathrm{d}} &= a, \quad a \in A, & (\neg\varphi)^{\mathrm{d}} &= \neg(\varphi^{\mathrm{d}}), \\ \top^{\mathrm{d}} &= \bot, & \bot^{\mathrm{d}} &= \top, \\ (\varphi \wedge \psi)^{\mathrm{d}} &= (\varphi^{\mathrm{d}} \vee \psi^{\mathrm{d}}), & (\varphi \vee \psi)^{\mathrm{d}} &= (\varphi^{\mathrm{d}} \wedge \psi^{\mathrm{d}}). \end{aligned}$$

Show that for all assignments $v \colon A \to \{0,1\}$, $v(\varphi^{\mathrm{d}}) = v_\neg(\neg\varphi)$, where $v_\neg \colon A \to \{0,1\}$ is the assignment defined by $v_\neg(a) = 1 - v(a)$ for each $a \in A$.

10. Let $\varphi, \psi \in \mathrm{Prop}(A)$.

(i) $\varphi \equiv \psi$ if and only if $\varphi^{\mathrm{d}} \equiv \psi^{\mathrm{d}}$.

(ii) $\varphi \vDash \psi$ if and only if $\psi^{\mathrm{d}} \vDash \varphi^{\mathrm{d}}$.