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ALGEBRAIC CLASSIFICATION OF THE WEYL TENSOR

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Abstract

Alignment classification of tensors on Lorentzian manifolds of arbitrary dimension is summarized. This classification scheme is then applied to the case of the Weyl tensor and it is shown that in four dimensions it is equivalent to the well known Petrov classification. The approaches using Bel-Debever criteria and principal null directions of the superenergy tensor are also discussed.

1. Introduction

The Einstein equations in an n-dimensional spacetime represent a set of second order non-linear PDEs for $n(n+1)/2$ unknown components of metric g_{ab}. In full generality it is hopeless to search for exact solutions of the system. However, there are approaches seeking to reduce its compexity. First, obvious method is to assume some kind of continuous symmetry (for example axial symmetry or staticity). Another approach, less known outside of the community studying Einstein field equations and Lorentzian differential geometry, is to make simplifying assumptions about the Weyl tensor (consisting of partial derivatives of the metric up to the second order) instead of assuming special properties of the metric itself. This approach is based on the algebraic (Petrov-Penrose) classification developed by Petrov [22], Debever [6], Penrose [20] and others and on the Newman-Penrose formalism [17] and it subsequently led to a discovery of many new exact solutions of the Einstein field equations, including the Kerr metric describing gravitational field of a rotating black hole.

Since 1980s there is a growing interest in theoretical physics and differential geometry in higher dimensional geometries with Lorentzian signature. Thus obviously it would be of great interest to have some sort of algebraic classification in higher dimensions than four. In four dimensions there are several equivalent methods leading to the Petrov-Penrose classification. This classification can be obtained using eigenbivectors [22], [1], using number and multiplicity of principal null directions (PNDs) of the Weyl tensor [6], using factorization of the symmetric Weyl spinor [20] or using principal directions of the Bel-Robinson tensor (see e.g. [2], [21]). In general these methods do not give equivalent results in higher dimensions.

Using alignment theory [15], [5] one arrives to a classification scheme valid in arbitrary dimension $n \geq 4$ which is equivalent to the Petrov-Penrose classification for $n = 4$. It turns out that similarly as in the four-dimensional case most of the known exact solutions to the Einstein field equations are algebraically special.

In this contribution we will focus on introducing the higher dimensional classification using alignment theory (Section 2), we briefly summarize two equivalent approaches using Bel-Debever conditions and Bel-Robinson tensor (Section 3) and we briefly discuss classification of several known metrics (Section 4). Various applications of this classification are discussed in Ref. [24] in this volume.

2. Weyl aligned null directions and their multiplicity

In a tangent space of an n-dimensional Lorentzian manifold we choose a null frame with two null vectors $\boldsymbol{\ell} = \boldsymbol{m^{(1)}} = \boldsymbol{m_{(0)}}$, $\boldsymbol{n} = \boldsymbol{m^{(0)}} = \boldsymbol{m_{(1)}}$ and $n-2$ spacelike vectors $\boldsymbol{m^{(i)}} = \boldsymbol{m_{(i)}}$ $(i, j, k = 2 \dots n-1)$, obeying

$$\ell^a \ell_a = n^a n_a = 0, \quad \ell^a n_a = 1, \quad m^{(i)a} m^{(j)}_a = \delta_{ij}, \quad a = 0 \dots n-1. \tag{2.1}$$

The metric then takes the form

$$g_{ab} = 2\ell_{(a} n_{b)} + \delta_{ij} m^{(i)}_a m^{(j)}_b. \tag{2.2}$$

Obviously, such frame is not unique - one can still perform Lorentz transformations. The group of ortochronous Lorentz transformations is generated by null rotations of one of the null frame vectors about the other one, e.g.

$$\hat{\boldsymbol{\ell}} = \boldsymbol{\ell} + z_i \boldsymbol{m^{(i)}} - \frac{1}{2} z^i z_i \, \boldsymbol{n}, \quad \hat{\boldsymbol{n}} = \boldsymbol{n}, \quad \hat{\boldsymbol{m}}^{(i)} = \boldsymbol{m^{(i)}} - z_i \boldsymbol{n}, \tag{2.3}$$

with parameters z_i, spins defined by an orthogonal matrix $X^i{}_j$

$$\hat{\boldsymbol{\ell}} = \boldsymbol{\ell}, \quad \hat{\boldsymbol{n}} = \boldsymbol{n}, \; \hat{\boldsymbol{m}}^{(i)} = X^i{}_j \boldsymbol{m^{(j)}} \tag{2.4}$$

and boosts with a parameter λ

$$\hat{\boldsymbol{\ell}} = \lambda \boldsymbol{\ell}, \quad \hat{\boldsymbol{n}} = \lambda^{-1} \boldsymbol{n}, \quad \hat{\boldsymbol{m}}^{(i)} = \boldsymbol{m^{(i)}}. \tag{2.5}$$

Let us now present a short summary of useful definitions based on [15], [5].

Definition 1. *A quantity q has a boost weight* bw *if it transforms under a boost according to*

$$\hat{q} = \lambda^{\mathrm{bw}} q. \tag{2.6}$$

Thus for frame components of a tensor $\boldsymbol{T}$ we obtain their boost weight

$$\hat{T}_{a \dots b} \equiv T(\hat{\boldsymbol{m}}_{(a)}, \dots \hat{\boldsymbol{m}}_{(b)}) = \lambda^{\mathrm{bw}_T (a \dots b)} T_{a \dots b}, \tag{2.7}$$

where $\mathrm{bw}_T(a \dots b)$ can be conveniently expressed as number of 0's minus number of 1's in frame component indices.

Definition 2. *Boost order of a tensor $\boldsymbol{T}$ with respect to the null frame $\boldsymbol{\ell}$, $\boldsymbol{n}$, $\boldsymbol{m}^{(i)}$ is the maximum boost weight of its frame components*

$$\mathrm{bo}_T = \max\left\{\mathrm{bw}_T(a\dots b) \mid T_{a\dots b} \neq 0\right\}. \tag{2.8}$$

Proposition 1. *Let $\boldsymbol{\ell}$, $\boldsymbol{n}$, $\boldsymbol{m}^{(i)}$ and $\hat{\boldsymbol{\ell}}$, $\hat{\boldsymbol{n}}$, $\hat{\boldsymbol{m}}^{(i)}$ be two null frames with $\boldsymbol{\ell}$ and $\hat{\boldsymbol{\ell}}$ being scalar multiples of each other. Then the boost order of a given tensor is the same relative to both frames.*

Thus boost order of a tensor depends only on the choice of null direction $\langle\boldsymbol{\ell}\rangle$ and we will denote it $\mathrm{bo}_T(\boldsymbol{\ell})$. Note that for components for which $\mathrm{bw}_T(a\dots b) = \mathrm{bo}_T(\ell)$ it follows

$$\hat{T} \equiv T(\hat{\boldsymbol{m}}_{(a)}\dots\hat{\boldsymbol{m}}_{(b)}) = T(\boldsymbol{m}_{(a)}\dots\boldsymbol{m}_{(b)}). \tag{2.9}$$

Definition 3. *Let* $\mathbf{T}$ *be a tensor and let $b_{\max}(T)$ denote the maximum value of* $\mathrm{bo}_T(\boldsymbol{\ell})$ *taken over all null vectors $\boldsymbol{\ell}$*

$$b_{\max}(T) = \max\{\mathrm{bo}_T(\ell) \mid \ \forall \ \ \mathrm{null} \ \ \langle\ell\rangle\}. \tag{2.10}$$

*We say that a vector $\boldsymbol{\ell}$ is aligned null direction (*AND*) of a tensor $\boldsymbol{T}$ whenever* $\mathrm{bo}_T(\ell) < b_{\max}(T)$ *and we will call integer $b_{\max}(T) - \mathrm{bo}_T(\ell)$ its multiplicity.*

Definition 4. *We will call a quantity*

$$b_{\max}(T) - b_{\min}(T), \tag{2.11}$$

where

$$b_{\min}(T) = \min\{\mathrm{bo}_T(\ell) \mid \forall \ \ \mathrm{null} \ \ \langle\ell\rangle\}, \tag{2.12}$$

*principal alignment type (*PAT*) of a tensor $\boldsymbol{T}$.*

Choosing $\boldsymbol{\ell}$ with maximal multiplicity (which is equal to $b_{\max}(T) - b_{\min}(T)$), we define secondary alignment type, SAT*, to be*

$$b_{\max}(T) - \tilde{b}_{\min}(T), \tag{2.13}$$

with

$$\tilde{b}_{\min}(T) = \min\{\mathrm{bo}_T(\boldsymbol{n}) \mid \forall \ \ \mathrm{null} \ \ \langle\boldsymbol{n}\rangle \ \ \mathrm{except} \ \langle\boldsymbol{\ell}\rangle\}. \tag{2.14}$$

Definition 5. *We can classify an arbitrary tensor according to its alignment type consisting of two integers* $(\mathrm{PAT}, \mathrm{SAT})$.

To determine an alignment type of a tensor one has to project the tensor $\boldsymbol{T}$ on the null frame and sort its components by their boost weight

$$\boldsymbol{T} = \sum_b (\boldsymbol{T})_{(b)}, \tag{2.15}$$

where

$$(\boldsymbol{T})_{(b)} = \sum T_{a\ldots b}\boldsymbol{m^{(a)}}\ldots\boldsymbol{m^{(b)}}, \quad \mathrm{bw}_T(a\ldots b) = b. \tag{2.16}$$

Then using null rotations (2.3) about $\boldsymbol{\ell}$ and $\boldsymbol{n}$ one has to set as many leading and trailing terms in (2.15) as possible to zero.

For arbitrary tensor in arbitrary dimension we define (in part we employ definitions of [12], [11]):

Definition 6. *A tensor $\boldsymbol{T}$ is of*

- *type G if for all frames some maximal boost-weight components do not vanish, i.e.* $(\boldsymbol{T})_{(b_{\max}(T))} \neq 0$,
- *type I if there exists a frame such that maximal boost-weight components do vanish, i.e.* $(\boldsymbol{T})_{(b_{\max}(T))} = 0$,
- *type II if there exists a frame such that all positive boost-weight components vanish, i.e.* $(\boldsymbol{T})_{(b>0)} = 0$ *and* $\boldsymbol{T} = \sum_{b\leq 0}(\boldsymbol{T})_{(b)}$,
- *type D if there exists a frame such that $\boldsymbol{T}$ has only zero boost-weight components,* $\boldsymbol{T} = (\boldsymbol{T})_{(0)}$,
- *type III if there exists a frame such that $\boldsymbol{T}$ has only negative boost-weight components, i.e.* $(\boldsymbol{T})_{(b\geq 0)} = 0$ *and* $\boldsymbol{T} = \sum_{b<0}(\boldsymbol{T})_{(b)}$,
- *type N if there exists a frame such that $\boldsymbol{T}$ has only components of boost-weight* $-b_{\max}(T)$, *i.e.* $\boldsymbol{T} = (\boldsymbol{T})_{(-b_{\max}(T))}$.

Note that according to this definition, type N is a subcase of type III which is again a subcase of type II, etc. Sometimes, the term *pure type* II is used meaning a spacetime of type II which is not of type III, etc.

Let us illustrate these definitions on some examples:

- A vector $\boldsymbol{v}$ can be decomposed as

  $$\boldsymbol{v} = v_0\boldsymbol{n} + v_i\boldsymbol{m^{(i)}} + v_1\boldsymbol{\ell}. \tag{2.17}$$

  It has $b_{\max}(v) = 1$. There are three classes of vectors:

  1. Timelike vector ($v^a v_a < 0$) is of alignment type $(0,0)$ (type G):
     In this case v_0 cannot be set to zero by null rotations, i.e. for all $\boldsymbol{\ell}$ $\mathrm{bo}_v(\boldsymbol{\ell}) = 1$, there are no ANDs.
  2. Spacelike vector ($v^a v_a > 0$) is of alignment type $(1,1)$ (type D):
     There exist $\boldsymbol{\ell}$ and $\boldsymbol{n}$ such that $v_0 = 0 = v_1$, i.e. with $\mathrm{bo}_v(\boldsymbol{\ell}) = 0 = \mathrm{bo}_v(\boldsymbol{n})$, both ANDs are of multiplicity 1.

3. Null vector ($v^a v_a = 0$) is of alignment type $(2,0)$ (type N):
   There exists $\boldsymbol{\ell}$ ($\boldsymbol{\ell} \parallel \boldsymbol{v}$) such that $v_0 = 0 = v_i$, i.e. with $\mathrm{bo}_v(\boldsymbol{\ell}) = -1$ and multiplicity is 2.

- A bivector (an antisymmetric tensor of rank two), $\boldsymbol{F}$, has in general decomposition

$$\boldsymbol{F} = (\boldsymbol{F})_{(+1)} + (\boldsymbol{F})_{(0)} + (\boldsymbol{F})_{(-1)}, \tag{2.18}$$

or equivalently

$$F_{ab} = \overbrace{2F_{0i}\, n_{[a} m^i{}_{b]}}^{1} + \overbrace{2F_{01}\, n_{[a}\ell_{b]} + F_{ij}\, m^i{}_{[a} m^j{}_{b]}}^{0} + \overbrace{2F_{1i}\, \ell_{[a} m^i{}_{b]}}^{-1}. \tag{2.19}$$

Thus $b_{\max}(F) = 1$.

There are cases

1. Case $(0,0)$ - type G: there are no ANDs.
2. Case $(1,0)$ - type II: this occurs when F_{0i} can be set to zero using null rotations, $\boldsymbol{\ell}$ is AND of multiplicity 1.
3. Case $(1,1)$ - type D: this occurs when F_{0i}, F_{1i} can be set to zero, both ANDs $\boldsymbol{\ell}$ and $\boldsymbol{n}$ are of multiplicity 1.
4. Case $(2,0)$ - type N: all components F_{0i}, F_{01}, F_{ij} can be set to zero, $\boldsymbol{\ell}$ is AND of multiplicity 2.

In even dimensions, there always exists an AND and thus a bivector is of type II or more special (see Prop. 4.4 in [7]).

In four dimensions, only the following two cases exist

1. Generic case $(1,1)$ with canonical form $F_{ab} = \lambda m^2{}_{[a} m^3{}_{b]} + \mu n_{[a}\ell_{b]}$.
2. Special case $(2,0)$ with canonical form $F_{ab} = \lambda \ell_{[a} m^2{}_{b]}$.

- In this paper, we are mainly interested in the algebraic classification of the Weyl tensor with the following symmetries

$$C_{abcd} = C_{\{abcd\}} \equiv \tfrac{1}{2}(C_{[ab][cd]} + C_{[cd][ab]}), \quad C^c{}_{acb} = 0, \quad C_{a[bcd]} = 0. \tag{2.20}$$

Decomposing the Weyl tensor in its frame components we obtain:

$$\boldsymbol{C} = (\boldsymbol{C})_{(+2)} + (\boldsymbol{C})_{(+1)} + (\boldsymbol{C})_{(0)} + (\boldsymbol{C})_{(-1)} + (\boldsymbol{C})_{(-2)}, \tag{2.21}$$

or more specifically

$$
\begin{aligned}
C_{abcd} \; = \; & \overbrace{4C_{0i0j}\, n_{\{a}m^{(i)}_{b} n_{c} m^{(j)}_{d\}}}^{\text{boost weight 2 -- type G}} + \overbrace{8C_{010i}\, n_{\{a}l_{b}n_{c}m^{(i)}_{d\}} + 4C_{0ijk}\, n_{\{a}m^{(i)}_{b}m^{(j)}_{c}m^{(k)}_{d\}}}^{1,\ \text{I}} \\
& \left.\begin{aligned} &+4C_{0101}\; n_{\{a}l_{b}n_{c}l_{d\}} \;+\; 4C_{01ij}\; n_{\{a}l_{b}m^{(i)}_{c}m^{(j)}_{d\}} \\ &+8C_{0i1j}\; n_{\{a}m^{(i)}_{b}l_{c}m^{(j)}_{d\}} + C_{ijkl}\; m^{(i)}_{\{a}m^{(j)}_{b}m^{(k)}_{c}m^{(l)}_{d\}} \end{aligned}\right\}_{0,\ \text{II,D}} \qquad (2.22) \\
& + \overbrace{8C_{101i}\, l_{\{a}n_{b}l_{c}m^{(i)}_{d\}} + 4C_{1ijk}\, l_{\{a}m^{(i)}_{b}m^{(j)}_{c}m^{(k)}_{d\}}}^{-1,\ \text{III}} + \overbrace{4C_{1i1j}\, l_{\{a}m^{(i)}_{b}l_{c}m^{(j)}_{d\}}}^{-2,\ \text{N}},
\end{aligned}
$$

so e.g. components $C_{0i0j} = C_{abcd}\ell^a m^b_{(i)} \ell^c m^d_{(j)}$ have boost weight $\mathrm{bw}_C(0i0j) = 2$. The Weyl tensor has $b_{\max}(C) = 2$.

For Weyl components, we will follow the notation of [8] which is together with additional identities (2.20) summarized in table 1.

bw	Compt.	Notation	Identities
2	C_{0i0j}	Ω_{ij}	$\Omega_{ij} = \Omega_{ji},\ \Omega_{ii} = 0$
1	C_{0ijk}	Ψ_{ijk}	$\Psi_{ijk} = -\Psi_{ikj},\ \Psi_{[ijk]} = 0$
	C_{010i}	Ψ_i	$\Psi_i = \Psi_{kik}.$
0	C_{ijkl}	Φ_{ijkl}	$\Phi_{ijkl} = \Phi_{[ij][kl]} = \Phi_{klij},\ \Phi_{i[jkl]} = 0$
	C_{0i1j}	Φ_{ij}	$\Phi_{(ij)} \equiv \Phi^{\mathrm{S}}_{ij} = -\frac{1}{2}\Phi_{ikjk}$
	C_{01ij}	$2\Phi^{\mathrm{A}}_{ij}$	$\Phi^{\mathrm{A}}_{ij} \equiv \Phi_{[ij]}$
	C_{0101}	Φ	$\Phi = \Phi_{ii}$
-1	C_{1ijk}	Ψ'_{ijk}	$\Psi'_{ijk} = -\Psi'_{ikj},\ \Psi'_{[ijk]} = 0$
	C_{101i}	Ψ'_i	$\Psi'_i = \Psi'_{kik}.$
-2	C_{1i1j}	Ω'_{ij}	$\Omega'_{ij} = \Omega'_{ji},\ \Omega'_{ii} = 0$

Table 1: Decomposition of the Weyl tensor by boost weight bw for dimensions $n > 4$ (c.f. Ref. [5]).

We classify the Weyl tensor according to the (non)existence of *Weyl aligned null directions (WANDs)* and their multiplicity. Note that a generic Weyl tensor for $n \geq 5$ does not possess any WAND [15]. All possible algebraical types are given in the table 2 (for the conformally flat case, type O, the Weyl tensor vanishes). Alignment type classification of the Weyl tensor in four dimensions is equivalent to the Petrov classification and WANDs coincide with principal null directions (PNDs) of the Weyl tensor. As in four dimensions the Weyl tensor is called *algebraically special* if it is of type II or more special.

Let us briefly summarize further refinement [5, 4] of the alignment type classification.

n >4 dimensions		4 dimensions
Petrov type	alignment type	Petrov type
G	(0,0)	
I	(1,0)	
I_i	(1,1)	I
II	(2,0)	
II_i	(2,1)	II
D	(2,2)	D
III	(3,0)	
III_i	(3,1)	III
N	(4,0)	N

Table 2: Possible Petrov/alignment types in higher dimensions compared to the four-dimensional case [5].

- Boost-weight +1 components Ψ_{ijk} can be decomposed as (see [4])

$$\Psi_{ijk} = -\tfrac{1}{n-3}(\delta_{ij}\Psi_k - \delta_{ik}\Psi_j) + T_{ijk}, \quad T_{(ijk)} = 0, \quad T_{iji} = 0, \quad T_{i(jk)} = 0. \tag{2.23}$$

  Thus there are two subcases of type I

  - Subcase I_a: $\Psi_i = 0 \Leftrightarrow \Psi_{iji} = 0$,
  - Subcase I_b: $T_{ijk} = 0 \Leftrightarrow \Psi_{ijk}\Psi_{ijk} = \frac{2}{n-3}\Psi_i\Psi_i$.

  Similar subclassification can be introduced for type III, i.e. for boost-weight -1 components, Ψ'_{ijk}.

- Zero boost-weight components Φ_{ijkl} can be decomposed in the same way as the Riemann tensor:

$$\Phi_{ijkl} = \bar{C}_{ijkl} + \frac{2}{d-2}\left(\delta_{i[k}\bar{R}_{l]j} - \delta_{j[k}\bar{R}_{l]i}\right) - \frac{2}{(d-1)(d-2)}\bar{R}\delta_{i[k}\delta_{l]j} \tag{2.24}$$

  with $d = n - 2$ and

$$\bar{R}_{ij} = \bar{S}_{ij} + \frac{\bar{R}}{d}\delta_{ij} = \Phi_{ikjk} = -2\Phi^{\mathrm{S}}_{ij}, \quad \bar{R} = -2\Phi. \tag{2.25}$$

  Therefore the following subclasses appear

  - Subcase II_a: $\bar{R} = 0$,
  - Subcase II_b: $\bar{S}_{ij} = 0$,
  - Subcase II_c: $\bar{C}_{ijkl} = 0$,
  - Subcase II_d: ${\Phi^{\mathrm{A}}}_{ij} = 0$.

  Some of their possible combinations are given in the table 3.

Type	Bel-Debever crit.	superenergy ten.	+2	+1	0	-1
G	$\ell_{[e}C_{a]bc[d}\ell_{f]}\ell^b\ell^c \neq 0$	$\mathcal{T}_{abcd}\ell^a\ell^b\ell^c\ell^d \neq 0$				
I	$\ell_{[e}C_{a]bc[d}\ell_{f]}\ell^b\ell^c = 0$	$\mathcal{T}_{abcd}\ell^a\ell^b\ell^c\ell^d = 0$	Ω_{ij}			
I_a	$\ell_{[e}C_{a]bcd}\ell^b\ell^c = 0$		Ω_{ij}	Ψ_i		
I_b			Ω_{ij}	T_{ijk}		
II	$\ell_{[e}C_{a]b[cd}\ell_{f]}\ell^b = 0$	$\mathcal{T}_{abcd}\ell^a\ell^b\ell^c = 0$	Ω_{ij}	Ψ_{ijk} (Ψ_i)		
II_a	$C_{abcd}\ell^b\ell^c = 0$		Ω_{ij}	Ψ_{ijk} (Ψ_i)	Φ	
II_b			Ω_{ij}	Ψ_{ijk} (Ψ_i)	$\bar{S}_{ij}$	
II_c			Ω_{ij}	Ψ_{ijk} (Ψ_i)	$\bar{C}_{ijkl}$	
II_d	$C_{ab[cd}\ell_{e]}\ell^b = 0$		Ω_{ij}	Ψ_{ijk} (Ψ_i)	Φ^{A}_{ij}	
II_{abc}	$\ell_{[e}C_{ab][cd}\ell_{f]} = 0$		Ω_{ij}	Ψ_{ijk} (Ψ_i)	Φ_{ijkl} $(\Phi,\ \Phi^{\mathrm{S}}_{ij})$	
II_{abd}	$C_{abc[d}\ell_{e]}\ell^c = 0$	$\mathcal{T}_{abcd}\ell^a\ell^b = 0$	Ω_{ij}	Ψ_{ijk} (Ψ_i)	Φ_{ij} $(\Phi,\ \Phi^{\mathrm{A}}_{ij})$	
$\mathrm{II'}_{abd}$	$C_{abcd}\ell^d = 0$		Ω_{ij}	Ψ_{ijk} (Ψ_i)	Φ_{ij} $(\Phi,\ \Phi^{\mathrm{A}}_{ij})$	Ψ'_i
III	$\ell_{[e}C_{ab][cd}\ell_{f]} = 0$ $C_{abc[d}\ell_{e]}\ell^c = 0$	$\mathcal{T}_{abcd}\ell^a\ell^c = 0$	Ω_{ij}	Ψ_{ijk} (Ψ_i)	$\Phi_{ijkl},\ \Phi^{\mathrm{A}}_{ij}$ $(\Phi,\ \Phi^{\mathrm{S}}_{ij})$	
III_a	$\ell_{[e}C_{ab][cd}\ell_{f]} = 0$ $C_{abcd}\ell^d = 0$		Ω_{ij}	Ψ_{ijk} (Ψ_i)	$\Phi_{ijkl},\ \Phi^{\mathrm{A}}_{ij}$ $(\Phi,\ \Phi^{\mathrm{S}}_{ij})$	Ψ'_i
III_b			Ω_{ij}	Ψ_{ijk} (Ψ_i)	$\Phi_{ijkl},\ \Phi^{\mathrm{A}}_{ij}$ $(\Phi,\ \Phi^{\mathrm{S}}_{ij})$	T'_{ijk}
N	$C_{ab[cd}\ell_{e]} = 0$	$\mathcal{T}_{abcd}\ell^a = 0$	Ω_{ij}	Ψ_{ijk} (Ψ_i)	$\Phi_{ijkl},\ \Phi^{\mathrm{A}}_{ij}$ $(\Phi,\ \Phi^{\mathrm{S}}_{ij})$	Ψ'_{ijk} (Ψ'_i)

Table 3: Summary of criteria for various algebraic classes of the Weyl tensor. Note that for some subtypes Bel-Debever criteria or conditions involving the superenergy tensor are not known. Last four columns show vanishing components of the Weyl tensor of the corresponding boost weight. Components that are automatically zero due to the identities given in table 1 are in brackets. Note that $C_{abc[d}\ell_{e]}\ell^c = 0$ which is equivalent with $\mathcal{T}_{abcd}\ell^a\ell^b = 0$ follows from $\mathcal{T}_{abcd}\ell^a\ell^c = 0$ which is equivalent with $\{\ell_{[e}C_{ab][cd}\ell_{f]} = 0 \ \wedge \ C_{ab[cd}\ell_{e]}\ell^b = 0\}$ [28, 29]. The same conditions can be applied in the case of secondary classification (e.g. condition for type II applied to a vector $\boldsymbol{n}$ in type D spacetimes). In four dimensions the following equivalences hold: I_a =II=II_b =II_c, II_{abc} =II_a, II_{abd} =III and $\mathrm{II'}_{abd}$ =III_a =N [19].

- Boost-weight -2 components are represented by a symmetric traceless matrix Ω'_{ij} and so type N spacetimes can be further classified according to multiplicities of eigenvalues of Ω'_{ij}. In four dimensions, there is only one case with two distinct non-vanishing eigenvalues, i.e. in Segre notation $\{11\}$. In five dimensions, there are three possible cases:

– Three distinct non-vanishing eigenvalues, i.e. in Segre notation $\{111\}$,
– Two distinct non-vanishing eigenvalues, one of them with multiplicity 2, i.e. in Segre notation $\{(11)1\}$,
– Two distinct non-vanishing eigenvalues and one vanishing, i.e. in Segre notation $\{110\}$.

One can similarly classify type N in each dimension. As was shown in [26], for type N Ricci flat spacetimes the only possible case in arbitrary dimension is $\{110\dots 0\}$. This result can be straightforwardly generalized to the case of Einstein spacetimes, see also Ref. [24] in this volume.

3. Equivalent approaches to the algebraic classifications of the Weyl tensor – superenergy tensor and Bel-Deber criteria

There are equivalent approaches to the algebraic classification of the Weyl tensor leading to the same classification scheme as in table 2, namely finding principal null directions of the superenergy tensor [28, 29] and classifying the Weyl tensor according to Bel-Deber criteria [19].

3.1. Classification based on principal null directions of the superenergy tensor

In four dimensions Petrov types can be defined using principal null directions of the completely symmetric and traceless Bel-Robinson tensor[1] [28, 29]

$$\mathcal{T}_{abcd} = C_{aecf}C_{b\ d}^{\ e\ f} - \frac{1}{8}g_{ab}g_{cd}C_{efgh}C^{efgh} \tag{3.1}$$

as follows[2]

1. Petrov type I $\Leftrightarrow$ there exists $\boldsymbol{\ell}$ such that $\mathcal{T}_{abcd}\ell^a\ell^b\ell^c\ell^d = 0$,
2. Petrov type II (or D) $\Leftrightarrow$ there exists $\boldsymbol{\ell}$ such that $\mathcal{T}_{abcd}\ell^b\ell^c\ell^d = 0$,
3. Petrov type III $\Leftrightarrow$ there exists $\boldsymbol{\ell}$ such that $\mathcal{T}_{abcd}\ell^c\ell^d = 0$,
4. Petrov type N $\Leftrightarrow$ there exists $\boldsymbol{\ell}$ such that $\mathcal{T}_{abcd}\ell^d = 0$.

In higher dimensions, superenergy tensor [28, 29], a generalization of the Bel-Robinsor tensor, can be defined as

$$\mathcal{T}_{abcd} = C_{aecf}C_{b\ d}^{\ e\ f} + C_{aedf}C_{b\ c}^{\ e\ f} - \tfrac{1}{2}g_{ab}C_{efcg}C^{ef\ g}_{\ \ d} - \tfrac{1}{2}g_{cd}C_{aefg}C_b^{\ efg} + \frac{1}{8}g_{ab}g_{cd}C_{efgh}C^{efgh}, \tag{3.2}$$

[1] Note that in four dimensions PNDs of the Bel-Robinson tensor coincide with PNDs of the Weyl tensor.

[2] Recall that in the sequence of algebraic types I, II, III, N, each type is considered as a special subcase of more general types.

having symmetries

$$\mathcal{T}_{abcd} = \mathcal{T}_{(ab)(cd)} = \mathcal{T}_{(cd)(ab)}. \tag{3.3}$$

The superenergy tensor is completely symmetric only in four and five dimensions [27] and in four dimensions it reduces to the Bel-Robinson tensor (3.1) [9, 27].

Algebraic classification using principal null directions of the superenergy tensor is summarized in the table 3.

3.2. Higher dimensional generalization of the Bel-Debever conditions

Generalization of the Bel-Debever approach towards classifying the Weyl tensor was developed in [19]. It is summarized in table 3.

4. Examples

Now let us briefly classify some known exact solutions of the Einstein equations in higher dimensions.

- Black ring solution representing spinning five-dimensional black hole with horizon topology $S^2 \times S^1$ [10] consists of regions of type I and G and it is of type II on the black hole horizon [25].

- Kerr metric describing gravitational field of a rotating black hole and its higher dimensional generalization - Myers-Perry rotating black hole metric [16] are of the algebraic type D in arbitrary dimension.

- Kerr-Schild spacetimes [13] are spacetimes with metric of the form

  $$g_{ab} = \eta_{ab} - 2\mathcal{H}k_a k_b, \tag{4.1}$$

  where $\mathcal{H}$ is a scalar function and $\boldsymbol{k}$ a null vector with respect to the background flat metric η_{ab} and also to the full metric g_{ab}. Thanks to the simple form of the metric these spacetimes can be analyzed in arbitrary dimension [18]. This class contains important solutions such as Kerr and Myers-Perry black holes [16] and type N pp-waves [3, 18]. Einstein Kerr-Schild spacetimes are of type II or more special in arbitrary dimension and they split in two groups [18, 14]:

  - Non-expanding solutions are always of type N and belong to the Kundt class [3, 23]. This case contains radiative solutions.
  - Expanding solutions are always of type II or D. This case contains black holes.

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