Method of lower and upper functions and the existence of solutions to singular periodic problems for second order nonlinear differential equations

Irena Rachůnková* and Milan Tvrdý

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Abstract. We construct nonconstant lower and upper functions for the periodic boundary value problem $u'' = f(t,u)$, $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$ and find their estimates. By means of these results we prove existence criteria for the problems $u'' \pm g(u) = e(t)$, $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$, where $\limsup_{x\to 0+} g(x) = \infty$ is allowed and $e \in \mathbb{L}[0,2\pi]$ need not be essentially bounded.

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1 . Introduction

In this paper we construct lower and upper functions to the periodic boundary value problem

$$u'' = f(t,u), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \tag{1.1}$$

By means of these results we prove existence criteria for the problems

$$u'' \pm g(u) = e(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi),$$

where $\limsup_{x\to 0+} g(x) = \infty$ is allowed and $e \in \mathbb{L}[0,2\pi]$ need not be essentially bounded. We assume that $f : [0,2\pi]\times\mathbb{R} \mapsto \mathbb{R}$ fulfils the Carathéodory conditions on $[0,2\pi]\times\mathbb{R}$, i.e. f has the following properties: (i) for each $x \in \mathbb{R}$ the function $f(.,x)$ is measurable on $[0,2\pi]$; (ii) for almost every $t \in [0,2\pi]$ the function $f(t,.)$ is continuous on $\mathbb{R}$; (iii) for each compact set $\mathrm{K} \subset \mathbb{R}$ the function $m_{\mathrm{K}}(t) = \sup_{x\in\mathrm{K}} |f(t,x)|$ is Lebesgue integrable on $[0,2\pi]$.

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For a given subinterval J of $\mathbb{R}$ (possibly unbounded), $\mathbb{C}(J)$ denotes the set of functions continuous on J. Furthermore, $\mathbb{L}[0,2\pi]$ stands for the set of functions Lebesgue integrable on $[0,2\pi]$, $\mathbb{L}_2[0,2\pi]$ is the set of functions square Lebesgue integrable on $[0,2\pi]$ and $\mathbb{AC}[0,2\pi]$ denotes the set of functions absolutely continuous on $[0,2\pi]$. For x bounded on $[0,2\pi]$, $y \in \mathbb{L}[0,2\pi]$ and $z \in \mathbb{L}_2[0,2\pi]$ we denote

$$\|x\|_{\mathbb{C}} = \sup_{t\in[0,2\pi]} |x(t)|, \quad \overline{y} = \frac{1}{2\pi}\int_0^{2\pi} y(s)\,\mathrm{d}s,$$

$$\|y\|_1 = \int_0^{2\pi} |y(t)|\mathrm{d}t \text{ and } \|z\|_2 = \Big(\int_0^{2\pi} z^2(t)\,\mathrm{d}t\Big)^{\frac{1}{2}}.$$

By a *solution of* (1.1) we mean a function $u : [0,2\pi] \mapsto \mathbb{R}$ such that $u' \in \mathbb{AC}[0,2\pi]$, $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$ and

$$u''(t) = f(t,u(t)) \quad \text{for a.e.} \quad t \in [0,2\pi].$$

1.1. Definition. A function σ_1 is said to be a *lower function of the problem* (1.1) if $\sigma_1' \in \mathbb{AC}[0,2\pi]$,

$$\sigma_1''(t) \geq f(t,\sigma_1(t)) \quad \text{for a.e.} \quad t \in [0,2\pi],$$
$$\sigma_1(0) = \sigma_1(2\pi), \quad \sigma_1'(0) \geq \sigma_1'(2\pi).$$

Similarly, a function σ_2 is said to be an *upper function of the problem* (1.1) if $\sigma_2' \in \mathbb{AC}[0,2\pi]$,

$$\sigma_2''(t) \leq f(t,\sigma_2(t)) \quad \text{for a.e.} \quad t \in [0,2\pi],$$
$$\sigma_2(0) = \sigma_2(2\pi), \quad \sigma_2'(0) \leq \sigma_2'(2\pi).$$

The lower and upper functions approach we will use here is based on the following theorem which is contained in [8, Theorems 4.1 and 4.2].

1.2. Theorem. *Let σ_1 and σ_2 be respectively a lower and an upper function of the problem* (1.1).

(I) *Suppose $\sigma_1(t) \leq \sigma_2(t)$ on $[0,2\pi]$. Then there is a solution u of the problem* (1.1) *such that $\sigma_1(t) \leq u(t) \leq \sigma_2(t)$ on $[0,2\pi]$.*

(II) *Suppose $\sigma_1(t) \geq \sigma_2(t)$ on $[0,2\pi]$ and there is $m \in \mathbb{L}[0,2\pi]$ such that*

$$f(t,x) \geq m(t) \quad \textit{for a.e.} \quad t \in [0,2\pi] \textit{ and all } x \in \mathbb{R}$$
$$(\textit{or } f(t,x) \leq m(t) \quad \textit{for a.e.} \quad t \in [0,2\pi] \textit{ and all } x \in \mathbb{R}.)$$

Then there is a solution u of the problem (1.1) *such that $\|u'\|_{\mathbb{C}} \leq \|m\|_1$ and*

$$\sigma_2(t_u) \leq u(t_u) \leq \sigma_1(t_u) \quad \textit{for some} \quad t_u \in [0,2\pi].$$

2 . Construction of lower and upper functions

2.1. Proposition. *Assume that there are* $A \in \mathbb{R}$ *and* $b \in \mathbb{L}[0, 2\pi]$ *such that*

$$\overline{b} = 0, \tag{2.1}$$

$$f(t, x) \le b(t) \text{ for a.e. } t \in [0, 2\pi] \text{ and all } x \in [A, B], \tag{2.2}$$

where

$$B = A + \frac{\pi}{3} \|b\|_1. \tag{2.3}$$

Then there exists a lower function σ *of the problem* (1.1) *such that*

$$A \le \sigma(t) \le B \text{ on } [0, 2\pi]. \tag{2.4}$$

Proof. Define

$$\sigma_0(t) = c_0 + \int_0^{2\pi} g(t, s) b(s) \, \mathrm{d}s \text{ for } t \in [0, 2\pi],$$

where

$$g(t, s) = \begin{cases} \dfrac{t(s - 2\pi)}{2\pi} & \text{if } 0 \le t \le s \le 2\pi, \\ \dfrac{(t - 2\pi)s}{2\pi} & \text{if } 0 \le s < t \le 2\pi \end{cases}$$

is the Green function of the problem $v'' = 0$, $v(0) = v(2\pi) = 0$ and

$$c_0 = -\frac{1}{2\pi} \int_0^{2\pi} \Big(\int_0^{2\pi} g(t, s) b(s) \, \mathrm{d}s \Big) \mathrm{d}t.$$

Then

$$\sigma_0''(t) = b(t) \quad \text{a.e. on } [0, 2\pi] \tag{2.5}$$

and

$$\sigma_0(0) = \sigma_0(2\pi). \tag{2.6}$$

Furthermore, in virtue of (2.1) we have also

$$\sigma_0'(0) = \sigma_0'(2\pi). \tag{2.7}$$

Multiplying the relation (2.5) by σ_0, integrating it over $[0, 2\pi]$ and using the Hölder inequality we get

$$\|\sigma_0'\|_2^2 \le \|b\|_1 \|\sigma_0\|_{\mathbb{C}}.$$

Further, as $\overline{\sigma_0} = 0$, the Sobolev inequality (see [5, Proposition 1.3]) yields

$$\|\sigma_0'\|_2^2 \le \sqrt{\frac{\pi}{6}}\,\|b\|_1\,\|\sigma_0'\|_2,$$

and so

$$\|\sigma_0'\|_2 \le \sqrt{\frac{\pi}{6}}\,\|b\|_1,$$

wherefrom using again the Sobolev inequality we get

$$\|\sigma_0\|_{\mathbb{C}} \le \frac{\pi}{6}\,\|b\|_1.$$

Thus, the function σ given by

$$\sigma(t) = \frac{\pi}{6}\,\|b\|_1 + A + \sigma_0(t) \text{ for } t \in [0, 2\pi] \tag{2.8}$$

satisfies (2.4). Furthermore, according to (2.1),(2.2) and (2.6)-(2.8) we have

$$\sigma''(t) = \sigma_0''(t) = b(t) \ge f(t, \sigma(t)) \text{ for a.e. } t \in [0, 2\pi] \tag{2.9}$$

and

$$\sigma(0) = \sigma(2\pi), \quad \sigma'(0) = \sigma'(2\pi), \tag{2.10}$$

i.e. σ is a lower function of (1.1). □

The following assertion is dual to Proposition 2.1 and its proof will be omitted.

2.2. Proposition. *Assume that there are $A \in \mathbb{R}$ and $b \in \mathbb{L}[0, 2\pi]$ such that*

$$\overline{b} = 0$$

and

$$f(t, x) \ge b(t) \text{ for a.e. } t \in [0, 2\pi] \text{ and all } x \in [A, B]$$

where B is given by (2.3). *Then there exists an upper function σ of the problem* (1.1) *with the property* (2.4). □

3 . Applications to Lazer-Solimini singular problems

In this section we will consider possibly singular problems of the attractive type

$$u'' + g(u) = e(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi) \tag{3.1}$$

and of the repulsive type

$$u'' - g(u) = e(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \tag{3.2}$$

where

$$g \in \mathbb{C}(0,\infty) \text{ and } e \in \mathbb{L}[0,2\pi] \tag{3.3}$$

and it is allowed that $\limsup_{x\to 0+} g(x) = \infty$.

The problem (3.1) has been studied by Lazer and Solimini in [6] for $e \in \mathbb{C}[0,2\pi]$ and g positive. In [9, Corollary 3.3], their existence result has been extended to $e \in \mathbb{L}[0,2\pi]$ essentially bounded from above. Here, we bring conditions for the existence of solutions to (3.1) without assuming boundedness of e.

3.1. Theorem. *Assume* (3.3) *and let there exist* $A_1, A_2 \in (0,\infty)$ *such that*

$$g(x) \ge \overline{e} \text{ for all } x \in [A_1, B_1], \tag{3.4}$$

$$g(x) \le \overline{e} \text{ for all } x \in [A_2, B_2], \tag{3.5}$$

where

$$B_1 - A_1 = B_2 - A_2 = \frac{\pi}{3}\,\|e - \overline{e}\|_1 \tag{3.6}$$

and $A_2 \ge B_1$.

Then the problem (3.1) *has a solution* u *such that* $A_1 \le u(t) \le B_2$ *on* $[0,2\pi]$.

Proof. Define, for a.e. $t \in [0,2\pi]$,

$$f(t,x) = e(t) - \begin{cases} g(A_1) & \text{if } x < A_1, \\ g(x) & \text{if } x \ge A_1. \end{cases}$$

Then f satisfies the Carathéodory conditions on $[0,2\pi]\times\mathbb{R}$. Furthermore, by (3.4) and (3.6), f satisfies (2.1)-(2.3) with $b(t) = e(t) - \overline{e}$ a.e. on $[0,2\pi]$ and $[A,B] = [A_1,B_1]$. Hence, by Proposition 2.1 there exists a lower function σ_1 of (1.1) such that $\sigma_1(t) \in [A_1,B_1]$ for all $t \in [0,2\pi]$. Similarly, (3.5), (3.6) and Proposition 2.2 yield the existence of an upper function σ_2 of (1.1) such that $\sigma_2(t) \in [A_2,B_2]$ on $[0,2\pi]$. Now, since $A_2 \ge B_1$, we have $\sigma_1(t) \le \sigma_2(t)$ on $[0,2\pi]$ and the assertion (I) of Theorem 1.2 gives the existence of the desired solution u to (1.1) which is also a solution to (3.1), of course. □

Classical Lazer and Solimini's considerations [6] of the repulsive problem (3.2) have been extended by several authors (see e.g. [1], [2], [3], [4], [7] and [10]). Here we present a related result, where e need not be essentially bounded.

3.2. Theorem. *Assume* (3.3),

$$\lim_{x\to 0+} \int_x^1 g(\xi)\,\mathrm{d}\xi = \infty, \tag{3.7}$$

and

$$g_* := \inf_{x\in(0,\infty)} g(x) > -\infty. \tag{3.8}$$

Furthermore, let there exist $A_1, A_2 \in (0,\infty)$ *such that*

$$g(x) \le -\overline{e} \text{ for all } x \in [A_1, B_1], \tag{3.9}$$

$$g(x) \ge -\overline{e} \text{ for all } x \in [A_2, B_2], \tag{3.10}$$

where (3.6) *is true and* $A_1 \ge B_2$.

Then the problem (3.2) *has a positive solution.*

Proof. Denote

$$K = \|e\|_1 + 2\pi\,|g_*|, \quad B = B_1 + 2\pi\,K \text{ and } K^* = K\,\|e\|_1 + \int_{A_2}^{B} |g(x)|\,\mathrm{d}x.$$

It follows from (3.7) that $\limsup_{x\to 0+} g(x) = \infty$ and there exists $\varepsilon \in (0, A_2)$ such that

$$\int_{\varepsilon}^{A_2} g(x)\,\mathrm{d}x > K^* \text{ and } g(\varepsilon) > 0. \tag{3.11}$$

Define

$$\widetilde{g}(x) = \begin{cases} g(x) & \text{if } x \ge \varepsilon, \\ g(\varepsilon) & \text{if } x < \varepsilon, \end{cases}$$

and

$$f(t,x) = e(t) + \widetilde{g}(x) \text{ for a.e. } t \in [0,2\pi] \text{ and all } x \in \mathbb{R}.$$

Now, we can argue as in the proof of Theorem 3.1 obtaining a lower function σ_1 and an upper function σ_2 of (1.1) such that $\sigma_1(t) \ge \sigma_2(t)$ on $[0,2\pi]$. The assertion (II) of Theorem 1.2 (with $m(t) = g_* + e(t)$ a.e. on $[0,2\pi]$) implies that (1.1) has a solution u such that $u(t_u) \in [A_2, B_1]$ for some $t_u \in [0,2\pi]$ and $\|u'\|_{\mathbb{C}} \le K$. It remains to show that $u(t) \ge \varepsilon$ holds on $[0,2\pi]$.

Let t_0 and $t_1 \in [0,2\pi]$ be such that

$$u(t_0) = \min_{t\in[0,2\pi]} u(t) \text{ and } u(t_1) = \max_{t\in[0,2\pi]} u(t).$$

Clearly, $A_2 \le u(t_1) \le B$. With respect to the periodic boundary conditions we have $u'(t_0) = u'(t_1) = 0$. Now, multiplying the differential relation $u''(t) = e(t) + \widetilde{g}(u(t))$ by $u'(t)$ and integrating over $[t_0, t_1]$ we get

$$0 = \int_{t_0}^{t_1} u''(t)\,u'(t)\,\mathrm{d}t = \int_{t_0}^{t_1} e(t)\,u'(t)\,\mathrm{d}t + \int_{t_0}^{t_1} \widetilde{g}(u(t))\,u'(t)\,\mathrm{d}t,$$

i.e.

$$\int_{u(t_0)}^{u(t_1)} \widetilde{g}(x)\,\mathrm{d}x = -\int_{t_0}^{t_1} e(t)\,u'(t)\,\mathrm{d}t \le K\,\|e\|_1.$$

Further,

$$\int_{u(t_0)}^{A_2} \widetilde{g}(x)\,\mathrm{d}x \le K\,\|e\|_1 + \int_{A_2}^{B} |\widetilde{g}(x)|\,\mathrm{d}x = K^*$$

which, with respect to (3.11), is possible only if $u(t_0) \ge \varepsilon$. Thus, u is a solution to (3.2). □

3.3. Example. Let $g(x) = \frac{1}{x^\gamma}$ on $(0,\infty)$. If $\gamma > 0$, then Theorem 3.1 ensures the existence of a positive solution to (3.1) for any $e \in \mathbb{L}[0,2\pi]$ such that

$$\overline{e} > 0 \text{ and } \frac{\pi}{3}\,\overline{e}^{\frac{1}{\gamma}}\,\|e-\overline{e}\|_{\mathbb{L}} < 1. \tag{3.12}$$

The function $e(t) = c + \frac{1}{\sqrt{2\pi t}} - \frac{1}{\pi}$ with $c \in \mathbb{R}$ is not essentially bounded from above on $[0,2\pi]$. However, it satisfies (3.12) if

$$0 < c < \left(\frac{3}{\pi}\right)^\gamma.$$

We should mention that provided $e \in \mathbb{C}[0,2\pi]$ or e is essentially bounded from above, the condition $\overline{e} > 0$ is sufficient for the existence of a solution to (3.1) (cf. [6] or [9], respectively).

3.4. Example. Let $e \in \mathbb{L}[0,2\pi]$ be essentially unbounded from below and let

$$g(x) = \frac{1+\sin(\frac{\pi}{x})}{x} - \arctan(x), \quad x \in (0,\infty).$$

Then g verifies the assumptions (3.3), (3.7) and (3.8) of Theorem 3.2. Let us suppose that $\overline{e} = -5$. Then the equation $g(x) = 5$ has exactly 5 roots in the interval $[0.125,\infty)$. In particular, we have (see Figures 1 and 2)

$$x_1 \approx 0.126804,\ x_2 \approx 0.141071,\ x_3 \approx 0.167853,\ x_4 \approx 0.200541,\ x_5 \approx 0.244461,$$
$$g(x) > 5 \text{ on } (x_2,x_3)\cup(x_4,x_5) \text{ and } g(x) < 5 \text{ on } (x_1,x_2)\cup(x_3,x_4)\cup(x_5,\infty).$$

Therefore, by Theorem 3.2, if

$$\|e-\overline{e}\|_{\mathbb{L}} \le \frac{3}{\pi}\,(x_5-x_4) \approx 0.0419392,$$

the problem

$$u'' = \frac{1+\sin(\frac{\pi}{u})}{u} - \arctan(u) + e(t), \quad u(0)=u(2\pi), \quad u'(0)=u'(2\pi) \tag{3.13}$$

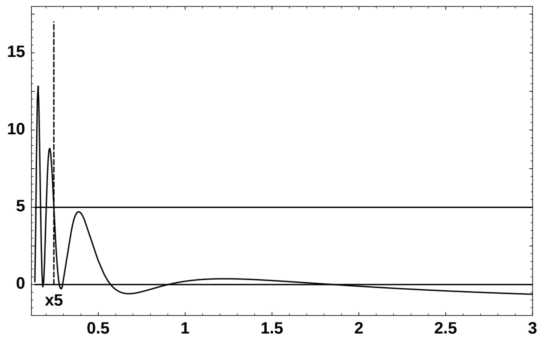

Figure 1

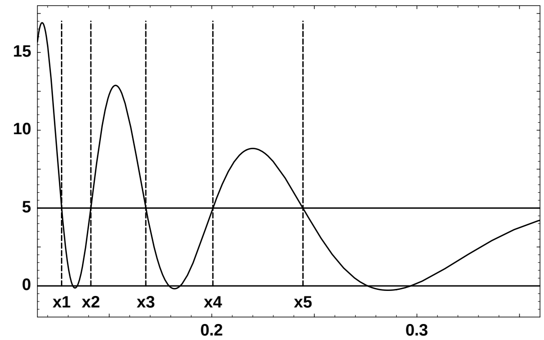

Figure 2

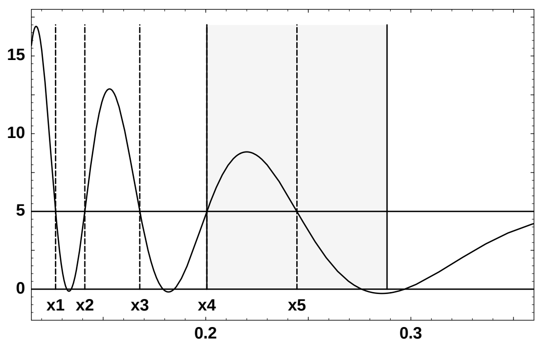

Figure 3

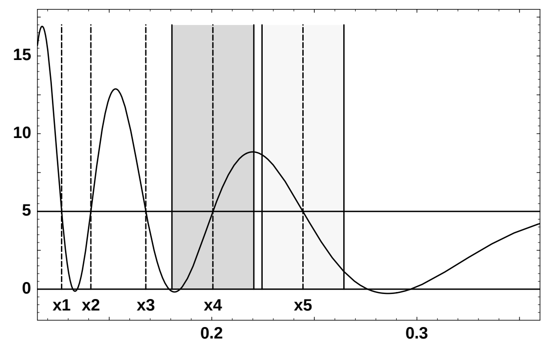

Figure 4

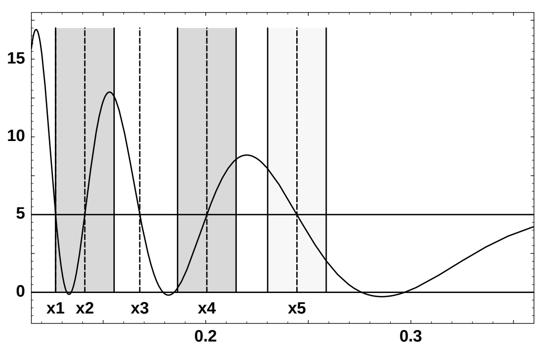

Figure 5

has a solution u_1 such that $u_1(t^*) \in [x_4, x_5 + d_1]$ for some $t^* \in [0, 2\pi]$, where $d_1 = x_5 - x_4$ (see Figure 3).

Similarly, by Theorems 3.1 and 3.2, if

$$\|e - \overline{e}\|_{\mathbb{L}} < \frac{3}{2\pi}(x_5 - x_4) \approx 0.0209699,$$

the problem (3.13) has at least 2 different solutions u_1 and u_2, where $u_1(t^*) \in (x_5 - d_2, x_5 + d_2)$ for some $t^* \in [0, 2\pi]$ and $u_2(t) \in (x_4 - d_2, x_4 + d_2)$ for all $t \in [0, 2\pi]$, where $d_2 = \frac{1}{2}(x_5 - x_4)$ (see Figure 4).

Finally, if

$$\|e - \overline{e}\|_{\mathbb{L}} \leq \frac{3}{\pi}(x_2 - x_1) \approx 0.0136238,$$

the problem (3.13) has at least 3 different solutions u_1, u_2 and u_3, where $u_1(t^*) \in [x_5 - d_3, x_5 + d_3]$ for some $t^* \in [0, 2\pi]$, $u_2(t) \in [x_4 - d_3, x_4 + d_3]$ for all $t \in [0, 2\pi]$ and $u_3(t) \in [x_1, x_2]$ for all $t \in [0, 2\pi]$, where $d_3 = x_2 - x_1$ (see Figure 5).

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Irena Rachůnková, Department of Mathematics, Palacký University, 779 00 OLOMOUC, Tomkova 40, Czech Republic
(`e-mail: rachunko@risc.upol.cz`)

Milan Tvrdý, Mathematical Institute, Academy of Sciences of the Czech Republic, 115 67 PRAHA 1, Žitná 25, Czech Republic
(`e-mail: tvrdy@math.cas.cz`, \http: //www.math.cas.cz/~tvrdy)