NEW RESULTS FOR THE LIEBAU PHENOMENON VIA FIXED POINT INDEX

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Abstract. We prove new results regarding the existence of positive solutions for a nonlinear periodic boundary value problem related to the Liebau phenomenon. As a consequence we obtain new sufficient conditions for the existence of a pump in a simple model. Our methodology relies on the use of classical fixed point index. Some examples are provided to illustrate our theory. We improve and complement previous results in the literature.

1. Introduction

During the experiments developed in the 1950s, the German cardiologist Gerhart Liebau observed (see [1]) that a periodic compression could produce the circulation of a fluid in a mechanical system without valves to ensure the direction of the flow. This valveless pumping effect is nowadays called the Liebau phenomenon. It was reported to occur for instance in embryonic blood circulation, in applications of nanotechnology and in oceanic currents, see e.g. [2, 3, 4, 5, 6] or [7, Chapter 8]. In particular, G. Propst [6] presented an explanation of the pumping effect for flow configurations of several rigid tanks that are connected by rigid pipes. He proved the existence of periodic solutions to the corresponding differential equations for systems of 2 or 3 tanks. However, the apparently simplest configuration consisting of 1 pipe and 1 tank turned out to be, from mathematical point of view, the most interesting one, as it leads to the singular periodic problem

$$\begin{cases} u''(t) + a\,u'(t) = \dfrac{1}{u(t)}\left(e(t) - b(u'(t))^2\right) - c, & t \in [0,T], \\ u(0) = u(T), \quad u'(0) = u'(T), \end{cases} \tag{1.1}$$

where u' is the fluid velocity in the pipe (oriented in the direction from the tank to the piston), $T > 0$,

$$a = \frac{r_0}{\rho}, \quad b = 1 + \frac{\zeta}{2}, \quad c = \frac{gA_\pi}{A_\tau}, \quad e(t) = \frac{g\,V_0}{A_\tau} - \frac{p(t)}{\rho},$$

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r_0 is the friction coefficient, ρ is the density of the fluid, $\zeta \geq 1$ is the junction coefficient (depending on the particular geometry and smoothness of the junction of the tank and the pipe), g is the acceleration of gravity, A_τ is the cross section of the tank, A_π is the cross section of the pipe (small in comparison with A_τ), V_0 is the total volume (assumed to be constant) of the fluid in the system and p is the T-periodic external force. As a result, from the fluid mechanics point of view, the assumptions

$$a \geq 0, \quad b > 1, \quad c > 0, \quad e \text{ continuous and } T-\text{periodic}$$

are quite natural, from $\zeta \geq 1$ we would even have $b \geq 3/2$. Of course, we are interested in the search of positive solutions of problem (1.1). A detailed justification of the model can be also found e.g. in [7, Chapter 8].

One can observe that if a periodic external force e produces a nonconstant periodic response u then the mean level of the fluid in the tank is higher than the level produced by a constant force with the same mean value. Moreover the increasing of the level is proportional to $\|u'\|^2$.

The change of variables $u = x^{\mu}$, where $\mu = \frac{1}{b+1}$, was used in [8] in order to overcome the singularity, transforming and simplifying problem (1.1) into the regular BVP

$$\text{(1.2)} \qquad \begin{cases} x''(t) + a\,x'(t) = \dfrac{e(t)}{\mu}\,x^{1-2\mu}(t) - \dfrac{c}{\mu}\,x^{1-\mu}(t), \quad t \in [0,T], \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases}$$

where $0 < \mu < \frac{1}{2}$. By means of the lower and upper solution technique Cid and co-authors [8] provided results on the existence and stability of a positive solution of (1.2).

In our recent paper [9] we considered a generalization of problem (1.2), namely

$$\text{(1.3)} \qquad \begin{cases} x''(t) + ax'(t) = r(t)\,x^{\alpha}(t) - s(t)\,x^{\beta}(t), \; t \in [0,T], \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases}$$

under the assumption

$$\text{(H0)} \qquad a \geq 0, \quad r, s : [0,T] \to \mathbb{R} \text{ are continuous and } \quad 0 < \alpha < \beta < 1.$$

Of course, to extend the obtained solution of the boundary value problem (1.3) (on $[0,T]$) to a T-periodic solution of the corresponding differential equation, we would have to assume that r and s are also T-periodic. Making use of a shifting argument and Krasnosel'skiĭ's expansion/compression fixed point theorem on cone, we succeeded in [9] to improve the existence results from [8].

Furthermore, Torres in [7, Chapter 8] obtained a priori bounds for the periodic solutions of (1.1), which together with the Brouwer degree theoretical arguments, led to an alternative existence result.

We point out that the assumption

$$\min_{t\in[0,T]} e(t) > 0$$

is a common feature of the existence results in [7, 8, 9]. The goal of this paper is twofold: first, to improve the main results from [9] and, second, to obtain explicit sufficient conditions for the existence of periodic solutions of problem (1.2) that allow, for the first time, the function e to be sign-changing. Similarly to the papers [10, 11, 12, 13, 14, 15], our main tool will be the classical fixed point index on cones.

The paper is organized as follows: in Section 2 we present the shifting argument and recall some known facts regarding the Green's function of the problem and some properties of the fixed point index. In Section 3 we perform the fixed point index calculations that we use to prove our main results. In Section 4 we present the main results, some consequences and illustrative examples.

2. Preliminaries

By means of a shifting argument (see [9]) problem (1.3) may be rewritten in the equivalent form

$$\begin{cases} x''(t) + a\,x'(t) + m^2x(t) = r(t)\,x^{\alpha}(t) - s(t)\,x^{\beta}(t) + m^2x(t), \;\; t \in [0,T], \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases} \tag{2.1}$$

with $m \in \mathbb{R}$. In the sequel, we denote the right-hand side of the differential equation in (2.1) by f_m, i.e.

$$f_m(t,x) = r(t)\,x^{\alpha} - s(t)\,x^{\beta} + m^2x \quad \text{for} \;\; t \in [0,T] \;\; \text{and} \; x \in [0,+\infty). \tag{2.2}$$

We will assume that the linearization

$$\begin{cases} x''(t) + a\,x'(t) + m^2x(t) = h(t), \quad t \in [0,T], \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases} \tag{2.3}$$

of (2.1) possesses a positive Green's function. Its existence and its further needed properties are given by the following lemma.

Lemma 1. *Assume that*

$$a \geq 0 \quad \text{and} \quad 0 < m < \sqrt{\left(\frac{\pi}{T}\right)^2 + \left(\frac{a}{2}\right)^2}. \tag{2.4}$$

Then there exists a unique function $G_m(t,s)$ defined on $[0,T] \times [0,T]$ such that

$$\text{(G1)} \qquad \begin{cases} \text{For each } h \text{ continuous on } [0,T], \text{ the function} \\ \qquad x(t) = \displaystyle\int_0^T G_m(t,s)\,h(s)\,\mathrm{d}s \quad \text{for } t \in [0,T] \\ \text{is the unique solution of (2.3).} \end{cases}$$

$$\text{(G2)} \qquad G_m > 0 \text{ on } [0,T] \times [0,T].$$

$$\text{(G3)} \qquad \int_0^T G_m(t,s)\,\mathrm{d}s = \frac{1}{m^2} \quad \text{for all } t \in [0,T].$$

$$\text{(G4)} \qquad \begin{cases} \text{There exists a constant } c_m \in (0,1) \text{ such that} \\ G_m(t,s) \geq G_m(s,s) \geq c_m\, G_m(t,s) \quad \text{for all } (t,s) \in [0,T] \times [0,T]. \end{cases}$$

Proof. The existence of a function G_m with the properties (G1) and (G2) is given by [16, Proposition 2.2] and for the rest, see [9, Appendix A]. □

Throughout, given T, a and m in (2.4), we will write G_m and c_m for the corresponding function G_m in (G1) and the corresponding constant c_m in (G4).

Furthermore, let us recall that a *cone* P in a Banach space X is a closed, convex subset of X such that $P \cap (-P) = \{0\}$ and $\lambda x \in P$ for $x \in P$ and $\lambda \geq 0$. Here we will work in the space $X = \mathcal{C}[0,T]$ of continuous functions on $[0,T]$ endowed with the usual maximum norm $\|x\| = \max\{|x(t)| : t \in [0,T]\}$. The properties of Green's function G_m of (2.3) stated in Lemma 1 enable us to use the cone

$$P = \{x \in X : x(t) \geq c_m\,\|x\| \ \text{ for } t \in [0,T]\}, \tag{2.5}$$

a type of a cone first used by M.A. Krasnosel'skiĭ and D. Guo, see for example [17, 18].

Whenever (2.4) is true, we define the operator $F : P \to X$ by

$$F\,x(t) = \int_0^T G_m(t,s)\,f_m(s, x(s))\,\mathrm{d}s \quad \text{for } \ x \in P \ \text{ and } \ t \in [0,T], \tag{2.6}$$

where G_m is the corresponding Green's function for (2.3). Then, any fixed point of F in P is a nonnegative solution of problem (2.1) and, simultaneously, of problem (1.3). In order to obtain the existence of a fixed point of this kind we make use of the classical fixed point index. For readers' convenience we formulate below the principles that we will need later.

If Ω is an open bounded subset of P (in the relative topology) we denote by $\overline{\Omega}$ and $\partial\,\Omega$ the closure and the boundary relative to P. When Ω is an open bounded subset of X we write $\Omega_P = \Omega \cap P$. Recall that for a given cone P, an open set Ω and a compact operator $F:\overline{\Omega_P} \to P$, the symbol $i_P(F,\Omega_P)$ stands for the fixed point index of F with respect to P and Ω. For the definition and more details concerning the properties of the fixed point index see e.g. [18, 19, 20, 21]. The following existence principle is well-known, but we include its short proof for completeness.

Theorem 2. *Let P be a cone in a Banach space X. Let Ω, $\Omega' \subset X$ be open bounded sets such that $0 \in \Omega'_P$ and $\overline{\Omega'_P} \subset \Omega_P$. Assume that $F:\overline{\Omega_P} \to P$ is a compact map such that $F\,x \neq x$ for $x \in \partial\,\Omega_P \cup \partial\,\Omega'_P$, $i_P(F,\Omega_P) = 1$ and $i_P(F,\Omega'_P) = 0$. Then F has a fixed point in $\Omega_P \setminus \overline{\Omega'_P}$.*

Proof It follows from the additivity and solution properties of the fixed point index, cf. e.g. [18, Theorem 2.3.1 and Theorem 2.3.2]. For an analogous argument, see also e.g. the proof of Theorem 12.3 in [19]. □

To verify the assumptions of Theorem 2, the following assertion will be helpful.

Theorem 3 ([18], Lemma 2.3.1 and Corollary 2.3.1)**.** *Let P be a cone in a Banach space X, Ω be an open bounded set such that $0 \in \Omega_P$ and let $F:\overline{\Omega_P} \to P$ be compact. Then*

(i) *If $F\,x \neq \lambda\,x$ for all $x \in \partial\,\Omega_P$ and all $\lambda \geq 1$, then $i_P(F,\Omega_P) = 1$.*

(ii) *If there exists $x_0 \in P \setminus \{0\}$ such that $x - F\,x \neq \lambda\,x_0$ for all $x \in \partial\,\Omega_P$ and all $\lambda \geq 0$, then $i_P(F,\Omega_P) = 0$.*

We will complete this section by introducing further notations needed later:

For a given continuous function $h:[0,T] \to \mathbb{R}$ we denote

$$\overline{h} = \frac{1}{T}\int_0^T h(s)\,\mathrm{d}s, \quad h_* = \min\{h(t): t \in [0,T]\}, \quad h^* = \max\{h(t): t \in [0,T]\},$$

and

$$h_+(t) = \max\{h(t),0\} \quad \text{for} \quad t \in [0,T].$$

3. Calculations of the fixed point index

We will assume that (H0) holds and

$$\text{(H1)} \qquad \begin{cases} \text{there are } m>0,\ R_2>0 \text{ and } R_1\in(0,R_2) \text{ such that} \\ m^2<\left(\dfrac{\pi}{T}\right)^2+\left(\dfrac{a}{2}\right)^2 \quad \text{and} \quad f_m(t,x)\geq 0 \quad \text{for} \quad t\in[0,T] \text{ and } x\in[c_m\,R_1,R_2], \end{cases}$$

where f_m is given by (2.2). Let $\widetilde{f}_m\colon[0,T]\times[0,R_2]\to\mathbb{R}$ be an arbitrary continuous and nonnegative function which coincides with f_m on $[0,T]\times[c_m\,R_1,R_2]$.

For $\rho>0$ and P as in (2.5), we define

$$B_\rho=\{x\in P\colon\|x\|<\rho\} \quad \text{and} \quad B'_\rho=\{x\in P\colon x_*<c_m\,\rho\}.$$

Let us note that the sets of the form B'_ρ were introduced and utilized already by Lan in [11]. Note that we have $B'_\rho\subset B_\rho$.

We will consider the operator

$$\widetilde{F}x(t)=\int_0^T G_m(t,s)\,\widetilde{f}_m(s,x(s))\,\mathrm{d}s \quad \text{for} \quad x\in\overline{B_{R_2}}\,. \tag{3.1}$$

where G_m is Green's function of (2.3) whose existence and properties are given by Lemma 1.

The main object of this Section is to yield sufficient conditions for the fixed point index of $\widetilde{F}$ to be 1 or 0. In order to do this, we utilize, in the spirit of the papers by Lan [12] and Webb [15], the explicit dependence of the nonlinearity f_m on t.

Lemma 4. *Assume* (H0) *and* (H1)*. Then the operator $\widetilde{F}$ maps $\overline{B_{R_2}}$ into P and is compact.*

Proof. For $x\in\overline{B_{R_2}}$ and $t\in[0,T]$, we have by Lemma 1 and construction of $\widetilde{f}_m$

$$\widetilde{F}x(t)=\int_0^T G_m(t,s)\,\widetilde{f}_m(s,x(s))\,\mathrm{d}s\geq\int_0^T G_m(s,s)\,\widetilde{f}_m(s,x(s))\,\mathrm{d}s$$

and

$$\int_0^T G_m(s,s)\,\widetilde{f}_m(s,x(s))\,\mathrm{d}s\geq c_m\int_0^T G_m(t,s)\,\widetilde{f}_m(s,x(s))\,\mathrm{d}s.$$

Therefore $\widetilde{F}x(t)\geq c_m\|\widetilde{F}x\|$ on $[0,T]$. Hence $\widetilde{F}(\overline{B_{R_2}})\subset P$. The compactness of $\widetilde{F}$ follows in a standard way from the Arzelà-Ascoli theorem. $\square$

Lemma 5. *Assume that* (H0) *and* (H1) *hold, $\widetilde{F}\,x\neq x$ for $x\in\partial B_{R_2}$ and there exists a continuous function g_1 such that*

$$f_m(t,x)\leq g_1(t) \quad \textit{for } t\in[0,T] \textit{ and } x\in[c_m\,R_2,R_2] \tag{H2}$$

and moreover either

(H3) $$\delta_* \leq c_m \, R_2,$$

or

(H4) $$\delta^* \leq R_2,$$

where

$$\delta(t) = \int_0^T G_m(t,s)\, g_1(s)\,\mathrm{d}s \quad \text{for} \quad t \in [0,T].$$

Then $i_P(\widetilde{F}, B_{R_2}) = 1$.

Proof. It follows from Lemma 4 that $\widetilde{F}$ maps $\overline{B_{R_2}}$ into P and is compact. We show that $\widetilde{F}x \neq \lambda\, x$ for $x \in \partial B_{R_2}$ and $\lambda \geq 1$, which by Theorem 3 (i) (where we put $\Omega_P = B_{R_2}$) yields $i_P(\widetilde{F}, B_{R_2}) = 1$.

If not, there exist $\lambda \geq 1$ and $x \in P$ with $\|x\| = R_2$ such that $\widetilde{F}\, x = \lambda\, x$. Since we suppose that $\widetilde{F}\, x \neq x$ for $x \in \partial B_{R_2}$, we may assume that $\lambda > 1$. Consequently, $c_m\, R_2 \leq x(t) \leq R_2$ for $t \in [0,T]$, and, since $G_m(t,s) > 0$, hypothesis (H2) implies

$$\lambda\, x(t) = \int_0^T G_m(t,s)\, \widetilde{f}_m(s, x(s))\,\mathrm{d}s \leq \int_0^T G_m(t,s)\, g_1(s)\,\mathrm{d}s = \delta(t).$$

If (H3) holds then we obtain

$$\lambda\, c_m\, R_2 \leq \lambda\, x_* = \min_{t\in[0,T]} \int_0^T G_m(t,s)\, \widetilde{f}_m(s,x(s))\,\mathrm{d}s \leq \delta_* \leq c_m\, R_2\,,$$

a contradiction since $\lambda > 1$. If (H4) holds, then taking the maximum on $[0,T]$ we obtain $\lambda\, R_2 = \lambda\, x^* \leq \delta^* \leq R_2$, a contradiction. □

Lemma 6. *Assume that* (H0) *and* (H1) *hold,* $\widetilde{F}\, x \neq x$ *for* $x \in \partial B'_{R_1}$ *and there exists a continuous function* g_0 *such that*

(H5) $$f_m(t,x) \geq g_0(t) \geq 0 \quad \text{for } t \in [0,T] \quad \text{and } x \in [c_m\, R_1, R_1],$$

and moreover either

(H6) $$\gamma_* \geq c_m\, R_1,$$

or

(H7) $$\gamma^* \geq R_1,$$

where

$$\gamma(t) = \int_0^T G_m(t,s)\, g_0(s)\,\mathrm{d}s \quad \text{for} \quad t \in [0,T].$$

Then $i_P(\widetilde{F}, B'_{R_1}) = 0$.

Proof. By Lemma 4, the operator $\widetilde{F}:\overline{B'_{R_1}}\subset\overline{B_{R_1}}\to P$ is compact. Put $x_0(t)=1$ for $t\in[0,T]$. Then $x_0\in P$ and we shall prove that $x-\widetilde{F}\,x\neq\lambda\,x_0$ for all $x\in\partial B'_{R_1}$ and $\lambda\geq 0$, which by Theorem 3 (ii) (where we put $\Omega'_P=B'_{R_1}$) implies that $i_P(\widetilde{F},B'_{R_1})=0$.

If not, there exist $x\in\partial B'_{R_1}$ and $\lambda\geq 0$ such that $x=\widetilde{F}\,x+\lambda\,x_0$. Due to our assumption $\widetilde{F}\,x\neq x$ for $x\in\partial B'_{R_1}$, it is enough to consider $\lambda>0$. Furthermore, note that $x\in\partial B'_{R_1}$ means that $x\in P$ and $x_*=c_m\,R_1$.

In particular, $x\in\partial B'_{R_1}$ implies that $c_m\,R_1\leq x(t)\leq R_1$ for $t\in[0,T]$. Then due to (H5) and the positivity of Green's function G_m we have for $t\in[0,T]$ that

$$x(t)=\widetilde{F}\,x(t)+\lambda x_0(t)=\int_0^T G_m(t,s)\widetilde{f}_m(s,x(s))\,\mathrm{d}s+\lambda$$
$$\geq\int_0^T G_m(t,s)\,g_0(s)\,\mathrm{d}s+\lambda=\gamma(t)+\lambda.$$

Now, if (H6) holds then, using Lemma 1 we get the contradiction

$$c_m\,R_1=x_*\geq\gamma_*+\lambda>\gamma_*\geq c_m\,R_1.$$

On the other hand, if (H7) holds, using again Lemma 1 we arrive at the contradiction

$$R_1\geq x^*\geq\gamma^*+\lambda>\gamma^*\geq R_1.$$ □

Remark 7. Changing slightly the proof of Lemma 6 we could prove that $i_P(\widetilde{F},B_{R_1})=0$ holds also under (H0),(H1),(H5) and (H7). Furthermore, it is worth mentioning that (H7) is equivalent to

(H7') $$\text{there exists } t_0\in[0,T] \text{ such that } \gamma(t_0)\geq R_1,$$

a condition that is, in general, more readily verifiable.

4. Main Results

Now we are ready to formulate and prove our main results.

Theorem 8. *If the hypotheses* (H0)-(H3), (H5) *and* (H6) *hold, then problem* (1.3) *has at least one solution x such that $c_m\,R_1\leq x(t)\leq R_2$ on $[0,T]$. The same conclusion remains true with* (H3) *replaced by* (H4) *and/or* (H6) *replaced by* (H7).

Proof. Let the operators F and $\widetilde{F}$ be respectively given by (2.6) and (3.1). Recall that F and $\widetilde{F}$ coincide on $\overline{B_{R_2}}\setminus B'_{R_1}$ and thus each fixed point of $\widetilde{F}$ in $\overline{B_{R_2}}\setminus B'_{R_1}$ is a solution of (1.3).

Now, if $\widetilde{F}\,x = x$ for some $x \in \partial\, B_{R_2} \cup \partial\, B'_{R_1}$, then this x is a solution to our problem. On the other hand, if $\widetilde{F}\,x \neq x$ for all $x \in \partial\, B_{R_2} \cup \partial\, B'_{R_1}$, then, $i_P(\widetilde{F}, B_{R_2}) = 1$ by Lemma 5 and $i_P(\widetilde{F}, B'_{R_1}) = 0$ by Lemma 6. Therefore, by Theorem 2 (where we put $\Omega_P = B_{R_2}$ and $\Omega'_P = B'_{R_1}$), the operator $\widetilde{F}$ has a fixed point in $B_{R_2} \setminus \overline{B'_{R_1}}$. To summarize, problem (1.3) has a solution $x \in \overline{B_{R_2}} \setminus B'_{R_1}$. Finally, notice that $x \in \overline{B_{R_2}} \setminus B'_{R_1}$ if and only if $x \in P$ and $c_m\, R_1 \leq x(t) \leq R_2$ on $[0, T]$. This completes the proof of the theorem. □

Remark 9. We would like to emphasize that the assumptions of Theorem 8 are weaker than those of Theorem 3.2 of [9]. In particular, if $f_m(t, x) \leq m^2 R_2$ for $t \in [0, T]$ and $x \in [c_m\, R_2, R_2]$ then (H2) and (H4) of Theorem 8 are satisfied with $g_1(t) = m^2 R_2$. Clearly, hypotheses (H2)–(H4) also hold if there exists a continuous function $\hat{g}$, such that $f_m(t, x) \leq \hat{g}(t) R_2$ for $t \in [0, T]$ and $x \in [c_m\, R_2, R_2]$ with

$$\min_{t\in[0,T]} \int_0^T G_m(t, s)\, \hat{g}(s)\, \mathrm{d}s \leq c_m$$

or

$$\max_{t\in[0,T]} \int_0^T G_m(t, s)\, \hat{g}(s)\, \mathrm{d}s \leq 1.$$

Moreover, if $f_m(t, x) \geq m^2 R_1$ for $t \in [0, T]$ and $x \in [c_m\, R_1, R_1]$ then (H5)-(H7) of Theorem 8 are fulfilled with $g_0(t) = m^2\, R_1$. For similar type of comparisons see [12, 15].

Remark 10. Notice that the nature of our approach does not allow us to derive direct conclusions regarding the uniqueness of the solution in Theorem 8. On the other hand, Theorem 8 provides a localization $x \in \overline{B_{R_2}} \setminus B'_{R_1}$ of the corresponding solution x to problem (1.3), which means that

$$x \in P \quad \text{and} \quad c_m\, R_1 \leq x(t) \leq R_2 \;\; \text{for } t \in [0, T]. \tag{4.1}$$

Having in mind Remark 7, it is not difficult to modify the proof of Theorem 8 so that, under the assumptions (H0), (H1), (H2), (H3) (or (H4)), (H5) and (H7), we obtain the existence of a solution to problem (1.3) in the set $\overline{B_{R_2}} \setminus B_{R_1}$, where in comparison with the localization from Theorem 8, the set B'_{R_1} is replaced by B_{R_1}. Observe that this means that the corresponding solution x of the given problem will satisfy in addition to (4.1) also the condition $x^* \geq R_1$.

Next, we will apply Theorem 8 to provide sufficient conditions for the existence of positive solutions of problem (1.2), which is a special case of (1.3) with

$$r(t) = \frac{e(t)}{\mu}, \quad s(t) = \frac{c}{\mu}, \quad \alpha = 1 - 2\mu \text{ and } \beta = 1 - \mu\,.$$

Recall that it was proved in [8] that the necessary condition for the existence of a positive solution of (1.2) is $\overline{e} > 0$, and all existence theorems known up to now (cf. [7, 8, 9]) required e to be strictly positive on $[0, T]$. However, it is easy to verify that for

$$e(t) = 0.1V_0 + 1.8 + (2.1 - V_0)\cos t - 3\cos^2 t, T = 2\,\pi, a = 0, b = 2 \text{ and } c = 0.1, \tag{4.2}$$

with $V_0 > 3$, we have $\overline{e} > 0$ and $e_* < 0$, while

$$u(t) = (V_0 - 2) + \cos t,$$

is a solution to the corresponding problem (1.1) (Example (4.2) is in fact, a slight modification of the example by G. Propst from [6, (19)]).This indicates that the positivity of e can not be a necessary condition for the existence of positive solutions to problem (1.1) and it should be weakened. To our knowledge, next existence principle allows to deal for the first time also with the case $e_* \le 0$.

Theorem 11. *Suppose that*

(C0) $\quad a \ge 0,\ 0 < \mu < \frac{1}{2},\ c > 0,\ T > 0,$ *and* $e : [0, T] \to \mathbb{R}$ *is continuous.*

Moreover, assume that there are $m > 0$, $\kappa > 0$, *and* $R_1, R_2 > 0$ *with* $R_1 < R_2$ *such that* (2.4) *holds and*

(C1) $$e_* \le 0 \quad \text{and} \quad (c_m\, R_1)^{\mu} \ge \frac{c + \sqrt{c^2 - 4\,\mu\, m^2 e_*}}{2\mu m^2}\,,$$

(C2) $$(c_m R_1)^{1-2\mu} \ge \kappa\, \mu\,,$$

(C3) $$\int_0^T G_m(s, s)\, e_+(s)\, \mathrm{d}s \ge \frac{c_m\, R_1}{\kappa}\,,$$

(C4) $$e^* \le c\, R_2^{\mu}\,.$$

Then there exists a positive solution of problem (1.2).

Proof. We have $f_m(t, x) = \dfrac{e(t)}{\mu}\, x^{1-2\mu} - \dfrac{c}{\mu}\, x^{1-\mu} + m^2 x$. We will check the hypotheses (H0)–(H2), (H4)–(H6) of Theorem 8. Put $g_1(t) = m^2 R_2$ and $g_0(t) = \kappa\, e_+(t)$ for $t \in [0, T]$.

Clearly, (H0) follows from (C0). We will show that (H1) holds, as well. Indeed, (C1) gives $\mu\, m^2 x^{2\mu} - c\, x^{\mu} + e_* \geq 0$ on $[c_m\, R_1, R_2]$. Hence $x^{1-2\mu}\, (\mu\, m^2 x^{2\mu} - c\, x^{\mu} + e_*) \geq 0$ on $[c_m\, R_1, R_2]$. Consequently, we have

$$f_m(t, x) \geq \frac{e_*}{\mu}\, x^{1-2\,\mu} - \frac{c}{\mu}\, x^{1-\mu} + m^2 x \geq 0 \quad \text{for} \quad t \in [0, T] \quad \text{and} \quad x \in [c_m R_1, R_2].$$

This, together with (2.4) means that (H1) holds.

Now, we will show that (H2) follows from $\mu \in (0, \frac{1}{2})$, (C1) and (C4). We need to prove that

$$e^* \leq \mu\, m^2 R_2\, x^{2\mu-1} + c\, x^{\mu} - \mu\, m^2 x^{2\mu}$$

for $x \in [c_m R_2, R_2]$. The function $\mu\, m^2 R_2\, x^{2\mu-1} + c x^{\mu} - \mu\, m^2 x^{2\mu}$ is decreasing on $[c_m R_2, R_2]$, since by $\mu \in (0, \frac{1}{2})$ and (C1) we get

$$(\mu\, m^2 R_2\, x^{2\mu-1} + c\, x^{\mu} - \mu\, m^2 x^{2\mu})' = \mu\, x^{\mu-1}\, ((2\mu - 1)\, m^2 R_2\, x^{\mu-1} + c - 2\mu\, m^2 x^{\mu}) < 0\, .$$

From (C4) we obtain

$$e^* \leq c\, R_2^{\mu} \leq \mu\, m^2 R_2\, x^{2\mu-1} + c\, x^{\mu} - \mu\, m^2 x^{2\mu},$$

which proves (H2). Due to Lemma 1, $\delta^* = R_2$ in this case. Hence, (H4) is fulfilled, as well.

Let $t \in [0, T]$ be such that $e(t) < 0$. Then, since $e_+(t) = 0$ in this case, the inequality in (H5) obviously holds. Next, let $t \in [0, T]$ be such that $e(t) \geq 0$. Then we need to show that

$$e(t)\, (x^{1-2\mu} - \kappa\, \mu) \geq x^{1-\mu}\, (c - \mu\, m^2 x^{\mu}) \tag{4.3}$$

for $x \in [c_m\, R_1, R_1]$. Since

$$\frac{c + \sqrt{c^2 - 4\mu\, m^2 e_*}}{2\mu\, m^2} \geq \frac{c}{\mu\, m^2}\, ,$$

from (C1) and (C2) we get $c - \mu\, m^2 x^{\mu} \leq 0$ and $x^{1-2\mu} - \kappa\, \mu\ \geq 0$ for $x \in [c_m\, R_1, R_1]$. This gives (4.3), and therefore (H5) holds. (H6) follows immediately from (C3). □

Remark 12. Notice that in Theorem 11 the smaller are the difference $e^* - e_*$ and the period T, the greater is the chance to find proper m, κ, R_1 and R_2.

Moreover, one can verify that example (4.2) mentioned above does not satisfy the assumptions of Theorem 11. Possible extensions of our existence principle so that they would cover also such examples remain an open problem.

Next example is an illustrative application of Theorem 11.

Example 13. Let us consider problem (1.2) with the parameter values $a = 1.6$, $\mu = 0.01$, $c = 0.005$, $T = 1$ and being e the continuous periodic function defined on $[0, 1]$ as follows

$$e(t) = \begin{cases} e_* + \dfrac{e^* - e_*}{t_1}\, t \quad \text{for} \quad t \in [0, t_1), \\ e^* \quad \text{for} \quad t \in [t_1, t_2), \\ e^* - \dfrac{e^* - e_*}{1 - t_2}(t - t_2) \quad \text{for} \quad t \in [t_2, 1], \end{cases}$$

where $t_1 = 0.0005$, $t_2 = 0.9995$, $e^* = 0.00548239$ and $e_* = -0.00005$.

Now, defining $m = 0.7$, $R_1 = 25$, $R_2 = 10000$ and $\kappa = 2200$, direct computations show that all the assumptions of Theorem 11 are satisfied and therefore there exists a positive solution of problem (1.2) (note that, using explicit formulas from Appendix A.1 of [9] and the Mathematica software, we can for $a = 1.6$ and $m = 0.7$ approximate the value of c_m by 0.94144. Similarly we get $G_m(s, s) \approx 1.96026$).

In [9, Example 3.7] we showed that our main result of [9], namely Theorem 3.2, is more general than [8, Theorem 1.8] (see also [9, Corollary 3.6]) when applied to problem (1.2). However, since the function $e(t) \equiv 1.54$ used in Example 3.7 is constant, problem (1.2) has a constant solution, namely

$$x(t) = \left(\frac{e(t)}{c}\right)^{\frac{1}{\mu}} \equiv \left(\frac{1.54}{1.49}\right)^{100} = 27.1297.$$

Although [9, Example 3.7] is correct, the case of e constant is not meaningful for the pumping effect in the studied configuration (see [8, Definition 1.2]). Now we present an application of Theorem 8 to problem (1.2) that allows a nonconstant positive function e which, as it will be shown in Example 15, does not satisfy the assumptions of [8, Theorem 1.8].

Theorem 14. *Suppose that* (C0) *of Theorem 11 holds, and moreover:*

(C5) $\qquad e_* > 0,$

(C6) $\qquad$ *there is* $m > 0$ *such that* $\dfrac{c^2}{\mu\, e_*^2}\left(\dfrac{1}{c_m^{\mu}}\, e^* - e_*\right) \le m^2 < \left(\dfrac{\pi}{T}\right)^2 + \left(\dfrac{a}{2}\right)^2.$

Then problem (1.2) *has a positive solution* x *such that*

$$0 < x(t) \le \frac{1}{c_m}\left(\frac{e^*}{c}\right)^{\frac{1}{\mu}} \quad \textit{for all} \;\; t \in [0, T].$$

Proof. We will verify that the assumptions of Theorem 8 are satisfied.

Claim 1. (H0) *and* (H1) *are true.*

Clearly, (H0) is a consequence of (C0). Furthermore, let m^2 be as in (C6) and let

$$R_2 = \frac{1}{c_m} \left(\frac{e^*}{c} \right)^{\frac{1}{\mu}}.$$

We have

$$f_m(t,x) = \frac{e(t)}{\mu}\, x^{1-2\mu} - \frac{c}{\mu}\, x^{1-\mu} + m^2 x \quad \text{for} \;\; t \in [0,T] \;\; \text{and} \;\; x \geq 0.$$

Notice that

$$(4.4) \qquad \begin{cases} f_m(t,x) \geq f_*(x) \quad \text{for all} \;\; t \in [0,T] \text{ and } \;\; x \geq 0, \quad \text{where} \\ f_*(x) = \dfrac{e_*}{\mu}\, x^{1-2\mu} - \dfrac{c}{\mu}\, x^{1-\mu} + m^2 x = \dfrac{x^{1-2\mu}}{\mu} \, \left(e_* - c\, x^{\mu} + m^2 \mu\, x^{2\mu} \right) . \end{cases}$$

Obviously, $f_*(x) \geq 0$ whenever $x \in [0, \left(\frac{e_*}{c}\right)^{\frac{1}{\mu}}]$. Let $x \in [\left(\frac{e_*}{c}\right)^{\frac{1}{\mu}}, R_2]$. Taking into account (C6), we get

$$\begin{aligned} \mu\, x^{2\mu-1} f_*(x) &= e_* - c\, x^{\mu} + m^2 \mu\, x^{2\mu} \geq e_* - c\, R_2^{\mu} + m^2 \mu \left(\left(\frac{e_*}{c} \right)^{\frac{1}{\mu}} \right)^{2\mu} \\ &= e_* - \frac{e^*}{c_m^{\mu}} + m^2 \mu \left(\frac{e_*}{c} \right)^2 = \mu \left(\frac{e_*}{c} \right)^2 \left(m^2 - \frac{c^2}{\mu\, e_*^2} \left(\frac{e^*}{c_m^{\mu}} - e_* \right) \right) \geq 0. \end{aligned}$$

Thus $f_*(x) \geq 0$ for all $x \in [0, R_2]$ and hence

$$f_m(t,x) \geq 0 \quad \text{for} \;\; t \in [0,T] \;\; \text{and} \;\; x \in [0, R_2].$$

In particular, we conclude that (H1) is true (with an arbitrary $R_1 \in (0, R_2)$).

Claim 2. (H2) *and* (H4) *are true.*

First, we will prove that $f_m(t,x) \leq m^2 R_2$ for $t \in [0,T]$ and $x \in [c_m\, R_2, R_2]$. Indeed, for $t \in [0,T]$ and $x \geq 0$ we have

$$f_m(t,x) \leq \frac{e^*}{\mu}\, x^{1-2\mu} - \frac{c}{\mu}\, x^{1-\mu} + m^2 x\, .$$

Furthermore, if $x \geq c_m\, R_2 = \left(\dfrac{e^*}{c} \right)^{\frac{1}{\mu}}$, then

$$\frac{e^*}{\mu}\, x^{1-2\mu} - \frac{c}{\mu}\, x^{1-\mu} = \frac{x^{1-2\mu}}{\mu} \left(e^* - c\, x^{\mu} \right) \leq \frac{x^{1-2\mu}}{\mu} \left(e^* - c\, \frac{e^*}{c} \right) = 0\, .$$

Therefore, for $t \in [0,T]$ and $x \in [c_m\, R_2, R_2]$ we have

$$f_m(t,x) \le \frac{e^*}{\mu}\, x^{1-2\mu} - \frac{c}{\mu}\, x^{1-\mu} + m^2 x \le m^2 x \le m^2 R_2.$$

Now, it is easy to show (see also Remark 9) that (H2) and (H4) are true with $g_1(t) = m^2 R_2$.

Claim 3. (H5) *and* (H6) *are true.*

First, we will show that there is $R_1 \in (0, R_2)$ such that $f_m(t,x) \ge m^2 R_1$ for $t \in [0,T]$ and $x \in [c_m R_1, R_1]$. Indeed, due to (C5) we can choose $R_1 \in (0, R_2)$ in such a way that

$$(1 - c_m)\, m^2 \mu R_1^{2\mu} + c R_1^{\mu} \le e_*\, c_m^{1-2\mu}. \tag{4.5}$$

On the other hand, for $t \in [0,T]$ and $x \in [c_m\, R_1, R_1]$ we deduce

$$\begin{aligned} f_*(x) &= \frac{e_*}{\mu} x^{1-2\mu} - \frac{c}{\mu} x^{1-\mu} + m^2 x \ge \frac{e_*}{\mu} (c_m\, R_1)^{1-2\mu} - \frac{c}{\mu} R_1^{1-\mu} + m^2 c_m\, R_1 \\ &= \frac{R_1^{1-2\mu}}{\mu} (e_*\, c_m^{1-2\mu} - c\, R_1^{\mu} + c_m m^2 \mu R_1^{2\mu}) \end{aligned}$$

and hence, by (4.5),

$$f_*(x) \ge m^2 R_1,$$

wherefrom, according to (4.4), the desired inequality immediately follows. Now, it is easy to show (see also Remark 9) that (H5) and (H6) are true with $g_0(t) = m^2 R_1$.

By Claims 1–3, all the assumptions of Theorem 8 are satisfied and the proof can be completed applying it. □

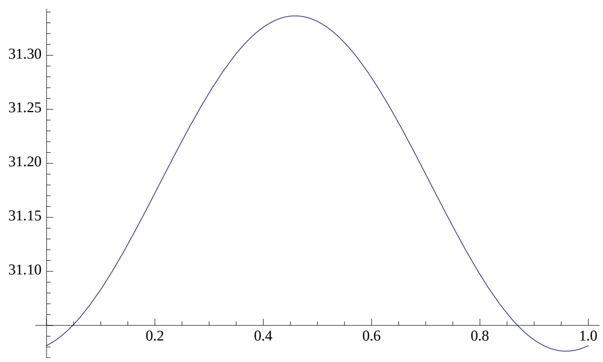
Example 15. Let us modify [9, Example 3.7] in order to admit a non-constant function e. Indeed, consider problem (1.2) with $a = 1.6$, $\mu = 0.01$, $c = 1.49$, $T = 1$ and e a continuous periodic function such that $e_* = 1.54$. Notice that in this case we cannot apply [8, Theorem 1.8] since

$$\frac{c^2}{4\,\mu\, e_*} \approx 36.0406 > 10.5096 \approx \left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4}\,.$$

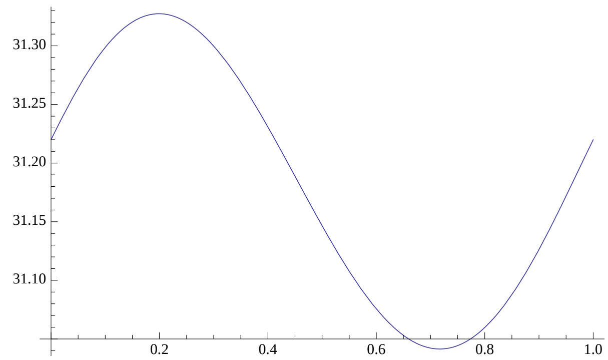
On the other hand, for $m = 0.7$ we have $c_m \approx 0.9414$ and then Theorem 14 implies that this problem has a positive solution x provided that $e^* < 1.5443$. Moreover,

$$0 < x(t) \le \frac{1}{c_m} \left(\frac{e^*}{c}\right)^{\frac{1}{\mu}} < 38.0844 \quad \text{for all} \;\; t \in [0,T].$$

Finally we show some numerical examples for particular choices of e:



Solution of problem (1.2) with $a = 1.6$, $\mu = 0.01$, $c = 1.49$, $T = 1$ and $e(t) = 1.54215 + 0.002097\cos(2\pi\, t)$



Solution of problem (1.2) with $a = 1.6$, $\mu = 0.01$, $c = 1.49$, $T = 1$ and $e(t) = 1.54215 - 0.02(t - 3t^2 + 2t^3)$

Some computations in Examples 13 and 15 were made with the help of the software system *Mathematica*.

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