Programs and Algorithms of Numerical Matematics 16
J. Chleboun, K. Segeth, J. Šístek, T. Vejchodský (Eds.)
Institute of Mathematics AS CR, Prague 2013

SUPERAPPROXIMATION OF THE PARTIAL DERIVATIVES IN THE SPACE OF LINEAR TRIANGULAR AND BILINEAR QUADRILATERAL FINITE ELEMENTS

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Abstract

A method for the second-order approximation of the values of partial derivatives of an arbitrary smooth function $u = u(x_1, x_2)$ in the vertices of a conformal and nonobtuse regular triangulation $\mathcal{T}_h$ consisting of triangles and convex quadrilaterals is described and its accuracy is illustrated numerically. The method assumes that the interpolant $\Pi_h(u)$ in the finite element space of the linear triangular and bilinear quadrilateral finite elements from $\mathcal{T}_h$ is known only.

1. Introduction

The problem to find second-order approximations of the first partial derivatives of smooth functions u in the vertices of triangulations by means of the interpolant $\Pi_h(u)$ only is actual since its formulation in [6] in the year 1967. Besides the widely acknowledged method [7] there exist successful methods like [5] and [3]. In this paper, we generalize the method of averaging from [2] to nonobtuse regular triangulations consisting of triangles as well as convex quadrilaterals in general. Numerical experiments indicate the second-order accuracy of this procedure. These high-order approximations of the partial derivatives have many applications. See [1] for some of them.

We denote $[a_1, a_2]$ the Cartesian coordinates of a point a and $|ab|$ the length of the segment $\overline{ab}$. For arbitrary points $a^1, \ldots, a^m$, operations „$+$“ and „$-$“ mean addition and subtraction modulo m on the set $\{1, \ldots, m\}$.

2. Bilinear quadrilateral finite elements

Besides the linear triangular finite elements, we work with the following bilinear quadrilateral ones.

Definition 1. A *reference bilinear finite element* consists of

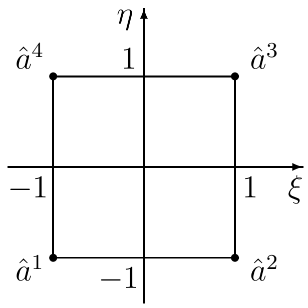

Figure 1: The reference square.

a) the *reference square* $\hat{K} = \overline{\hat{a}^1\hat{a}^2\hat{a}^3\hat{a}^4}$ from Fig. 1,

b) the *local space* $\mathbb{Q}^{(1)} = \{a + b\xi + c\eta + d\xi\eta \,|\, a, b, c, d \in \mathbb{R}\}$ and of

c) the *parameters* $\hat{p}(\hat{a}^1), \ldots, \hat{p}(\hat{a}^4)$ related to every function $\hat{p} \in \mathbb{Q}^{(1)}$. The parameters determine the function $\hat{p}$ uniquely.

Definition 2. A *bilinear quadrilateral finite element* consists of

a) an image $K = \overline{a^1a^2a^3a^4}$ of $\hat{K}$ by the injective bilinear mapping

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = F_K(\xi, \eta) \equiv \sum_{i=1}^{4} \hat{N}^i(\xi, \eta) \begin{bmatrix} a_1^i \\ a_2^i \end{bmatrix} \tag{1}$$

with the *Lagrange base functions*

$$\hat{N}^1(\xi, \eta) = (1 - \xi)(1 - \eta)/4, \qquad \hat{N}^2(\xi, \eta) = (1 + \xi)(1 - \eta)/4,$$
$$\hat{N}^3(\xi, \eta) = (1 + \xi)(1 + \eta)/4, \qquad \hat{N}^4(\xi, \eta) = (1 - \xi)(1 + \eta)/4$$

in the space $\mathbb{Q}^{(1)}$ related to the nodes $\hat{a}^1, \ldots, \hat{a}^4$ consecutively. Then $F_K(\hat{a}^i) = a^i$ for $i = 1, \ldots, 4$ obviously and F_K is an injection if and only if K is a *convex quadrilateral*, i.e. the inner angle $\angle a^{i-1}a^ia^{i+1}$ of K is less than π for $i = 1, \ldots, 4$ due to [4], Section 3.3,

b) the *local space* $\mathbb{Q}_K^{(1)} = \{q \,|\, q = \hat{q} \circ F_K^{-1}$ for some $\hat{q} \in \mathbb{Q}^{(1)}\}$ and of

c) the *parameters* $q(a^1), \ldots, q(a^4)$ related to every $q \in \mathbb{Q}_K^{(1)}$. The parameters determine the function q uniquely.

Lemma 1. *The functions* $1, x_1, x_2$ *belong to* $\mathbb{Q}_K^{(1)}$ *for every convex quadrilateral* K.

Proof. If $K = \overline{a^1a^2a^3a^4}$ is a convex quadrilateral then $\mathbb{Q}_K^{(1)} = \{q \,|\, q \circ F_K \in \mathbb{Q}^{(1)}\}$ is a direct consequence of Definition 2. This and

$$\begin{aligned} 1 \circ F_K &= 1 \in \mathbb{Q}^{(1)} \\ x_1 \circ F_K &= \hat{N}^1(\xi, \eta)a_1^1 + \ldots + \hat{N}^4(\xi, \eta)a_1^4 \in \mathbb{Q}^{(1)} \\ x_2 \circ F_K &= \hat{N}^1(\xi, \eta)a_2^1 + \ldots + \hat{N}^4(\xi, \eta)a_2^4 \in \mathbb{Q}^{(1)} \end{aligned}$$

give us the statement.

Definition 3. If K is a triangle and convex quadrilateral then we denote by $\Pi_K(u)$ the linear and bilinear interpolant of a function $u \in C(K)$ in the vertices of K, respectively.

Lemma 2. *Let us consider a bilinear quadrilateral finite element* $K = \overline{a^1a^2a^3a^4}$, $l = 1, 2$ *and a linear triangular finite element* $T_j = \overline{a^{j-1}a^ja^{j+1}}$. *Then the graph of* $\Pi_{T_j}(u)$ *is the tangent plane to that of* $\Pi_K(u)$ *at the point* a^j, *so that*

$$\frac{\partial \Pi_K(u)}{\partial x_l}(a^j) = \frac{\partial \Pi_{T_j}(u)}{\partial x_l} \quad \forall\; u \in C(K)$$

for $j = 1, \ldots, 4$.

Proof. As the functions from $\mathbb{Q}_K^{(1)}$ are linear on every side of K, $\Pi_K(u)$ is linear on the segments $\overline{a^{j-1}a^j}$ and $\overline{a^ja^{j+1}}$. Hence the segments $\overline{p^{j-1}p^j}$ and $\overline{p^jp^{j+1}}$ for $p^i = [a_1^i, a_2^i, u(a^i)]$, $i = j-1, j, j+1$, are subsets of graph$(\Pi_K(u))$. These segments belong to a unique plane. This one is the tangent plane of graph$(\Pi_K(u))$ at a^j and it contains graph$\big(\Pi_{T_j}(u)\big)$ as well. Lemma 2 follows immediately.

3. Nonobtuse regular triangulations

The symbols $\mathbb{P}^{(1)}$ and $\mathbb{P}^{(2)}$ are reserved for the spaces of real linear and quadratic polynomials in two variables and Ω for a non-empty bounded connected polygonal domain in the plane. We say that K is an *element* when K is a triangle or a convex quadrilateral, denote $|K|$ the area of K, h_K the diameter of K and ϱ_K the maximal diameter of the circles inside of K.

A system $\mathcal{T}_h$ of elements is said to be a *triangulation* of Ω when $\cup_{K\in\mathcal{T}_h} K = \overline{\Omega}$, any two different elements have disjoint interiors and any side of an element is either a side of another element or a subset of the boundary $\partial\Omega$. Let us consider a *vertex* a of (an element from) a triangulation $\mathcal{T}_h$. We call b a *neighbour* of a (in $\mathcal{T}_h$) when the segment $\overline{ab}$ is a side of an element from $\mathcal{T}_h$ and denote $\mathcal{N}_h(a)$ the set of neighbours of a in $\mathcal{T}_h$. We say that a is an *inner* and *boundary* vertex when $a \in \Omega$ and $a \in \partial\Omega$, respectively.

Definition 4. A system $\mathbf{T}$ of triangulations of Ω is said to be

a) a *family* when for every $\varepsilon > 0$ there exists $\mathcal{T}_h \in \mathbf{T}$ satisfying $h_K < \varepsilon$ for all $K \in \mathcal{T}_h$.

b) *shape-regular* when there is $\sigma > 0$ such that $\varrho_K/h_K > \sigma$ for all elements K of any triangulation from $\mathbf{T}$.

We work with a shape-regular family $\mathbf{T}$ of triangulations of Ω such that all inner angles of the triangles from any triangulation in $\mathbf{T}$ are less than or equal to the right angle. We call these triangulations *nonobtuse regular.*

4. The method of averaging

It is well-known that $\partial u/\partial x_l(a) = \partial\Pi_K(u)/\partial x_l(a) + O(h_K)$ for a vertex a of an element K from a nonobtuse regular triangulation, function $u \in C^2(K)$ and for $l = 1, 2$. We construct a weight vector such that the corresponding weighted average of the values of $\partial\Pi_K(u)/\partial x_l$ in various vertices of the elements K with vertex a approximates $\partial u/\partial x_l(a)$ with an error of the second order. A special case of this construction has been analysed in [2] for the nonobtuse regular triangulations consisting of triangles only.

Calculating the approximations of $\partial u/\partial x_l(a)$, we use local Cartesian coordinates with origin a.

Defrinition 5. Let $\mathcal{T}_h$ be a nonobtuse regular triangulation. We say that $r = (b^1, \dots, b^n)$ is a *ring* around

a) an inner vertex a of $\mathcal{T}_h$ when

a1) $\{b^1, \dots, b^n\} \supseteq \mathcal{N}_h(a)$ and

$$b^i \notin \mathcal{N}_h(a) \implies K = \overline{ab^{i-1}b^ib^{i+1}} \in \mathcal{T}_h \text{ and } \angle b^{i-1}ab^{i+1} > \pi/2,$$

a2) $\angle b^nab^1, \dots, \angle b^{n-1}ab^n$ have the same orientation and

a3) $\angle b^nab^1 + \dots + \angle b^{n-1}ab^n = 2\pi$.

b) a boundary vertex a of $\mathcal{T}_h$ when there is an inner vertex b^j such that

b1) $(b^1, \dots, b^{j-1}, a, b^{j+1}, \dots, b^n)$ is a ring around b^j with $n \geq 5$ or

b2) $\overline{ab^{j+1}b^jb^{j-1}} \in \mathcal{T}_h$ and $(b^1, \dots, b^{j-1}, b^{j+1}, \dots, b^n)$ is a ring around b^j.

We say that the triangles $U_1 = \overline{b^nab^1}, \dots, U_n = \overline{b^{n-1}ab^n}$ are *related* to r and set $H(a) = \max_{1\leq i\leq n} |ab^i|$.

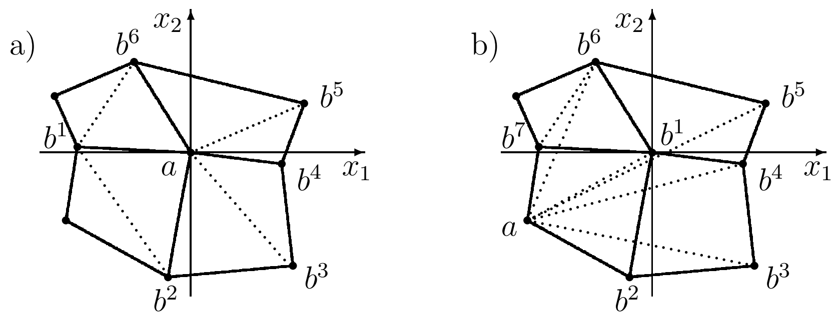

Figure 2: A ring around a) an inner vertex a and b) a boundary one.

In Fig. 2, the thick lines denote the quadrilaterals from the given triangulation and the dotted lines indicate triangles $U_1, \dots, U_6$ in the case a) and $U_1, \dots, U_7$ in b).

Definition 6. Let $l = 1, 2$, $r = (b^1, \ldots, b^n)$ be a ring around a vertex a of a nonobtuse regular triangulation and let $u \in C(\overline{\Omega})$. Then we set

$$\mathrm{B}_l[u](a) = f_1 \frac{\partial \Pi_1(u)}{\partial x_l} + \cdots + f_n \frac{\partial \Pi_n(u)}{\partial x_l}. \tag{2}$$

Here $\Pi_1(u), \ldots, \Pi_n(u)$ are the linear interpolants of u in the vertices of the triangles $U_1, \ldots, U_n$ related to r and the *weight vector* $f = [f_1, \ldots, f_n]^\top$ is the minimal 2-norm vector such that $\mathrm{B}_l[u](a)$ is *consistent*, i.e. $\mathrm{B}_l[u](a) = \partial u/\partial x_l(a)$ for all $u \in \mathbb{P}^{(2)}$. Due to [2], f is the minimal 2-norm solution of the equations $M(r)f = d$ with

$$M(r) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \frac{x_n^2 y_1 - x_1^2 y_n}{D_1} & \frac{x_1^2 y_2 - x_2^2 y_1}{D_2} & \cdots & \frac{x_{n-1}^2 y_n - x_n^2 y_{n-1}}{D_n} \\ \frac{y_n y_1 (x_n - x_1)}{D_1} & \frac{y_1 y_2 (x_1 - x_2)}{D_2} & \cdots & \frac{y_{n-1} y_n (x_{n-1} - x_n)}{D_n} \\ \frac{y_n y_1 (y_n - y_1)}{D_1} & \frac{y_1 y_2 (y_1 - y_2)}{D_2} & \cdots & \frac{y_{n-1} y_n (y_{n-1} - y_n)}{D_n} \end{bmatrix}, \qquad d = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$[x_i, y_i] = b^i$ and $D_i = D(a, b^{i-1}, b^i)$ for $i = 1, \ldots, n$.

Definition 5 is in agreement with Lemma 2 and with the following statement:

Lemma 3. *The system of equations $M(r)f = d$ related to the ring $r = (b^1, \ldots, b^4)$ around a vertex a is*

a) unsolvable if a is a boundary vertex and

b) solvable if and only if the vertices b^1, a, b^3 as well as b^2, a, b^4 are situated on one straight-line if a is an inner vertex.

We omit the proof of Lemma 3.

Example. For $a = [0, 0]$, we approximate the partial derivative $\partial u/\partial x_1(a) = -0.5403023$ of $u(x_1, x_2) = \sin(1 + 2x_1 + x_2)/(x_2 - 2)$ by $B_1[u](a)$. In Table 1, we use the ring from Fig. 2 a) with $H(a) = 1.3453624/2^i$ for $i = 1, \ldots, 8$.

i	$H(a)$	$B_1[u](a)$	$\partial u/\partial x_1(a) - B_1[u](a)$
1	6.72681 e-1	-0.460947	-7.93549 e-2
2	3.36341 e-1	-0.519906	-2.03960 e-2
3	1.68170 e-1	-0.535183	-5.11974 e-3
4	8.40852 e-2	-0.539023	-1.27939 e-3
5	4.20426 e-2	-0.539983	-3.19584 e-4
6	2.10213 e-2	-0.540222	-7.98508 e-5
7	1.05106 e-2	-0.540282	-1.99563 e-5
8	5.25532 e-3	-0.540297	-4.98822 e-6

Table 1

i	$H(a)$	$B_1[u](a)$	$\partial u/\partial x_1(a) - B_1[u](a)$
1	1.15244	-0.	-0.104569 e-1
2	5.76222 e-1	-0.577975	3.76723 e-2
3	2.88111 e-1	-0.556928	1.66261 e-2
4	1.44055 e-1	-0.545228	4.92589 e-3
5	7.20277 e-2	-0.541620	1.31737 e-3
6	3.60138 e-2	-0.540642	3.39385 e-4
7	1.80069 e-2	-0.540388	8.60568 e-5
8	9.00346 e-3	-0.540324	2.16627 e-5

Table 2

In Table 2, we use the ring from Fig. 2 b) with $H(a) = 2.3048861/2^i$ for $i = 1, \dots, 8$.

This example indicates the second order of error of the approximations $\mathrm{B}_l[u](a)$ both for the inner and the boundary vertices a, but an analysis of the accuracy of this averaging operator is necessary.

Acknowledgements

This outcome has been achieved with the financial support of the Ministry of Education, Youth and Sports, project No. 1M0579, within the activities of the CIDEAS Research Centre.

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