Effective Multiplication by Wavelet Matrix

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1. Problem and methods
2. 1D problem
3. Higher dimensional problem – design of the implementation
    - Approximate evaluation of the right-hand side
    - Data structures
    - Preconditioning
    - Approximate matrix multiplication

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Dirichlet problem

$$-\sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} + cu = f \qquad \text{on } \Omega = (0,1)^d$$

$$u = 0 \qquad \text{on } \partial\Omega$$

Galerkin method with a wavelet basis

$$\{\psi_i\}_{i=0}^{N-1}$$

Preconditioning

$$Au = f \rightarrow \quad (D^{-\frac{1}{2}} A D^{-\frac{1}{2}})(D^{\frac{1}{2}} u) = D^{-\frac{1}{2}} f$$

with D be a diagonal of A.

Iteration process

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$$D^{\frac{1}{2}} u^{n+1} = D^{\frac{1}{2}} u^{n} + \omega \left(D^{-\frac{1}{2}} f - (D^{-\frac{1}{2}} A D^{-\frac{1}{2}})(D^{\frac{1}{2}} u^{n}) \right)$$

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1D wavelet basis of quadratic splines, $N = 2^l$, l is a number of levels

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higher dimensional basis (anisotrophic wavelet basis)

$$\{\psi_{i_1} \times \psi_{i_2} \times \cdots \times \psi_{i_d}\}_{i_1,i_2,\ldots,i_d=0}^{N-1}$$

$\psi_0, \ldots, \psi_7$ are so called scaling functions

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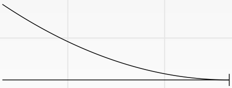
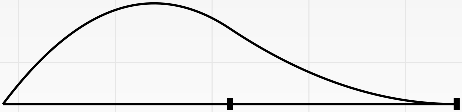
$$\psi_0(x) = \varphi_{sc\_bd}(8x), \quad \psi_7(x) = \varphi_{sc\_bd}(8(1-x))$$



$\frac{1}{2} - x + \frac{1}{2}x^2$



$2x - \frac{3}{2}x^2$ $\quad$ $\frac{1}{2} - (x-1) + \frac{1}{2}(x-1)^2$

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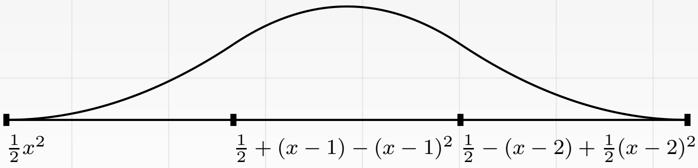
$$\text{for } i = 1..6 \qquad \psi_i(x) = \varphi_{sc\_in}(8x - i + 1)$$



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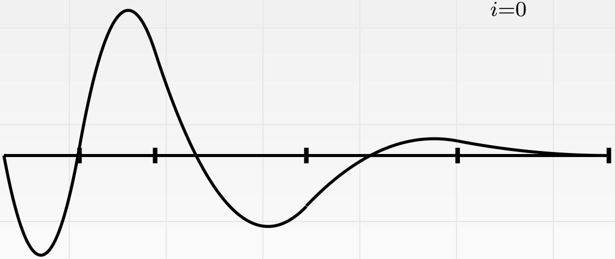
$$\psi_8(x) = a_0 \varphi_{sc\_bd}(16x) + \sum_{i=0}^{7} a_i \varphi_{sc\_in}(16x - i)$$



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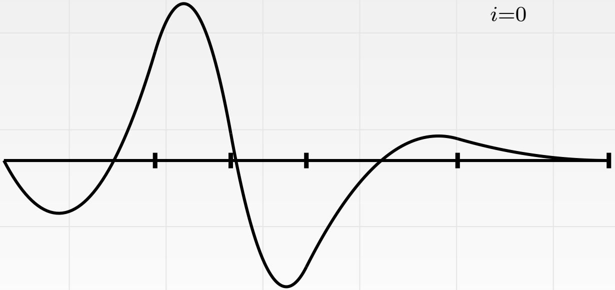
$$\psi_9(x) = b_0\varphi_{sc\_bd}(16x) + \sum_{i=0}^{7} b_i\varphi_{sc\_in}(16x - i)$$



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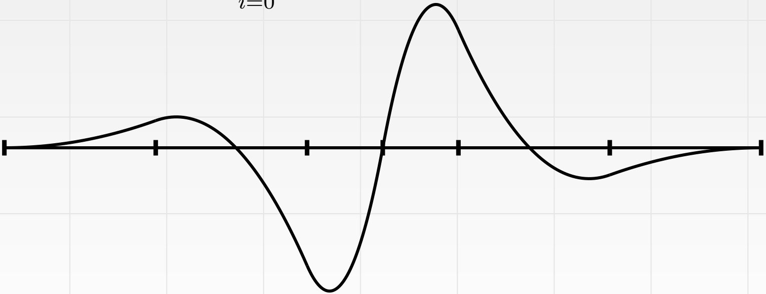
$$\psi_k(x) = \sum_{i=0}^{9} c_i \varphi_{sc\_in}(16x - i - 2k) \quad \text{for } k = 10..13$$



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\begin{aligned}
\psi_{14}(x) &= \psi_9(1-x) \\
\psi_{15}(x) &= \psi_8(1-x) \\
\psi_{16}(x) &= \psi_8(2x) \\
\psi_{17}(x) &= \psi_9(2x) \\
\psi_k(x) &= \psi_{10}(2x-k+18) \quad \text{for } k = 18..29 \\
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Moment of a function f of n-th order is

$$\int_{\mathbb{R}} x^n f(x)\,\mathrm{d}x$$

For $n = 0, 1, 2,\ i = 8, \ldots$

$$\int_{\mathbb{R}} x^n \psi_i(x)\,\mathrm{d}x = 0.$$

Corollary:

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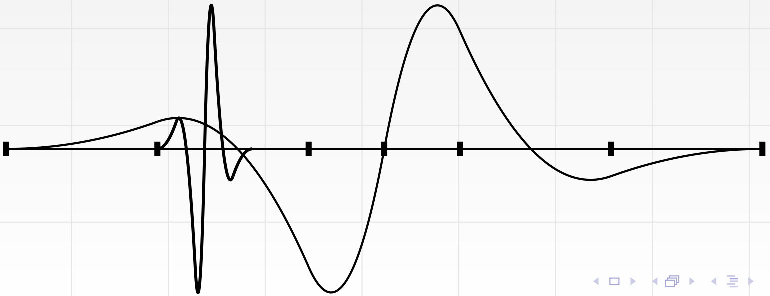
Corollary: $\int \psi_i(x)\psi_j(x)\,\mathrm{d}x = 0.$



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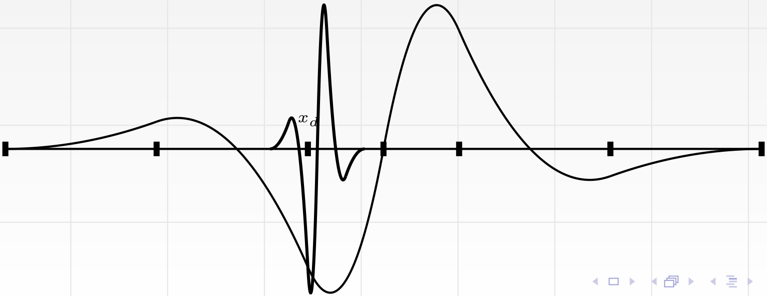
Corollary: $\int_{\mathbb{R}} \psi_i(x)\psi_j(x)\,\mathrm{d}x = a\int_{[x_d,+\infty)} (x - x_d)^2 \psi_j(x)\,\mathrm{d}x$.



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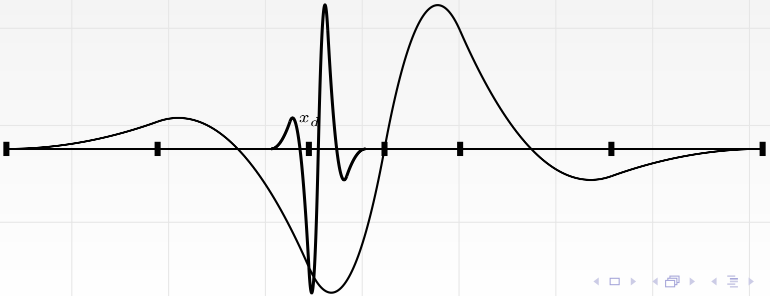

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$$d_{ij} = \int_0^1 \psi_i'(x)\psi_j'(x)\,\mathrm{d}x \qquad g_{ij} = \int_0^1 \psi_i(x)\psi_j(x)\,\mathrm{d}x$$

k is the length of a wavelet,

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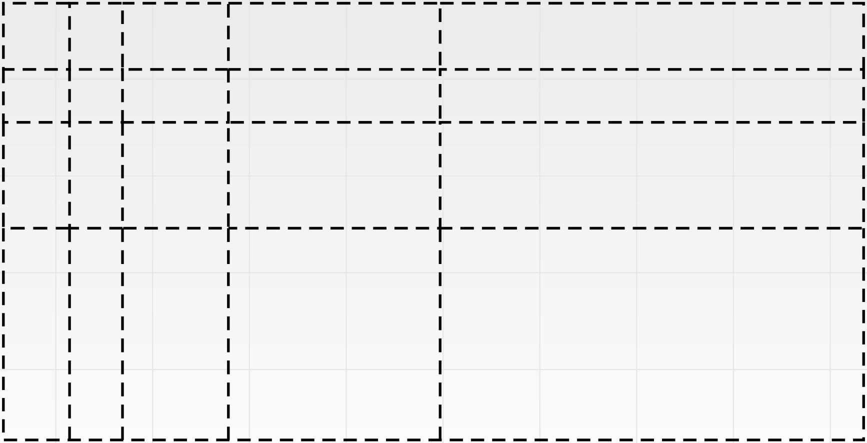

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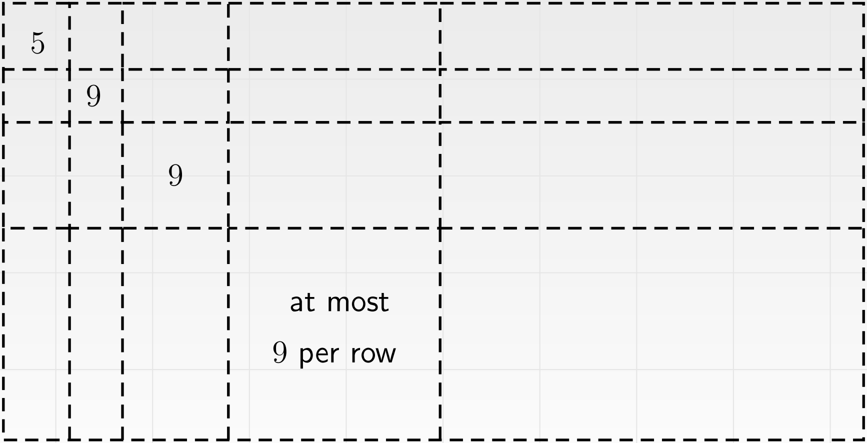

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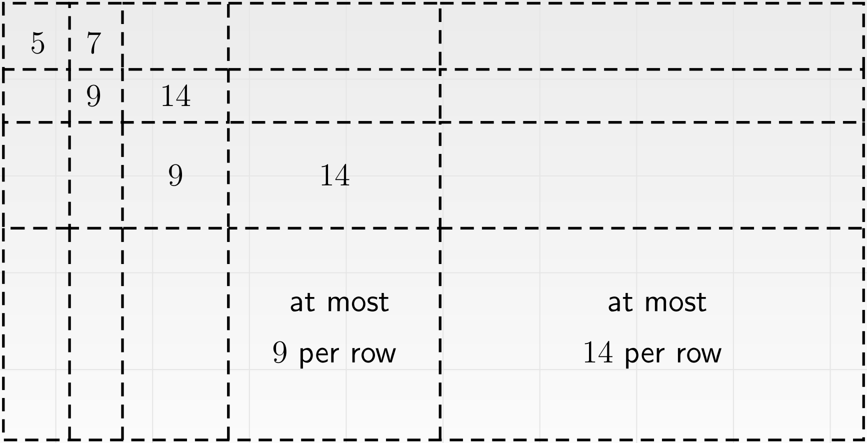

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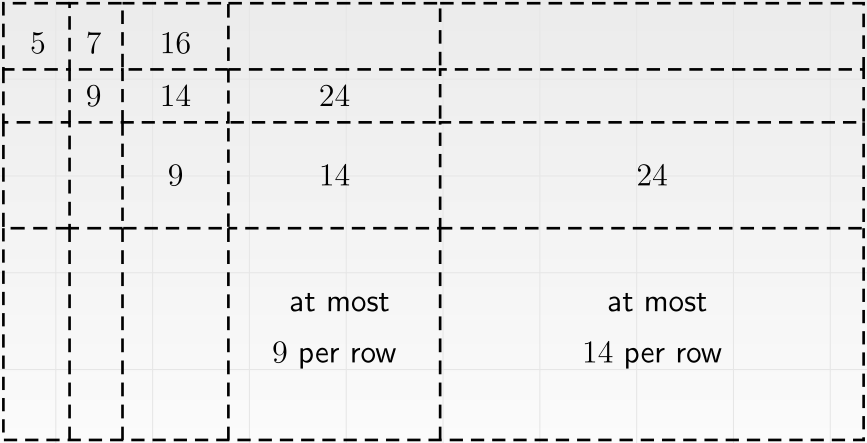

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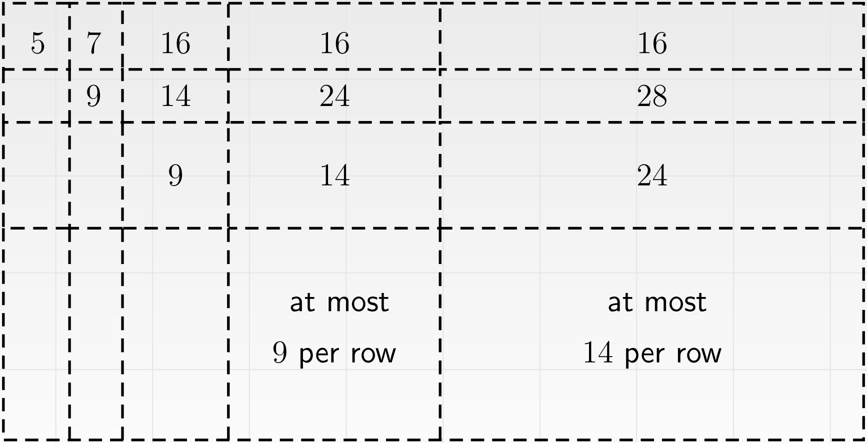

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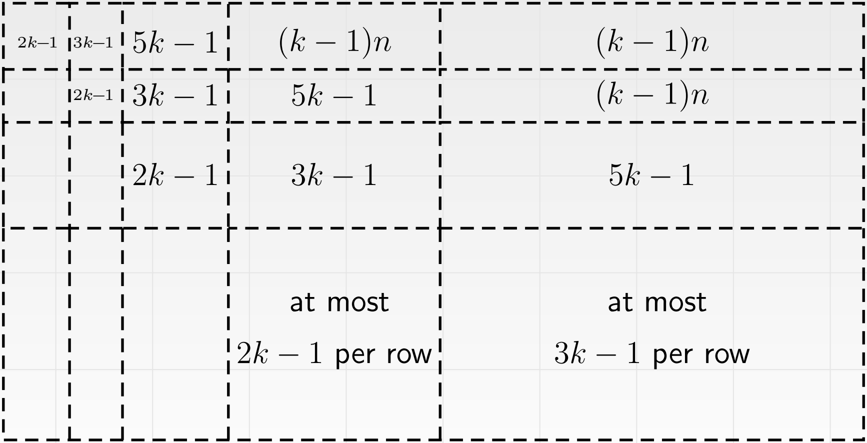

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Theorem. Matrices D and G of the order $N = 2^n$ have at most $0.5(15k - 5 + kl - l)N$ number of nonzero coefficients.

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Given the number of levels l and ε as an order of accuracy

for $i_1, \ldots, i_d = 0..(2^l - 1)$ we evaluate

$$f_{i_1 \ldots i_d} = \int_{\mathrm{supp}\psi_{i_1} \times \cdots \times \mathrm{supp}\psi_{i_d}} f(x_1, \ldots, x_d)\psi_{i_1}(x_1) \ldots \psi_{i_d}(x_d)\,\mathrm{d}x$$

by Simpson rule for two divisions

- $k = 2^5$ and $2k$ nodes at each of $\mathrm{supp}\psi_{i_k}$, $k = 1..d$
- estimate error
- repeat $k \to 2k$ until error $< \varepsilon$.

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During evaluation of right-hand side we

- calculate $\|f\|_{l_2}$,
- sort f_i by heapsort (with limited heap size and merging them) sort them according their absolute value from smallest store them as a couple (value, index)

Smallest values we put zero

- $sum \leftarrow 0$, $i \leftarrow 0$
- $sum \leftarrow sum + v_i^2$
- while $sum < \varepsilon^2 \|f\|_{l_2}^2$ do $v_i \leftarrow 0$, $i \leftarrow i + 1$, $sum \leftarrow sum + v_i^2$
- while $i \leq maximal\_value$ put c_i to a linked list, $i \leftarrow i + 1$

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We use 1D stifness matrix for multiplication. We store it in a relatively small structure (approximately 1000 elements).

Possibilities how to store right-hand side and iterations:

1. To store them in an associative C++ container.
2. To store them in blocks in d-dimensional arays.
   To store pointers to blocks in an aray of size l^d.
   To store coefficients of nonzero elements of an iteration in a linked list.
3. To store couples (index, value) in a smaller structure with index hashed.

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We normalize basis functions

$$\int_0^1 (\psi_i(x))^2 \, \mathrm{d}x = 1$$

Diagonal element of the matrix (we display it for the dimension $d = 3$)

$$D \times G \times G \; + \; G \times D \times G \; + \; G \times G \times D \; + \; c \, G \times G \times G$$

is then

$$a_{i_1 i_1 \ldots i_d i_d} = c + \sum_{k=1}^{d} d_{i_d i_d}$$

We store d_{ii} for first two levels – scaling functions and first level of wavelets (16 elements). Others can be easily calculated – they grows 4 times from one to finer level.

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$$\int_0^1 (\psi_i(x))^2 \, \mathrm{d}x = 1$$

Diagonal element of the matrix (we display it for the dimension $d = 3$)

$$D \times G \times G \; + \; G \times D \times G \; + \; G \times G \times D \; + \; c\, G \times G \times G$$

is then

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Strategy: given ε as an order of accuracy
multiply an element of a value v with blocks with elements $\geq \frac{\varepsilon}{|v|}$.

We evaluate max to every block (maximal absolute value of its elements) of matrices D and G before the iteration process starts. From them we calculate max for tenzor products of blocks.

Blocks B_{ij} we for now index by one-dimensional indeces $i, j = 0..(l^d - 1)$ and we store $max(B_{ij})$ in two-dimensional $array[l^d][l^d]$ – every row is sorted – $array[i][0]$ is $\max\{max(B_{ij}) : j = 0..(l^d - 1)\}$.

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