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Simulation of fluid flow in lower urinary tract

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PANM 16, 3.6. – 8.6. 2012

Outline

- Fluid flow through the urethra
- High resolution method
- General steady state
- Numerical experiments
- Complex model of lower urinary tract
- Numerical experiment

Equations for 1D fluid flow through the male urethra

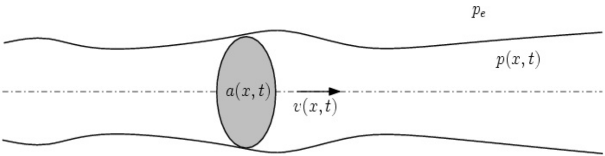
$$\begin{aligned} a_t + q_x &= 0, \\ q_t + \left(\frac{q^2}{a} + \frac{a^2}{2\rho\beta}\right)_x &= \frac{a}{\rho}\left(\frac{a_0}{\beta}\right)_x + \frac{a^2}{2\rho\beta^2}\beta_x - \frac{q^2}{4a^2}\sqrt{\frac{\pi}{a}}\lambda(Re), \end{aligned} \tag{1}$$

- $a = a(x,t)$ ... cross section of the tube
- $q = q(x,t)$ ... the flow rate in the concrete place ($q = av$, where $v = v(x,t)$ is flow velocity)
- $a_0 = a_0(x), \beta = \beta(x)$ ... cross section under zero pressure and tube compliance
- $\lambda(Re)$ ... the Mooney-Darcy friction factor ($\lambda(Re) = 64/Re$ for the laminar flow)
- Re ... Reynolds number

Constitutive relation between the pressure and the cross section of the tube

$$p = \frac{a - a_0}{\beta} + p_e, \tag{2}$$

where p_e is surrounding pressure.



Finite volume discretization

We have the system in conservation nonhomogeneous form

$$\mathbf{u}_t + [\mathbf{f}(\mathbf{u}, x)]_x = \boldsymbol{\psi}(\mathbf{u}, x) \tag{3}$$

For the following consideration, we reformulate this problem to the nonconservative homogeneous form.

Nonconservative problem

$$\begin{aligned} \mathbf{u}_t + \mathbf{A}(\mathbf{u})\mathbf{u}_x &= \mathbf{0},\ x \in \mathbf{R},\ t \in (0, T), \\ \mathbf{u}(x, 0) &= \mathbf{u}_0(x),\ x \in \mathbf{R}, \end{aligned} \tag{4}$$

The numerical high-resolution scheme for solving problems (4) can be written in fluctuation form

$$\frac{\partial \mathbf{U_j}}{\partial t} = -\frac{1}{\Delta x}[\mathbf{A}^-(\mathbf{U}^-_{j+1/2}, \mathbf{U}^+_{j+1/2}) + \mathbf{A}(\mathbf{U}^-_{j+1/2}, \mathbf{U}^+_{j-1/2}) + \mathbf{A}^+(\mathbf{U}^-_{j-1/2}, \mathbf{U}^+_{j-1/2})], \tag{5}$$

where $\mathbf{A}^{\pm}(\mathbf{U}^-_{j+1/2}, \mathbf{U}^+_{j+1/2})$ are so called fluctuations. They can be defined by the sum of waves moving to the right or to the left. The directions are dependent on the signs of the speeds of these waves, which are related to the eigenvalues of matrix $\mathbf{A}(\mathbf{u})$.
$\mathbf{U}^+_{j+1/2}$ and $\mathbf{U}^-_{j+1/2}$ are reconstructed values represent the approximations of limit values at the points $x_{j+1/2}$.

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Reconstruction

The reconstruction can be applied to each component of $\mathbf{u}$. But this approach does not work well in general. It is better to apply the reconstruction to the characteristic field of $\mathbf{u}$. It means that each jump is decomposed to the eigenvectors $\mathbf{r}$ of Jacobian matrix $\mathbf{A}(\mathbf{u})$.

$$\mathbf{U}_{j+1} - \mathbf{U}_j = \sum_{p=1}^{m} \alpha^p_{j+1/2} \mathbf{r}^p_{j+1/2}. \tag{6}$$

Arbitrary reconstruction ENO, WENO, ... the reconstruction based on minmod function can be defined by following

$$\begin{aligned} \mathbf{U}^+_{j+1/2} &= \mathbf{U}_{j+1} + \sum_p \phi^{p,+}_{I+1/2} \alpha^p_{j+1/2} \mathbf{r}^p_{j+1/2}, \\ \mathbf{U}^-_{j+1/2} &= \mathbf{U}_j + \sum_p \phi^{p,-}_{I+1/2} \alpha^p_{j+1/2} \mathbf{r}^p_{j+1/2}, \end{aligned} \tag{7}$$

where

$$\phi^{p,\pm}_{I+1/2} = \mp \frac{1}{2} \left(1 + \mathrm{sgn}(\theta^p_{I+1/2})\right) \min(1, |\theta^p_{I+1/2}|) \tag{8}$$

$$I = \begin{cases} j - 1/2, & \text{if } s^p_{j+1/2} \geq 0, \\ j + 3/2, & \text{if } s^p_{j+1/2} < 0. \end{cases} \tag{9}$$

The function $\theta^p_{j+1/2}$ can be determined by the following way

$$\theta^p_{j+1/2} = \frac{\alpha^p_{j+1/2} \mathbf{r}^p_{j+1/2} \cdot \mathbf{r}^p_{I+1/2}}{\alpha^p_{I+1/2} \mathbf{r}^p_{I+1/2} \cdot \mathbf{r}^p_{I+1/2}}. \tag{10}$$

Augmented system

Extension of the system by other equations. The advantage of this step is in the conversion of the nonhomogeneous system to the homogeneous one.
Augmented vector of unknown functions is then $\mathbf{w} = [a, q, \frac{a_0}{\beta}, \beta]^T$. Furthermore we formally augment this system by adding components of the flux function $\mathbf{f}(\mathbf{u})$ to the vector of the unknown functions. We multiply balance law by Jacobian matrix $\mathbf{f}'(\mathbf{u})$ and obtain following relation

$$\mathbf{f}'(\mathbf{u})\mathbf{u}_t + \mathbf{f}'(\mathbf{u})[\mathbf{f}(\mathbf{u})]_x = \mathbf{f}'(\mathbf{u})\boldsymbol{\psi}(\mathbf{u}, x). \quad (11)$$

Because of $\mathbf{f}'(\mathbf{u})\mathbf{u}_t = [\mathbf{f}(\mathbf{u})]_t$ we obtain hyperbolic system for the flux function

$$[\mathbf{f}(\mathbf{u})]_t + \mathbf{f}'(\mathbf{u})[\mathbf{f}(\mathbf{u})]_x = \mathbf{f}'(\mathbf{u})\boldsymbol{\psi}(\mathbf{u}, x). \quad (12)$$

In the case of the urethra fluid flow modelling we add only one equation for the second component of the flux function i.e. $\phi = av^2 + \frac{a^2}{2\rho\beta}$ (the first component q is unknown function of the original balance law), which has the form

$$\phi_t + (-v^2 + \frac{a}{2\rho\beta})(av)_x + 2v\phi_x - \frac{2av}{\rho}\left(\frac{a_0}{\beta}\right)_x - \frac{a^2 v}{\rho\beta^2}\beta_x = 0. \quad (13)$$

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Augmented system

Finally augmented system can be written in the nonconservative form

$$\left[\begin{array}{c} a \\ q \\ \phi \\ \frac{a_0}{\beta} \\ \beta \end{array}\right]_t + \left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ -\frac{q^2}{a^2}+\frac{a}{\rho\beta} & 2\frac{q}{a} & 0 & -\frac{a}{\rho} & -\frac{a^2}{\rho\beta^2} \\ 0 & -\frac{q^2}{a^2}+\frac{a}{\rho\beta} & 2\frac{q}{a} & 2\frac{q}{\rho} & -\frac{aq}{\rho\beta^2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} a \\ q \\ \phi \\ \frac{a_0}{\beta} \\ \beta \end{array}\right]_x = \mathbf{0}, \qquad (14)$$

briefly $\mathbf{w}_t + \mathbf{B}(\mathbf{w})\mathbf{w}_x = \mathbf{0}$, where matrix $\mathbf{B}(\mathbf{w})$ has following eigenvalues

$$\lambda^1 = v - \sqrt{\frac{a}{\rho\beta}}, \lambda^2 = v + \sqrt{\frac{a}{\rho\beta}}, \lambda^3 = 2v, \lambda^4 = \lambda^5 = 0 \qquad (15)$$

and corresponding eigenvectors

$$\mathbf{r}^1 = \left[\begin{array}{c} 1 \\ \lambda^1 \\ (\lambda^1)^2 \\ 0 \\ 0 \end{array}\right], \mathbf{r}^2 = \left[\begin{array}{c} 1 \\ \lambda^2 \\ (\lambda^2)^2 \\ 0 \\ 0 \end{array}\right], \mathbf{r}^3 = \left[\begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{array}\right], \mathbf{r}^4 = \left[\begin{array}{c} \frac{-a}{\rho\lambda^1\lambda^2} \\ 0 \\ \frac{a}{\rho} \\ 1 \\ 0 \end{array}\right], \mathbf{r}^5 = \left[\begin{array}{c} \frac{-a^2}{\rho\beta^2\lambda^1\lambda^2} \\ 0 \\ \frac{a^2}{2\rho\beta^2} \\ 0 \\ 1 \end{array}\right]. \qquad (16)$$

General steady state

General steady state $\mathbf{u}_t = \mathbf{0}$, therefore $[\mathbf{f}(\mathbf{u})]_x = \boldsymbol{\psi}(\mathbf{u}, x)$. In the concrete $a_t = 0$, $q_t = 0$.

$$\begin{aligned} q_x &= 0, \\ \left(\frac{q^2}{a} + \frac{a^2}{2\rho\beta}\right)_x &= \frac{a}{\rho}\left(\frac{a_0}{\beta}\right)_x + \frac{a^2}{2\rho\beta^2}\beta_x. \end{aligned} \tag{17}$$

It can be derived

$$\left(-v^2 + \frac{a}{\rho\beta}\right) a_x = \frac{a}{\rho}\left(\frac{a_0}{\beta}\right)_x + \frac{a^2}{\rho\beta^2}\beta_x. \tag{18}$$

Bernoulli equation

$$\left(\frac{v^2}{2} + \frac{a - a_0}{\rho\beta}\right)_x = 0. \tag{19}$$

Discrete form

$$\left(\frac{V^2}{2} + \frac{A - A_0}{\rho\beta}\right)_j = \left(\frac{V^2}{2} + \frac{A - A_0}{\rho\beta}\right)_{j+1}. \tag{20}$$

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General steady state

The steady state for the augmented system means $\mathbf{B}(\mathbf{w})\mathbf{w}_x = \mathbf{0}$, therefore $\mathbf{w}_x$ is a linear combination of the eigenvectors corresponding to the zero eigenvalues. The discrete form of the vector $\Delta\mathbf{w}$ corresponds to the certain approximation of these eigenvectors.

$$\Delta \begin{bmatrix} A \\ Q \\ \Phi \\ \frac{a_0}{\beta} \\ \beta \end{bmatrix} = \begin{bmatrix} \frac{\bar{A}}{\rho} \frac{1}{\overline{\lambda^1 \lambda^2}} \\ 0 \\ \frac{\bar{A}}{\rho} \frac{\widetilde{\lambda^1 \lambda^2}}{\overline{\lambda^1 \lambda^2}} \\ 1 \\ 0 \end{bmatrix} \Delta \left(\frac{a_0}{\beta} \right) + \begin{bmatrix} \frac{\bar{A}^2}{\rho \beta_{j+1} \beta_j} \frac{1}{\overline{\lambda^1 \lambda^2}} \\ 0 \\ \frac{\bar{A}^2}{\rho \beta_{j+1} \beta_j} \frac{\widetilde{\lambda^1 \lambda^2}}{\overline{\lambda^1 \lambda^2}} - \frac{\tilde{A}^2}{2 \rho \beta_{j+1} \beta_j} \\ 0 \\ 1 \end{bmatrix} \Delta\beta, \qquad (21)$$

where $\bar{A} = \frac{A_j + A_{j+1}}{2}$, $\bar{\beta} = \frac{\beta_j + \beta_{j+1}}{2}$, $\tilde{A}^2 = \frac{A_j^2 + A_{j+1}^2}{2}$, $\tilde{V}^2 = |V_j V_{j+1}|$, $\bar{V}^2 = \left(\frac{V_j + V_{j+1}}{2} \right)^2$ and

$$\widetilde{\lambda^1 \lambda^2} = -\tilde{V}^2 + \frac{\bar{A}\bar{\beta}}{\rho \beta_{j+1} \beta_j}, \qquad \overline{\lambda^1 \lambda^2} = -\bar{V}^2 + \frac{\bar{A}\bar{\beta}}{\rho \beta_{j+1} \beta_j}. \qquad (22)$$

Therefore we use vectors on the RHS of (21) as approximations of the fourth and fifth eigenvectors of the matrix $\mathbf{B}(\mathbf{w})$ to preserve general steady state.

Augmented system method

Eigenvectors matrix $\tilde{\mathbf{R}} = [\tilde{\mathbf{r}}^{\mathbf{1}}, \tilde{\mathbf{r}}^{\mathbf{2}}, \tilde{\mathbf{r}}^{\mathbf{3}}, \tilde{\mathbf{r}}^{\mathbf{4}}, \tilde{\mathbf{r}}^{\mathbf{5}}]$

$$\tilde{\mathbf{R}} = \begin{bmatrix} 1 & 1 & 0 & \frac{\bar{A}}{\rho}\frac{1}{\overline{\overline{\lambda^1\lambda^2}}} & \frac{\bar{A}^2}{\rho\beta_{j+1}\beta_j}\frac{1}{\overline{\overline{\lambda^1\lambda^2}}} \\ \lambda^1 & \lambda^2 & 0 & 0 & 0 \\ (\lambda^1)^2 & (\lambda^2)^2 & 1 & \frac{\bar{A}}{\rho}\frac{\widetilde{\lambda^1\lambda^2}}{\overline{\overline{\lambda^1\lambda^2}}} & \frac{\bar{A}^2}{\rho\beta_{j+1}\beta_j}\frac{\widetilde{\lambda^1\lambda^2}}{\overline{\overline{\lambda^1\lambda^2}}} - \frac{\tilde{A}^2}{2\rho\beta_{j+1}\beta_j} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (23)$$

Augmented system method - Positive semidefiniteness

Eigenvectors matrix $\tilde{\mathbf{R}} = [\tilde{\mathbf{r}}^{\mathbf{1}}, \tilde{\mathbf{r}}^{\mathbf{2}}, \tilde{\mathbf{r}}^{\mathbf{3}}, \tilde{\mathbf{r}}^{\mathbf{4}}, \tilde{\mathbf{r}}^{\mathbf{5}}]$

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where the approximation of eigenvalues of the $\mathbf{B}(\mathbf{w})$ are replaced by Einfeld speeds

$$s_\varepsilon^1 = \min_p \min\left\{\lambda_j^p, \lambda_{j+1/2}^p\right\}, \qquad s_\varepsilon^2 = \max_p \max\left\{\lambda_{j+1}^p, \lambda_{j+1/2}^p\right\},$$
$$s_\varepsilon^3 = s_\varepsilon^1 + s_\varepsilon^2, \qquad s_\varepsilon^4 = 0, \qquad s_\varepsilon^5 = 0. \tag{25}$$

The other necessary assumptions to the approximations of the eigenvectors.

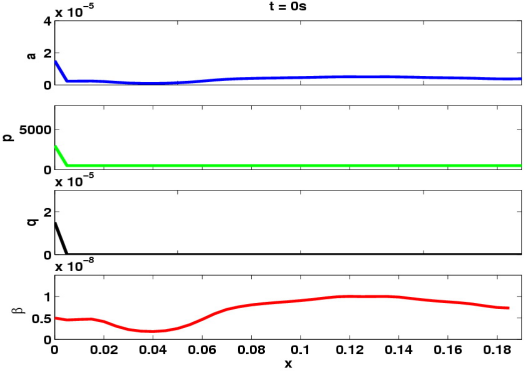
Numerical scheme

We have five linearly independent eigenvectors. The approximation is chosen to be able to prove the consistency and provide the stability of the algorithm. In some special cases this scheme is conservative and we can guarantee the positive semidefiniteness, but only under the additional assumptions.
The fluctuations are then defined by

$$\begin{aligned}
\mathbf{A}^-(\mathbf{U}^-_{j+1/2}, \mathbf{U}^+_{j+1/2}) &= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \cdot \sum_{p=1, s^{p,n}_{j+1/2}<0}^{m} \gamma^p_{j+1/2} \mathbf{r}^p_{j+1/2}, \\
\mathbf{A}^+(\mathbf{U}^-_{j+1/2}, \mathbf{U}^+_{j+1/2}) &= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \cdot \sum_{p=1, s^{p,n}_{j+1/2}>0}^{m} \gamma^p_{j+1/2} \mathbf{r}^p_{j+1/2}, \\
\mathbf{A}(\mathbf{U}^+_{j-1/2}, \mathbf{U}^-_{j+1/2}) &= \mathbf{f}(\mathbf{U}^-_{j+1/2}) - \mathbf{f}(\mathbf{U}^+_{j-1/2}) - \mathbf{\Psi}(\mathbf{U}^-_{j+1/2}, \mathbf{U}^+_{j-1/2}),
\end{aligned} \tag{26}$$

where $\mathbf{\Psi}(\mathbf{U}^-_{j+1/2}, \mathbf{U}^+_{j-1/2})$ is a suitable approximation of the source term and $\mathbf{r}^p_{j+1/2}$ are suitable approximations of the eigenvectors (16).

Urethra flow



Initial condition:

$$p(x,0) = \begin{cases} 3000 \text{ [Pa]}, & \text{pro } x = 0 \\ 500 \text{ [Pa]}, & \text{jinak}, \end{cases}, v(x,0) = \begin{cases} 1 \text{ [m/s]}, & \text{pro } x = 0 \\ 0 \text{ [m/s]}, & \text{jinak}, \end{cases}$$

Boundary condition:

$$q(0,t) = \text{konst.}$$

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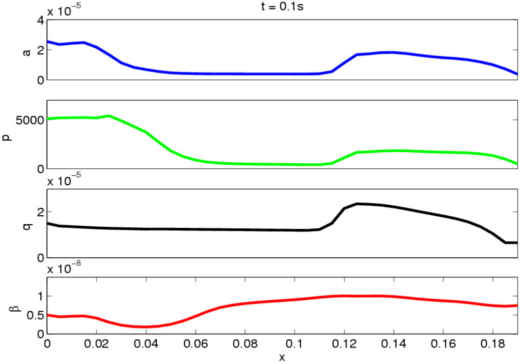

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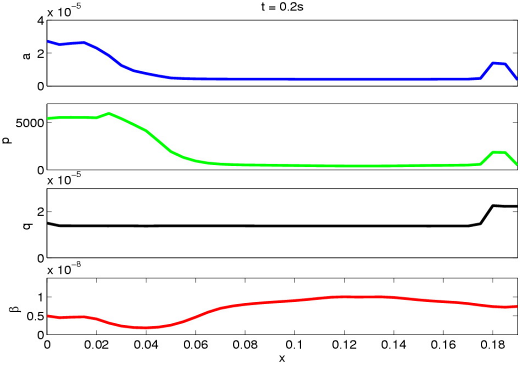

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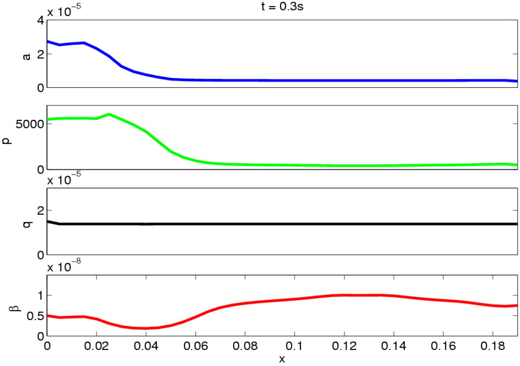
Urethra flow



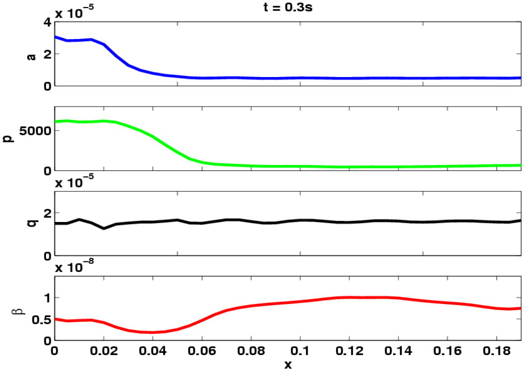
Initial condition:

$$p(x,0) = \begin{cases} 3000 \text{ [Pa]}, & \text{pro } x = 0 \\ 500 \text{ [Pa]}, & \text{jinak}, \end{cases}, v(x,0) = \begin{cases} 1 \text{ [m/s]}, & \text{pro } x = 0 \\ 0 \text{ [m/s]}, & \text{jinak}, \end{cases}$$

Boundary condition:

$$q(0,t) = \text{konst.}$$

Urethra flow - unsteady



Initial condition:

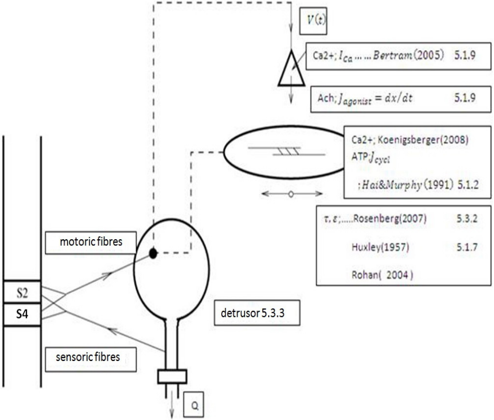
$$p(x,0) = \begin{cases} 3000\ [\mathrm{Pa}], & \text{pro } x = 0 \\ 500\ [\mathrm{Pa}], & \text{jinak}, \end{cases} \quad , v(x,0) = \begin{cases} 1\ [\mathrm{m/s}], & \text{pro } x = 0 \\ 0\ [\mathrm{m/s}], & \text{jinak}, \end{cases}$$

Boundary condition:

$$q(0,t) = \text{konst.}$$

Bladder contraction

Mechano-chemical coupling of the smooth muscle cell contraction.
The product of the chemical reaction affinity (the ATP hydrolysis) with its rate plays an important role in the discussed model (sliding between actin and myosin). Further it can be assumed that the rate of the ATP hydrolysis depends on the ATP consumption.



Ca_2^+ dynamics

$$\frac{dc}{dt} = J_{IP3} - J_{VOCC} + J_{Na/Ca} - J_{SRuptake} + J_{CICR} - J_{extrusion} + J_{leak} + 0.1J_{stretch}$$

Rate of change calcium concentration in cytoplasm - cation release from the IP3 (sensitive reservoir), flow of calcium through the membrane, flow the mechanical-sensitive channels,...

$$\frac{ds}{dt} = J_{SRuptake} - J_{CICR} - J_{leak}$$

Rate of change calcium concentration in ER/SR.

$$\frac{dv}{dt} = \gamma(-J_{Na/K} - J_{Cl} - 2J_{VOCC} - J_{Na/Ca} - J_K - J_{stretch})$$

Rate of change of membrane tension - flow Na^+/K^+, chloride flow,potassium flow,...

$$\frac{dw}{dt} = \lambda K_{activate}$$

Rate of change of probability of opening channels activated by Ca_2^+ - activation of K^+ channels.

$$\frac{dI}{dt} = J_{agonist} - J_{degrad}$$

Rate of change IP3 concentration in cytoplasm

Isotonic contraction

Mechano-chemical coupling of the smooth muscle cell contraction

$$\frac{dx}{dt} = k_1 \left(\tau - z(x-1)\right),$$

$$\frac{dy}{dt} = \frac{y}{k_2} \left(x\tau - \frac{1}{2} z(x-1)^2 + C'\right),$$

$$\frac{dz}{dt} = \mathrm{sgn}(m) \left(r - \frac{1}{2} z(x-1)^2\right).$$

Volume of the bladder

$$V = \kappa (xy)^3.$$

Model of phosphorilation of light myosin chain

The muscle cell contraction is caused by the relative movement of the myosin and actin filaments

$$\frac{dA_M}{dt} = k_5 A_{M_p} - (k_7 + k_6) A_M,$$

$$\frac{dA_{M_p}}{dt} = k_3 M_p + k_6 A_M - (k_4 + k_5) A_{M_p},$$

$$\frac{dM_p}{dt} = k_1(1 - A_M) + (k_4 - k_1) A_{M_p} - (k_1 + k_2 + k_3) M_p,$$

$$\frac{dY}{dt} = -Q_a Y + L J_{cycle}.$$

Y ... ATP concentration

Interconnection variables

The bladder pressure

$$p = \frac{V_{sh}}{3V}\tau,$$

τ... tension in the fiber
V_{sh}... volume of the bladder wall
V... internal volume
The outflow from the bladder

$$q = \frac{dV}{dt},$$

$$\tau = \frac{\frac{-q}{3\kappa(x\cdot y)^2} + \left[k_1 zy(x-1) + \frac{zyx}{2k_2}(x-1)^2 - \frac{xy}{k_2}C'\right]}{k_1 y + \frac{x^2 y}{k_2}}. \tag{27}$$

Mathematical model

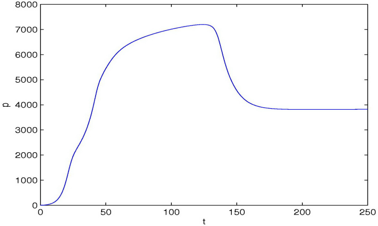
System of ordinary differential equations

- 12 equations describing the bladder model and the detrusor contraction during voiding
- $2J$ equations of urethra flow, where J is the number of finite volumes which divide the urethra region

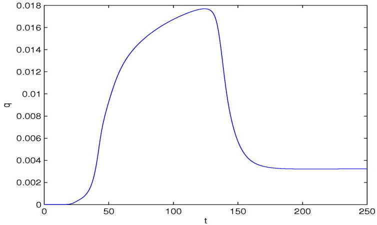
Properties of the method describing urethra flow

- preserving general steady states
- positive semidefiniteness
- high order of accuracy

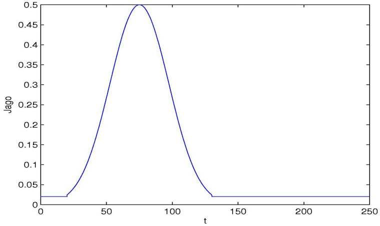
Numerical experiment - quantities at the bladder neck



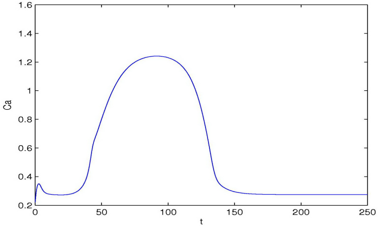
pressure



outflow



mediator flux



Ca++ concentration

Conclusion

- complex model of the voiding
- high resolution discretization with preserving general steady state
- properties of the results?