On selection of interface weights in domain decomposition methods

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joint work with

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Outline

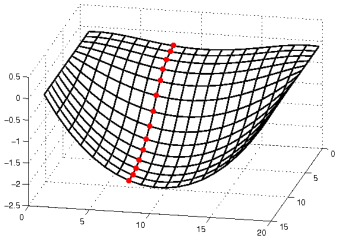
Outline

Model problem

Abstract problem:

$$A\,u = f$$

Example: 2D Poisson problem, 2 subdomains

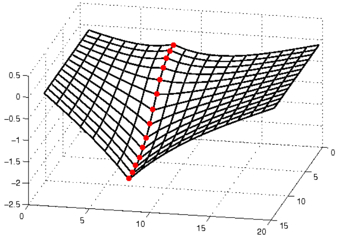


Reduction to the interface

Schur complement with respect to the interface:

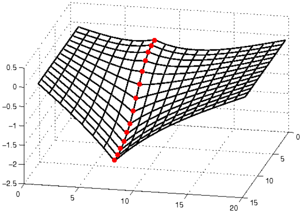
$$A\,u = f \quad \Rightarrow \quad \widehat{S}\,\widehat{u} = \widehat{g}$$

- $\widehat{S} : U \to U'$ symmetric positive definite
- $\dim\ U < \infty$
- visualization of function from U on the whole domain:



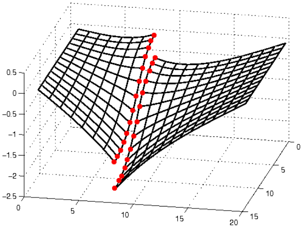
Primal methods (one level)

U



$R \downarrow \quad \uparrow E$

W



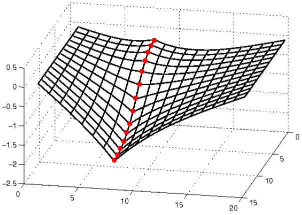
$$\widehat{S}\,\widehat{u} = \widehat{g}$$

$$S\,u = g \qquad S = \begin{bmatrix} S^1 & 0 \\ 0 & S^2 \end{bmatrix}$$

- $R \ldots$ operator of injection
- $E \ldots$ averaging
- $E\,R = I, \quad R\,E \ldots$ projection
- $\widehat{S} = R^T S\,R, \quad \widehat{g} = R^T g$

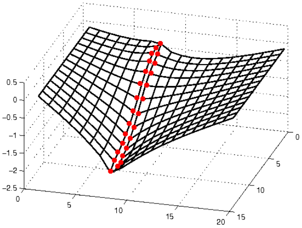
Primal methods - BDDC

U



$R \downarrow \quad \uparrow E$

$\widetilde{W}$



$$\widehat{S}\,\widehat{u} = \widehat{g}$$

$$\widetilde{S}\,u = \widetilde{g} \qquad \widetilde{S} = \begin{bmatrix} S^1 & 0 & \widetilde{S}^{1c} \\ 0 & S^2 & \widetilde{S}^{2c} \\ \widetilde{S}^{c1} & \widetilde{S}^{c2} & \widetilde{S}^{c} \end{bmatrix}$$

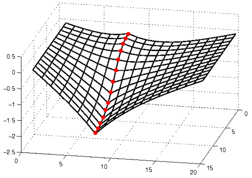
- $R \ldots$ operator of injection
- $E \ldots$ averaging
- $E\,R = I$, $\;R\,E \;\ldots$ projection
- $\widehat{S} = R^T \widetilde{S}\,R$, $\;\widehat{g} = R^T \widetilde{g}$

Abstract primal preconditioner

$$
\begin{array}{ccccc}
 & U & \xrightarrow{\widehat{S}} & U' & \\
E \uparrow & R \downarrow & & \uparrow R^T & \downarrow E^T \\
 & \widetilde{W} & \xleftarrow{\widetilde{S}^{-1}} & \widetilde{W}' &
\end{array}
$$

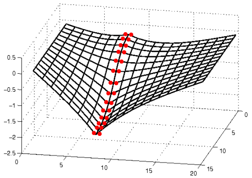
- $R \dots$ operator of injection
- $E \dots$ averaging, $\quad E\,R = I, \quad R\,E \dots$ projection
- $\widehat{S} = R^T \widetilde{S} R$, $\quad \widehat{S}, \widetilde{S} \dots$ symmetric positive definite
- $M := E \widetilde{S}^{-1} E^T \approx \widehat{S}^{-1}$
- $cond\,(M\widehat{S}) \leq ||RE||^2_{\widetilde{S}}$ [Mandel, Sousedík 2007]

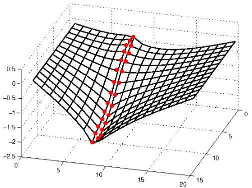
Notation

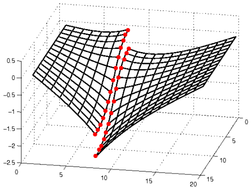


U

$R \downarrow \quad \uparrow E$







$$\widehat{W} \quad \subset \quad \widetilde{W} \quad \subset \quad W$$

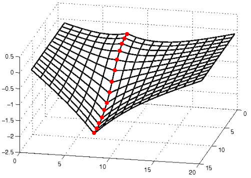
$$Bu = 0 \qquad\qquad Cu = 0$$

$B : W \longrightarrow \Lambda$ ... a jump across the interface

C ... a jump at the coarse nodes

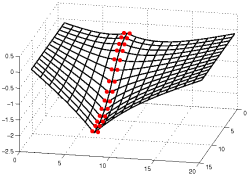
Formulation of the dual problem

U



$$\widehat{S}\,\widehat{u} = \widehat{g} \quad , \quad \widehat{S} : U \rightarrow U'$$

$\widehat{W}$

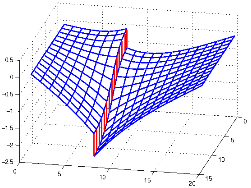


$$Su = g\,, \quad u \in W,\; Bu = 0$$

$\Rightarrow$ saddle point problem:

$$\begin{bmatrix} S & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda' \end{bmatrix} = \begin{bmatrix} g \\ 0 \end{bmatrix}$$

Λ



$$BS^{-1}B^T\lambda' = BS^{-1}g$$

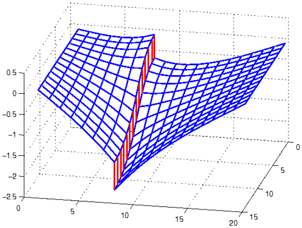
$S : W \rightarrow W'$

$B : W \rightarrow \Lambda$ ... a jump

$B^T : \Lambda' \rightarrow W'$

Dual method (one level)

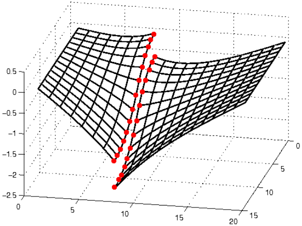
Λ



$$F\,\lambda' = v, \quad F : \Lambda' \to \Lambda$$

$$B_D^T \downarrow \quad \uparrow B$$

W



- $B \ldots$ a jump
- $B_D^T \ldots$ representation of a jump
- $B\,B_D^T = I,\;\; B_D^T\,B\;\ldots$ projection
- $F = BS^{-1}B^T,\;\; v = BS^{-1}g$

Abstract dual preconditioner

$$
\begin{array}{ccccc}
 & \Lambda & \overset{F}{\longleftarrow} & \Lambda' & \\
B_D^T \downarrow & B \uparrow & & \downarrow B^T & \uparrow B_D \\
 & \widetilde{W} & \overset{\widetilde{S}}{\longrightarrow} & \widetilde{W}' &
\end{array}
$$

- $B_D^T \ldots$ jump representation
- $B \ \ldots$ jump operator, $\quad B\,B_D^T = I, \ \ B_D^T\,B \ldots$ projection
- $F = B\widetilde{S}^{-1}B^T \ , \quad F, \widetilde{S} \ldots$ symmetric positive definite
- $M := B_D\,\widetilde{S}\,B_D^T \approx F^{-1}$
- $cond\,(MF) \leq ||B_D^T\,B||_{\widetilde{S}}^2$ [Mandel, Sousedík 2007]

Relationship between primal and dual methods

Let operators determining primal and dual method satisfy a relationship

$$B_D^T B + RE = I_{\widetilde{W}} \ .$$

Then both the preconditioned linear systems are spectrally equivalent.

[Mandel, Dohrmann, Tezaur 2005]

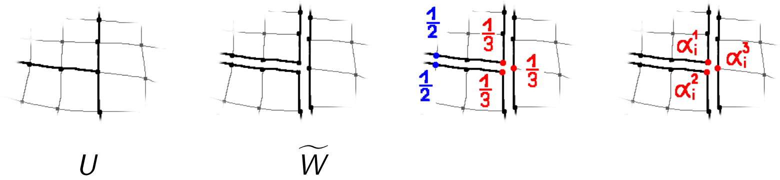
We have $||B_D^T B||_{\widetilde{S}} = ||I - RE||_{\widetilde{S}} = ||RE||_{\widetilde{S}}$

Outline

Standard choices of E

α_i^k = weight of the i-th node in the k-th subdomain

$\sum_k \alpha_i^k = 1$



Standard choices of α_i^k:

- arithmetic average
- weighted average with weights derived from:
  - diagonal entries of local and global Schur compl.: $\alpha_i^k = s_{ii}^k / \widehat{s}_{ii}$
  - jumps at coefficients of the equation across the interface
  - diagonal entries of K, $K^{(i)}$

Our proposition of the choice of E

Let $\mathbf{d}$ is a probing vector, $\mathbf{d}^k$ its restriction to k-th subdomain.

Weights for the k-th subdomain:

$$\alpha^k = (\mathbf{d}^k)^T \mathbf{S}^k \mathbf{d}^k \ / \ \mathbf{d}^T \widehat{\mathbf{S}} \, \mathbf{d}$$

Different probing vectors $\mathbf{d}$:

$\mathbf{d} = \mathbf{e}_i = (0, \dots 0, 1, 0 \dots 0) \quad \Longrightarrow \quad \alpha_i^k = s_{ii}^k / \widehat{s}_{ii}$

$\mathbf{d} = (1, \dots 1) \quad \Longrightarrow \quad \alpha^k = \sum_{ij} s_{ij}^k / \sum_{ij} (\widehat{s}_{ij})$

$\mathbf{d}$ = a jump at the last approximation

...

(for more subdomains, $\mathbf{d}$ is supposed to be nonzero only for selected face or edge or a part of it)

Derivation of the formula for 2 subdomains

minimization of *RE* in energetic norm for given vector *u*
with respect to node weights α_i^1, for 2 subdomains:

$$||\mathbf{RE}\,\mathbf{u}||_S^2 \longrightarrow \min$$

$$\begin{bmatrix} \widehat{s}_{11} d_1 d_1 & \widehat{s}_{12} d_1 d_2 & \dots & \widehat{s}_{1\mathrm{n}} d_1 d_\mathrm{n} \\ \dots & \dots & \dots & \dots \\ \widehat{s}_{i1} d_i d_1 & \widehat{s}_{i2} d_i d_2 & \dots & \widehat{s}_{i\mathrm{n}} d_i d_\mathrm{n} \\ \dots & \dots & \dots & \dots \\ \widehat{s}_{\mathrm{n}1} d_\mathrm{n} d_1 & \widehat{s}_{\mathrm{n}2} d_\mathrm{n} d_2 & \dots & \widehat{s}_{\mathrm{nn}} d_\mathrm{n} d_\mathrm{n} \end{bmatrix} \begin{bmatrix} \alpha_1^1 \\ \dots \\ \dots \\ \dots \\ \alpha_\mathrm{n}^1 \end{bmatrix} = \begin{bmatrix} \sum_j s_{1j}^1 d_j d_1 \\ \dots \\ \sum_j s_{ij}^1 d_j d_i \\ \dots \\ \sum_j s_{\mathrm{n}j}^1 d_j d_\mathrm{n} \end{bmatrix},$$

where $\mathbf{d} = (d_1, d_2, \dots d_\mathrm{n})^T$ is a jump in *u* across the interface.

Simplifying choices:

$$\mathbf{d} = (0, \dots 0, d_i, 0 \dots 0) \quad \Longrightarrow \quad \alpha_i^1 = s_{ii}^1 / \left(s_{ii}^1 + s_{ii}^2\right) = s_{ii}^1 / \widehat{s}_{ii}$$

$$\alpha_1^1 = \dots = \alpha_\mathrm{n}^1 = \alpha^1 \quad \Longrightarrow \quad \alpha^1 = \mathbf{d}^T \mathbf{S}^1\, \mathbf{d} / \mathbf{d}^T (\mathbf{S}^1 + \mathbf{S}^2)\, \mathbf{d}$$

Outline

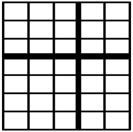
Substructuring DD methods

Choice of the averaging operator

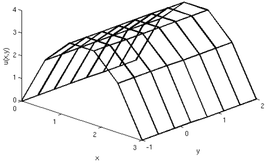
Numerical results

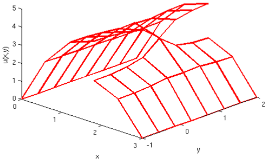
Test problem: 2D Poisson equation (stationary heat problem)

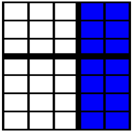
bilinear elements, 4 subdomains, 1 coarse node (at crosspoint)



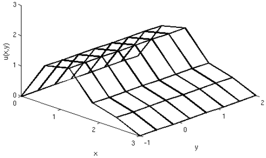
no jumps

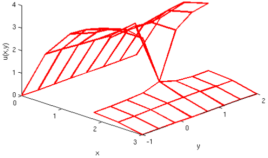






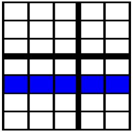
jump 1:9



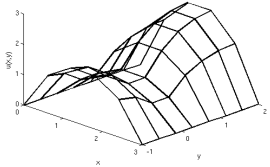


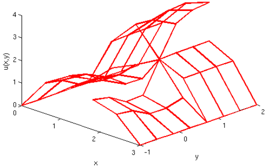
Test problem: 2D Poisson equation (stationary heat problem)

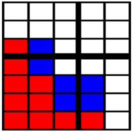
bilinear elements, 4 subdomains, 1 coarse node (at crosspoint)



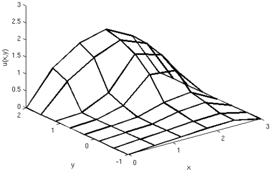
jumps 1:10

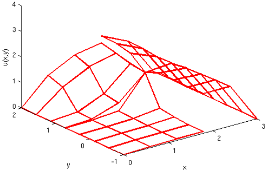






jumps 1:10:100





Errors in the first 5 iterations

Richardson method: $u_{i+1} = u_i + M\, r_i$

no jumps			jump 1:9		
$\alpha = 1/2$	α_i diag. Sch.	$d = (1, ..1)$	$\alpha = 1/2$	α_i diag. Sch.	$d = (1, ..1)$
9.67e-01	9.52e-01	8.62e-01	1.91e+01	1.14	1.08
2.46e-01	2.42e-01	2.18e-01	5.57e+01	3.47e-01	3.28e-01
6.18e-02	6.04e-02	5.44e-02	1.62e+02	1.05e-01	9.88e-02
1.55e-02	1.51e-02	1.36e-02	4.68e+02	3.15e-02	2.97e-02
3.89e-03	3.78e-03	3.41e-03	1.35e+03	9.50e-03	8.96e-03

jumps 1:10			jumps 1:10:100		
$\alpha = 1/2$	α_i diag. Sch.	$d = (1, ..1)$	$\alpha = 1/2$	α_i diag. Sch.	$d = (1, ..1)$
1.08	9.35e-01	8.06e-01	1.76e+01	1.44e+01	5.44e-01
3.25e-01	2.46e-01	1.85e-01	6.33e+01	4.14e+01	1.17e-01
9.58e-02	6.37e-02	4.23e-02	2.27e+02	1.19e+02	5.66e-02
2.82e-02	1.65e-02	9.70e-03	8.16e+02	3.44e+02	3.75e-02
8.30e-03	4.29e-03	2.23e-03	2.93e+03	9.91e+02	2.58e-02

Errors in the first 5 iterations

PCG

no jumps			jump 1:9		
$\alpha = 1/2$	α_i diag. Sch.	$d = (1, ..1)$	$\alpha = 1/2$	α_i diag. Sch.	$d = (1, ..1)$
8.81e-01	8.70e-01	8.00e-01	6.16	1.07	9.85e-01
7.24e-03	6.87e-03	9.40e-03	9.05e-01	4.20e-03	6.46e-03
8.50e-04	7.22e-04	9.21e-04	1.22e-02	4.21e-04	5.08e-04
1.64e-06	1.17e-06	1.64e-06	7.50e-04	2.51e-07	3.76e-07
1.09e-09	1.05e-09	1.94e-09	1.00e-04	1.55e-10	1.55e-10

jumps 1:10			jumps 1:10:100		
$\alpha = 1/2$	α_i diag. Sch.	$d = (1, ..1)$	$\alpha = 1/2$	α_i diag. Sch.	$d = (1, ..1)$
1.00	8.82e-01	8.01e-01	5.95	5.65	6.17e-01
1.35e-02	1.34e-02	8.85e-03	2.57e-01	2.56e-01	5.48e-02
1.05e-03	7.49e-04	7.48e-04	7.82e-03	3.63e-03	1.09e-02
2.23e-06	1.73e-06	4.86e-06	1.29e-04	6.94e-05	1.33e-04
8.35e-10	9.52e-10	3.22e-08	1.24e-06	8.84e-08	3.29e-05

Conclusions

- a new form of the averaging operator E was proposed
- 3 choices of E were compared:
  1. arithmetic average $\alpha = 1/2$
  2. $\alpha_i^k = s_{\mathrm{ii}}^k / \widehat{s}_{\mathrm{ii}}$ (fractions of Schur diagonal entries)
  3. $\alpha^k = (\mathbf{d}^k)^T \mathbf{S}^k \mathbf{d}^k \; / \; \mathbf{d}^T \widehat{\mathbf{S}} \, \mathbf{d}$

     with $\mathbf{d}$ chosen as **ones** for a given edge, **zeros** otherwise
- best choice of E can differ for PCG and Richardson method

– for Richardson method: choice of 3 is the best

– for PCG:

  - choice of 1 is better than 2 or 3 for constant coefficients
  - 2 is the best choice for jumps in coefficients
  - 2 can be replaced by 3 for jump in coeff. across the interface

- future work: tests for large numbers of subdomains