Corson compact semilattices

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Motivations

Theorem (Gruenhage 1986)

Let T be a tree and let $P(T)$ denote the space of all initial segments of T. Then $P(T)$ is Eberlein compact if and only if T is special.

A partially ordered set $\langle T, < \rangle$ is special if

$$T = \bigcup_{n \in \omega} T_n$$

where each T_n consists of pairwise incomparable elements.

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A compact space K is Eberlein if it is homeomorphic to a weakly compact subset of some Banach space.

Proposition

Let K be a 0-dimensional compact space. Then K is Eberlein if and only if the space

$$\mathcal{C}_p(K, 2)$$

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A tree is a partially ordered set $\langle T, \leqslant \rangle$ which is a meet semilattice, i.e.

$$x \wedge y = \inf\{x, y\}$$

exists for every $x, y \in T$ and for each $y \in T$ the set

$$\{x \in T : x < y\}$$

is well ordered.

Fact

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Compact semilattices

A topological semilattice is a structure of the form

$$\mathbb{X} = \langle X, \wedge, 0, \tau \rangle,$$

such that $\langle X, \wedge \rangle$ is a semilattice, 0 is the minimal element of X and τ is a Hausdorff topology on X for which $\wedge$ is continuous.

Theorem

Let $\langle K, \wedge, 0, \tau \rangle$ be as above with $\langle K, \tau \rangle$ compact, assuming that $\wedge$ is only separately continuous. Then $\wedge$ is continuous and the topology τ is uniquely determined by the semilattice operation $\wedge$.

Duality

Let $\mathbb{K} = \langle K, \wedge, 0, \tau \rangle$ be a topological 0-dimensional semilattice. Define

$$\mathbb{K}^* = \hom(\mathbb{K}, 2),$$

where

$$2 = \langle \{0, 1\}, \wedge, 0, \tau_2 \rangle$$

is the unique discrete two-element semilattice.
Endow $\mathbb{K}^*$ with the obvious semilattice operation and with the pointwise topology.

Claim

- *If $\mathbb{K}$ is discrete then $\mathbb{K}^*$ is compact.*
- *If $\mathbb{K}$ is either discrete or compact then $\mathbb{K}^{**} = \mathbb{K}$.*

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The duality comes from

K.H. Hofmann, M. Mislove, A. Stralka:
The Pontryagin duality of compact 0-dimensional semilattices and its applications,
Lectures Notes in Mathematics, Vol. **396**, Springer-Verlag, Berlin-New York, 1974.

A semilattice $\mathbb{K} = \langle K, \wedge, 0, \tau \rangle$ is modest if for every $p \in K$ such that $[p, \to)$ is clopen, the set of immediate predecessors of p is finite.

Proposition

Let $\mathbb{K}$ be a modest compact semilattice. Then $\mathbb{K}^* \setminus \{0\}$ is discrete.

Theorem

Let $\mathbb{K}$ be a modest 0-dimensional compact semilattice. Then $\mathbb{K}$ is Eberlein compact if and only if

$$\mathbb{K}^* \setminus \{0\} = \bigcup_{n \in \omega} S_n,$$

where for each $n \in \omega$:

- *no infinite subset of S_n is centered.*

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- *no infinite subset of S_n is centered.*

Proof.

Assume $\mathbb{K}$ is Eberlein.

- $\mathbb{K}^* \subseteq \mathcal{C}_p(\mathbb{K}, 2)$ is closed, hence σ-compact.
- An infinite compact subset of $\mathbb{K}^*$ is of the form

$$A \cup \{0\}$$

  where for each $x \in K$ the set $\{a \in A \colon a(x) = 1\}$ is finite.
- Let $\mathbb{K}^* = \bigcup_{n \in \omega} S_n$, where each S_n is compact.
- Then no infinite subset of S_n is centered.

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Proposition

Let K be a 0-dimensional compact. Then K is Eberlein iff there exists a T_0-separating family of clopen sets

$$\mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n$$

such that each $\mathcal{U}_n$ is point-finite.

Trees

Let $\langle T, \leqslant\rangle$ be a tree. Define

$$S(T) = T \cup \{\infty\},$$

where $\infty \notin T$ and consider the following ordering $\preceq$ on $S(T)$:

- $s \preceq t$ iff either $s = \infty$ or $s \geqslant t$.

Claim

$\langle S(T), \wedge, \infty\rangle$ is a semilattice and

$$S(T)^* = P(T).$$

Claim

$P(T)$ is a modest semilattice.

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Claim

$P(T)$ *is a modest semilattice.*

Corollary

Let T be a tree. Then $P(T)$ is Eberlein compact if and only if

$$T = \bigcup_{n\in\omega} S_n,$$

where each S_n is an antichain.

Adequate compacta

An adequate compact is a space $K \subseteq \mathcal{P}(\kappa)$ satisfying

$$x \in K \iff [x]^{<\omega} \subseteq K.$$

Claim

Let $K \subseteq \mathcal{P}(\kappa)$ be adequate. Then K^* is isomorphic to

$$\langle K \cap [\kappa]^{<\omega}, \cap, \emptyset, \tau \rangle,$$

where all nonempty sets are isolated and a basic neighborhood of $\emptyset$ is of the form

$$K^* \setminus \{x \colon x \subseteq a\},$$

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Corollary

Let $K \subseteq \mathcal{P}(\kappa)$ be an adequate compact. Then K is Eberlein if and only if

$$\kappa = \bigcup_{n\in\omega} S_n$$

where $\mathcal{P}(S_n) \cap K \subseteq [\kappa]^{<\omega}$ for every $n \in \omega$.

Spaces of chains

Let P be a partially ordered set. Denote by $K(P)$ the family of all chains of P.

Claim

$K(P)$ is an adequate compact.

Corollary (Leiderman & Sokolov)

Let P be a partially ordered set. Then $K(P)$ is Eberlein if and only if P is special.

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Assume $K(P)$ is Eberlein. Write $P = \bigcup_{n\in\omega} P_n$ so that no P_n contains an infinite chain.

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Example (Alster & Pol)

Let $P \subseteq \mathbb{R}$ be uncountable and let $\preceq$ be a well order on P.
Define $x \leqslant' y$ iff both $x \leqslant y$ and $x \preceq y$. Then P is a poset in which all chains are countable.

Claim

$K(P)$ is Corson and not Eberlein compact.

Proof.

Suppose $K(P)$ is Eberlein. Then $P = \bigcup_{n\in\omega} P_n$ where each P_n is an antichain.
Let P_k be uncountable. There is $t_0 < t_1 < t_2 < \dots$ in P_k.
But then $\dots \preceq t_2 \preceq t_1 \preceq t_0$, which contradicts the fact that $\preceq$ is a well order. □

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