Fractional vector analysis based on invariance requirements

(Critique of coordinate approaches)

In memory of Walter Noll

Miroslav Šilhavý
Institute of Mathematics, CAS
Žitná 25
115 67 Prague 1
Czech Republic
e-mail: silhavy@math.cas.cz

Abstract The paper discusses the fractional operators

$$\nabla^\alpha, \quad \operatorname{div}^\alpha, \quad (-\Delta)^{\alpha/2},$$

where α is a real number, the order of the operator. A frequently encountered definition of the fractional gradient uses an orthogonal basis $e_1, \dots, e_n$ in the physical space V and one-dimensional "partial" fractional derivatives $\mathrm{D}^\alpha_{\xi_i} f$ of a function f to lay down the formula

$$\nabla^\alpha f(x) = \mathrm{D}^\alpha_{\xi_1} f(x) e_1 + \dots + \mathrm{D}^\alpha_{\xi_n} f(x) e_n.$$

It will be shown that this definition is wrong: unlike the classical case $\alpha = 1$, it depends on the chosen basis, i.e., $\nabla^\alpha f$ does not transform as a vector under rotations. The same objection applies to similarly constructed fractional divergence and laplacean. The paper presents a novel approach to the operators of fractional vector analysis based on elementary requirements, viz.,

- translational invariance,
- rotational invariance,
- homogeneity of degree $\alpha \in \mathbb{R}$ under isotropic scaling;
- certain weak requirement of continuity.

Using methods of the theory of homogeneous distributions the paper

- proves that these requirements determine the fractional operators uniquely to within a multiplication by a scalar factor;

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- derives explicit formulas for these operators.

For $(-\Delta)^{\alpha/2}$ we recover the standard formulas for the fractional laplacean. For the fractional gradient the requirements lead to

$$\nabla^\alpha f(x) = \begin{cases} \mu_\alpha \lim\limits_{\varepsilon \downarrow 0} \int\limits_{|h| \geq \varepsilon} \dfrac{h f(x+h)}{|h|^{n+\alpha+1}}\, dh & \text{if } 0 \leq \alpha < 1, \\ \nabla f(x) & \text{if } \alpha = 1, \\ \mu_\alpha \int\limits_{\mathbb{R}^n} \dfrac{h\big(f(x+h) - f(x) - \nabla f(x) \cdot h\big)}{|h|^{n+\alpha+1}}\, dh & \text{if } 1 < \alpha \leq 2, \end{cases}$$

$x \in \mathbb{R}^n$, where μ_α is a normalization factor to be determined below from extra additional requirements. (The general case $-\infty < \alpha < \infty$ is treated in Section 4.) The paper then proceeds to prove some basic properties of the fractional operators, such as, e.g., the identity

$$\operatorname{div}^\alpha(\nabla^\beta f) = -(-\Delta)^{(\alpha+\beta)/2} f,$$

which generalizes the classical case $\operatorname{div}(\nabla f) = \Delta f$.

Keywords fractional gradient, divergence, and laplacean, translation invariance, rotation invariance, positive homogeneity of degree α, fractional vector identities, distributions

Contents

1 A frequent fundamental error in the definition of the fractional gradient

The isotropy of space plays a central role in all branches of physics. The success and importance of the classical differential operators of vector analysis, i.e., the gradient, divergence, and laplacean,

$$\nabla, \quad \operatorname{div}, \quad \Delta,$$

derives from their invariance under rotations, a concrete manifestation of isotropy.*

* The importance of symmetry and invariance principles for continuum mechanics was fully recognized for the first time by *Walter Noll.*

Recent works on long-range forces and on other nonlocal aspects of continuum mechanics employ the fractional analogs

$$\nabla^\alpha, \quad \mathrm{div}^\alpha, \quad (-\Delta)^{\alpha/2}$$

of the classical operators of non-integral orders α.★ The Riesz-Bochner-Feller fractional laplacean is classical; the literature is extremely large. In contrast, the fractional gradient and divergence are relatively new and only in a preliminary stage of development. Frequent approaches to the fractional gradient are based on a blind analogy with the integral-order: choose an orthonormal basis in the physical space V and represent the fractional gradient of a scalar function f on V by an n-tuple of one-dimensional "partial" fractional derivatives along the co-ordinate axes.★★ This definition has a fatal flaw: *the gradient* $\nabla^\alpha f$ *constructed in this manner depends on the chosen coordinate system.*

To see it, let f be a scalar-valued function on the n-dimensional real inner product space V. In an orthonormal basis $\{e_1, \dots, e_n\}$ in V, the function f is represented by a function $\tilde{f}$ of the coordinates $\xi_1, \dots, \xi_n$ of a general point x in V,

$$f(x) = \tilde{f}(\xi_1, \dots, \xi_n).$$

The fractional gradient of f is then defined by

$$\nabla^\alpha f = \mathrm{D}^\alpha_{\xi_1} \tilde{f} e_1 + \dots + \mathrm{D}^\alpha_{\xi_n} \tilde{f} e_n$$

where $\mathrm{D}^\alpha_{\xi_i} f$ are the one-dimensional partial fractional derivatives with respect to ξ_i, $1 \le i \le n$. For example, for $0 < \alpha < 1$ one can use Marchaud's derivative [36; Chapter 2]:

$$\mathrm{D}^\alpha_{\xi_i} f(\xi_1, \dots, \xi_n) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{\varphi(\xi_i) - \varphi(\xi_i - \eta)}{\eta^{\alpha+1}}\, d\eta$$

where Γ is the gamma function and for the given i, φ is a function of one scalar variable ξ, defined by

$$\varphi(\xi) = \tilde{f}(\xi_1, \dots, \xi_{i-1}, \xi, \xi_{i+1}, \dots, \xi_n)$$

with ξ_j for $j \neq i$ frozen. *No further formula needs to be written to see that this definition depends on the basis.* Indeed, to evaluate, say, $\nabla^\alpha f(0)$, one needs to know only the behavior of f along the *coordinate axes given by the basis,* the rest of the landscape of f is irrelevant. In a different basis, the same recipe sees the behavior of f in a different, totally unrelated n-tuple of directions, which, thanks to the nonlocality of $\mathrm{D}^\alpha_{\xi_i}$, leads to a different result in general. For example, one can have $\nabla^\alpha f(0) = 0$ in one basis and $\nabla^\alpha f(0) \neq 0$ in a different basis. Paradoxically, the reason for this flaw is not that the adopted definition is nonlocal, but that it is *not nonlocal enough* to encompass the behavior of f in all directions.★

★ There are many papers on the subject in the past 10–15 years, which differ in rigor and clarity. See, e.g., [28, 27, 32, 18, 7, 6] for some applications and [2–4, 10, 24, 5, 25, 41, 26, 42, 9, 1, 8, 43, 29–30, 33] for the theory and discussions of general aspects.

★★ The exceptions are [24–26, 8, 33].

★ An analogous coordinate definition of the fractional laplacean, In this paper I formulate natural "qualitative" requirements on the fractional operators: the invariance under translations and rotations, homogeneity of order α under isotropic scaling (to fix the order of the operator), and some mild continuity. The paper

- proves that these requirements determine the fractional operators uniquely to within a multiplication by a scalar factor;
- derives explicit formulas for these operators,

see Sections 2 and 4. The derivation uses aspects of the theory of homogeneous of distributions; references are given in Sections 4 to 8.

Outline of the paper. Section 2 describes the fractional operators of orders $0 \leq \alpha \leq 2$ on infinitely differentiable functions with compact support. Section 3 introduces two kinds of function spaces for the fractional operators used here, the spaces of type $\mathscr{D}$ and $\mathscr{T}$. The central section of the paper, Section 4, defines the fractional operators of all orders $-\infty < \alpha < \infty$ (or even of all complex orders), and establishes their position as the only operators satisfying our requirements. Section 5 presents fractional generalizations of the main classical vector identities and analytic dependence of the fractional operators on the order $\alpha \in \mathbb{C}$. Section 6 outlines weak definitions of fractional operators which extend them to less regular functions (actually to distributions from the dual of $\mathscr{T}$). The last two sections are devoted to proofs. Section 7 establishes the convergence of the integrals occurring in the definitions fractional operators and other analytical aspects. Finally, Section 8 shows that every operator meeting our qualitative requirements is a multiple of the fractional operators defined in Section 4.

2 Fractional gradient, divergence, and laplacean: the case $0 \leq \alpha \leq 2$

We work in a real inner product space of dimension n, which we represent by $\mathbb{R}^n$. For each $a \in \mathbb{R}^n$, $\lambda > 0$, and $q \in \mathrm{O}(n)$ (= the group of orthogonal transformations in $\mathbb{R}^n$) we denote by τ_a, η_λ and ρ_q the transformations in $\mathbb{R}^n$ given by

$$\tau_a x = x - a, \quad \eta_\lambda x = \lambda x, \quad \rho_q x = q^{-1} x, \quad x \in \mathbb{R}^n.$$

We use the symbol $\circ$ to denote the composition of mappings, defining the composite map $\phi \circ \psi$ by $(\phi \circ \psi)(x) = \phi(\psi(x))$ for any two mappings ϕ and ψ.

We characterize the fractional gradient of order $\alpha \in \mathbb{R}$ as a linear transformation G which associates with each "nice" complex-valued function f on $\mathbb{R}^n$ a complex-vector-valued function $Gf \equiv G(f)$ on $\mathbb{R}^n$ satisfying the following requirements

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$$(-\Delta)^{\alpha/2} u := -(\mathrm{D}^{\alpha}_{\xi_1} u + \ldots + \mathrm{D}^{\alpha}_{\xi_n} u)$$

results in a highly coordinate-dependent fractional wave equation

$$u_{tt} - (-\Delta)^{\alpha/2} u = f,$$

with obvious undesirable consequences. Fractional Maxwell's equations occurring in the literature are subject to the same criticism.

$$G(f \circ \tau_a) = (Gf) \circ \tau_a, \tag{2.1}$$

$$G(f \circ \rho_q) = q(Gf) \circ \rho_q, \tag{2.2}$$

$$G(f \circ \eta_\lambda) = \lambda^\alpha (Gf) \circ \eta_\lambda \tag{2.3}$$

for any nice f, any $q \in \mathrm{O}(n)$, $a \in \mathbb{R}^n$, and $\lambda > 0$. The translation invariance (2.1) need not be commented; Condition (2.2) is the independence on the basis in $\mathbb{R}^n$. Condition (2.3) fixes the *order* of the operator in analogy with the derivatives of integer orders. When augmented with a mild continuity requirement, Conditions (2.1)–(2.3) determine the operator G uniquely to within a constant multiplicative factor, as shown in Theorems 2.2 and 4.2. Analogous results for the fractional divergence and laplacean are presented in Theorems 2.4, 2.6 and 4.6.

We shall define the fractional operators on three types of spaces, viz., on scalar- and vector-valued versions of the space $\mathscr{D}$, on scalar- and vector-valued versions of the space $\mathscr{T}$, and on dual of $\mathscr{T}'$ of the space $\mathscr{T}$. The definitions of $\mathscr{D}$ and $\mathscr{T}$ are given in Section 3; the differences in the behavior of the fractional operators are described at the beginning of Section 4. We use the terms scalar- or vector-valued operator on $\mathscr{D}$ or $\mathscr{T}$ for any linear transformation from these domains into the spaces of all complex-valued or complex-vector-valued (= $\mathbb{C}^n$-valued) functions on $\mathbb{R}^n$. We use the modifiers 'translationally invariant' and 'α-homogeneous' for any scalar- or vector-valued operator satisfying (2.1) and (2.3), respectively. We use the symbols ∇ and div (without exponents) for the classical differential operators; the symbol $(-\Delta)^k$ with k a nonnegative integer is the power of the classical laplacean.

For nonnegative values of α the formulas for fractional operators display certain *periodicity with period* 2. In this section we treat only the first period, $0 \le \alpha \le 2$. Throughout this section, we assume tacitly this restriction on α.

2.1 Definition The fractional gradient ∇^α of order α is a vector-valued operator on $\mathscr{D}(\mathbb{R}^n)$ defined by

$$\nabla^\alpha f(x) = \begin{cases} \mu_\alpha \lim\limits_{\varepsilon \downarrow 0} \int\limits_{|h| \ge \varepsilon} \dfrac{hf(x+h)}{|h|^{n+\alpha+1}}\, dh & \text{if } 0 \le \alpha < 1, \\ \nabla f(x) & \text{if } \alpha = 1, \\ \mu_\alpha \int\limits_{\mathbb{R}^n} \dfrac{h\big(f(x+h) - f(x) - \nabla f(x) \cdot h\big)}{|h|^{n+\alpha+1}}\, dh & \text{if } 1 < \alpha \le 2 \end{cases}$$

for any $f \in \mathscr{D}(\mathbb{R}^n)$ and any $x \in \mathbb{R}^n$, where

$$\mu_\alpha := 2^\alpha \pi^{-n/2} \mathbf{\Gamma}\big((n+\alpha+1)/2\big) / \mathbf{\Gamma}\big((1-\alpha)/2\big). \tag{2.4}$$

The normalization factor μ_α (as well as the factor ν_α to be introduced below) is not important for most of the discussion in this paper. It is important only for the factional vector identities and for the analyticity to be discussed in Section 5.

It will be shown below that the definition of ∇^α is consistent in the sense that the involved integrals converge and the limits exist. The same remarks apply to Definitions 2.3 and 2.5, below.

Examples 6.2 and 6.3 evaluate the fractional gradient of Dirac's delta function $\nabla^\alpha \delta(x)$ and the one-dimensional fractional gradient $\mathrm{D}^\alpha \theta(x)$ of the Heaviside function θ for $0 < \alpha < 1$.

2.2 Theorem *The operator* $G := \nabla^\alpha$ *is*
(i) $\mathscr{D}$*-continuous,*★
(ii) *translationally invariant,*
(iii) *rotationally invariant in the sense of* (2.2)*, and*
(iv) α*-homogeneous.*
Conversely, any vector-valued operator G on $\mathscr{D}(\mathbb{R}^n)$ *satisfying Conditions* (i)–(iv) *is a scalar multiple of* ∇^α*, i.e.,*

$$Gf(x) = c\nabla^\alpha f(x)$$

for all $f \in \mathscr{D}(\mathbb{R}^n)$*, all* $x \in \mathbb{R}^n$ *and some* $c \in \mathbb{C}$.

2.3 Definition The fractional divergence $\operatorname{div}^\alpha$ of order α is a scalar-valued operator on $\mathscr{D}(\mathbb{R}^n, \mathbb{C}^n)$ defined by

$$\operatorname{div}^\alpha v(x) = \begin{cases} \mu_\alpha \lim\limits_{\varepsilon \downarrow 0} \int_{|h| \ge \varepsilon} \dfrac{h \cdot v(x+h)}{|h|^{n+\alpha+1}}\, dh & \text{if } 0 \le \alpha < 1, \\ \operatorname{div} v(x) & \text{if } \alpha = 1, \\ \mu_\alpha \int_{\mathbb{R}^n} \dfrac{h \cdot \big(v(x+h) - v(x) - \nabla v(x) \cdot h\big)}{|h|^{n+\alpha+1}}\, dh & \text{if } 1 < \alpha \le 2 \end{cases}$$

for any $v \in \mathscr{D}(\mathbb{R}^n, \mathbb{C}^n)$ and any $x \in \mathbb{R}^n$.

2.4 Theorem *The operator* $S := \operatorname{div}^\alpha$ *is*
(i) $\mathscr{D}$*-continuous,*
(ii) *translationally invariant,*
(iii) *rotationally invariant in the sense that*

$$S\big(q(v \circ \rho_q)\big) = S(v) \circ \rho_q \tag{2.5}$$

for any $v \in \mathscr{D}(\mathbb{R}^n, \mathbb{C}^n)$ *and any* $q \in \mathrm{O}(n)$*, and*
(iv) α*-homogeneous.*
Conversely, any scalar-valued operator S on $\mathscr{D}(\mathbb{R}^n, \mathbb{C}^n)$ *satisfying Conditions* (i)–(iv) *is a scalar multiple of* $\operatorname{div}^\alpha$*, i.e.,*

$$Sv(x) = c \operatorname{div}^\alpha v(x)$$

for all $v \in \mathscr{D}(\mathbb{R}^n, \mathbb{C}^n)$*, all* $x \in \mathbb{R}^n$ *and some* $c \in \mathbb{C}$.

2.5 Definition The fractional laplacean $(-\Delta)^{\alpha/2}$ of order $\alpha/2$ is a scalar-valued operator on $\mathscr{D}(\mathbb{R}^n)$ defined by

$$(-\Delta)^{\alpha/2} f(x) = \begin{cases} f(x) & \text{if } \alpha = 0, \\ \nu_\alpha \int_{\mathbb{R}^n} \dfrac{f(x+h) - f(x)}{|h|^{n+\alpha}}\, dh & \text{if } 0 < \alpha < 1, \\ \nu_\alpha \lim\limits_{\varepsilon \downarrow 0} \int_{|h| \ge \varepsilon} \dfrac{f(x+h) - f(x)}{|h|^{n+\alpha}}\, dh & \text{if } 1 \le \alpha < 2, \\ -\Delta f(x) & \text{if } \alpha = 2 \end{cases}$$

★ That is, for any $x \in \mathbb{R}^n$, the function $f \mapsto G(f)(x)$ is a Schwartz distribution. See Section 3.

for any $f \in \mathscr{D}(\mathbb{R}^n)$ and any $x \in \mathbb{R}^n$, where

$$\nu_\alpha := 2^\alpha \pi^{-n/2} \mathbf{\Gamma}\big((n+\alpha)/2\big)/\mathbf{\Gamma}(-\alpha/2). \tag{2.6}$$

2.6 Theorem *The operator* $L := (-\Delta)^{\alpha/2}$ *is*
(i) $\mathscr{D}$*-continuous,*
(ii) *translationally invariant,*
(iii) *rotationally invariant in the sense that*

$$L(f \circ \rho_q) = (Lf) \circ \rho_q, \tag{2.7}$$

for any $f \in \mathscr{D}(\mathbb{R}^n)$ *any* $q \in \mathrm{O}(n)$, *and*
(iv) α*-homogeneous.*
Conversely, any scalar-valued operator L *on* $\mathscr{D}(\mathbb{R}^n)$ *satisfying Conditions* (i)–(iv) *is a scalar multiple of* $(-\Delta)^{\alpha/2}$ *i.e.,*

$$Lf(x) = c(-\Delta)^{\alpha/2} f(x)$$

for all $f \in \mathscr{D}(\mathbb{R}^n)$, *all* $x \in \mathbb{R}^n$ *and some* $c \in \mathbb{C}$.

2.7 Remark (Tensorisation) The tensorial nature of functions for which we have defined the fractional differential operators can be generalized in the following way. Let V be a finite-dimensional complex vector space. We then define the tensor product $V \otimes \mathbb{C}^n$ as the linear space of all complex-linear transformations T from $\mathbb{C}^n$ into V. If $b \in \mathbb{C}^n$, we denote by $T \cdot b := T[b]$ the value of T on b, alternatively called the contraction of T by b. If $a \in V$ and $b \in \mathbb{C}^n$, we define the tensor product $a \otimes b \in V \otimes \mathbb{C}^n$ by $(a \otimes b)[c] = a(b \cdot c)$ for every $c \in \mathbb{C}^n$, where $b \cdot c = b_1 c_1 + \ldots + b_n c_n$ for every $b = (b_1, \ldots, b_n)$, $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$.

The fractional gradient ∇^α is then a map from the space $\mathscr{D}(\mathbb{R}^n, V)$ of V-valued testfunctions on $\mathbb{R}^n$ into the space of $V \otimes \mathbb{C}^n$-valued functions on $\mathbb{R}^n$ defined by

$$\nabla^\alpha f(x) = \begin{cases} \mu_\alpha \lim\limits_{\varepsilon \downarrow 0} \int\limits_{|h| \ge \varepsilon} \dfrac{f(x+h) \otimes h}{|h|^{n+\alpha+1}}\, dh & \text{if } \ 0 \le \alpha < 1, \\ \nabla f(x) & \text{if } \ \alpha = 1, \\ \mu_\alpha \int\limits_{\mathbb{R}^n} \dfrac{\big(f(x+h) - f(x) - \nabla f(x)[h]\big) \otimes h}{|h|^{n+\alpha+1}}\, dh & \text{if } \ 1 < \alpha \le 2 \end{cases}$$

for any $f \in \mathscr{D}(\mathbb{R}^n, V)$ and any $x \in \mathbb{R}^n$, where $\nabla f(x)[h]$ is the value of the linear transformation $\nabla f(x)$ on $h \in \mathbb{R}^n$.

Similarly, the fractional divergence $\operatorname{div}^\alpha$ is a map from the space $\mathscr{D}(\mathbb{R}^n, V \otimes \mathbb{C}^n)$ of $V \otimes \mathbb{C}^n$-valued testfunctions on $\mathbb{R}^n$ into the space of V-valued functions on $\mathbb{R}^n$ defined by

$$\operatorname{div}^\alpha v(x) = \begin{cases} \mu_\alpha \lim\limits_{\varepsilon \downarrow 0} \int\limits_{|h| \ge \varepsilon} \dfrac{v(x+h) \cdot h}{|h|^{n+\alpha+1}}\, dh & \text{if } \ 0 \le \alpha < 1, \\ \operatorname{div} v(x) & \text{if } \ \alpha = 1, \\ \mu_\alpha \int\limits_{\mathbb{R}^n} \dfrac{\big(v(x+h) - v(x) - \nabla v(x)[h]\big) \cdot h}{|h|^{n+\alpha+1}}\, dh & \text{if } \ 1 < \alpha \le 2 \end{cases}$$

for any $v \in \mathscr{D}(\mathbb{R}^n, V \otimes \mathbb{C}^n)$ and any $x \in \mathbb{R}^n$, where '$\cdot$' is the contraction defined above and $\nabla v(x)[h]$ is the value of the linear transformation $\nabla v(x)$ on $h \in \mathbb{R}^n$.

Finally, the fractional laplacean $(-\Delta)^{\alpha/2}$ of order $\alpha/2$ is a map from the space $\mathscr{D}(\mathbb{R}^n, V)$ of V-valued testfunctions on $\mathbb{R}^n$ into the space of V-valued functions on $\mathbb{R}^n$ defined by formulas from Definition 2.5 (no modification is needed in this case).

The same generalizations apply to fractional operators of arbitrary order as introduced in Section 4. We also note that the vector space V may be the space of tensors of arbitrary order, thus defining the fractional operators of tensors etc.

3 The spaces $\mathscr{D}(\mathbb{R}^n, Z)$ and $\mathscr{T}(\mathbb{R}^n, Z)$

Throughout, Z denotes a finite-dimensional complex normed vector space, mostly a placeholder of $\mathbb{C}$ or $\mathbb{C}^n$. Further, we denote by $\mathbb{N}_0$ and $\mathbb{N}$ the sets of all nonnegative and all positive integers, respectively.

We denote by $\mathscr{D}(\mathbb{R}^n, Z)$ the Schwartz space of infinitely differentiable complex-valued functions $f : \mathbb{R}^n \to Z$ with compact support. We abbreviate $\mathscr{D}(\mathbb{R}^n) := \mathscr{D}(\mathbb{R}^n, \mathbb{C})$ and simplify the notation in the same way also for other function spaces below. A sequence f_k of elements of $\mathscr{D}(\mathbb{R}^n, Z)$ is said to $\mathscr{D}$-converge to an element f of $\mathscr{D}(\mathbb{R}^n, Z)$ if there is a compact subset K of $\mathbb{R}^n$ such that the supports of all f_k are contained in K and all gradients $\nabla^i f_k$ uniformly converge to $\nabla^i f$, $i = 0, 1, \ldots$ Let X be another finite-dimensional complex normed vector space. A linear map U from $\mathscr{D}(\mathbb{R}^n, Z)$ into the space of X-valued functions on $\mathbb{R}^n$ is said to be $\mathscr{D}$-continuous if for any fixed $x \in \mathbb{R}^n$ one has

$$U(f_k)(x) \to U(f)(x) \tag{3.1}$$

for any sequence f_k that $\mathscr{D}$-converges to f.

Following [22–23], we introduce the space $\mathscr{T}(\mathbb{R}^n, Z)$ of all infinitely differentiable maps $f : \mathbb{R}^n \to Z$ such that the gradient $\nabla^i f$ of any order $i \in \mathbb{N}_0$ satisfies

$$\int_{\mathbb{R}^n} |\nabla^i f(x)|\, dx < \infty \quad \text{and} \quad \nabla^i f(x) \to 0 \quad \text{as} \quad |x| \to \infty.$$

We endow $\mathscr{T}(\mathbb{R}^n, Z)$ with a countable system of norms $|\cdot|_m$, $m \in \mathbb{N}_0$, defined by

$$|f|_m = \max\{\|\nabla^i f\|_1, \|\nabla^i f\|_\infty : i \in \mathbb{N}_0,\ 0 \le i \le m\}$$

where $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are the standard norms on the spaces $L^1(\mathbb{R}^n, Z)$ and $L^\infty(\mathbb{R}^n, Z)$. Here $L^p(\mathbb{R}^n, Z)$ is the Lebesgue space of Z-valued maps on $\mathbb{R}^n$ integrable with p th power, $1 \le p \le \infty$. We have the following embeddings:

$$\mathscr{T}(\mathbb{R}^n, Z) \subset L^p(\mathbb{R}^n, Z) \quad \text{for each} \quad p \in [1, \infty].$$

A sequence f_k of elements of $\mathscr{T}(\mathbb{R}^n, Z)$ is said to $\mathscr{T}$-converge to an element $f \in \mathscr{T}(\mathbb{R}^n, Z)$ if

$$|f_k - f|_m \to 0 \quad \text{as} \quad k \to \infty \quad \text{for each} \quad m \in \mathbb{N}_0.$$

A linear map U from a subset $\mathscr{A}$ of $\mathscr{T}(\mathbb{R}^n, Z)$ into the space of functions on $\mathbb{R}^n$ with values in a finite dimensional vector space X is said to be $\mathscr{T}$-continuous if for any fixed $x \in \mathbb{R}^n$ we have (3.1) for any sequence f_k of elements of $\mathscr{A}$ that $\mathscr{T}$-converges to an element $f \in \mathscr{A}$.

4 Fractional gradient, divergence, and laplacean: the general case

The objective of this section is to extend previous results on ∇^α, $\operatorname{div}^\alpha$, and $(-\Delta)^{\alpha/2}$ to all orders $\alpha \in \mathbb{R}$, eventually satisfying $\alpha \geq -n$. We shall consider these operators on the spaces of type $\mathscr{D}$ and $\mathscr{T}$. There are two main differences in the behavior of the fractional operators on $\mathscr{D}$ and on $\mathscr{T}$. First, while there are nontrivial translationally and rotationally invariant, α-homogeneous, and continuous operators on the space $\mathscr{D}$ for all $\alpha \in \mathbb{R}$, on the larger space $\mathscr{T}$ such nontrivial operators exist only for $\alpha \geq -n$. The second, and more important difference is that for $\alpha \geq 0$ the fractional operators map the space $\mathscr{T}$ into $\mathscr{T}$; there is no counterpart of that for the spaces of the type $\mathscr{D}$. The mapping property for $\mathscr{T}$ is necessary for the weak definitions of the fractional operators on irregular functions in Section 6.

If $\varphi : \mathbb{R}^n \to Z$ is a Lebesgue measurable map, we define the principal value of the integral $\int_{\mathbb{R}^n} \varphi(h)\, dh$ by

$$\operatorname{Pv} \int_{\mathbb{R}^n} \varphi(h)\, dh := \lim_{\varepsilon \downarrow 0} \int_{|h|>\varepsilon} \varphi(h)\, dh$$

provided the integrals on the right-hand side converge for all $\varepsilon > 0$ and the limit exists and is finite. If φ is absolutely integrable on $\mathbb{R}^n$, then

$$\operatorname{Pv} \int_{\mathbb{R}^n} \varphi(h)\, dh = \int_{\mathbb{R}^n} \varphi(h)\, dh$$

but of course the principal value can exist even when the integral does not exist.

If $f \in \mathscr{T}(\mathbb{R}^n, Z)$, $x \in \mathbb{R}^n$, $i \in \mathbb{N}_0$, and $h \in \mathbb{R}^n$, we write

$$\nabla^i f(x) \cdot h^i := \nabla^i f(x) \underbrace{(h, \dots, h)}_{i \text{ times}} \tag{4.1}$$

where on the right-hand side, $\nabla^i f(x)$ is interpreted as a symmetric Z-valued i-linear form on $\mathbb{R}^n$. We denote by $[\alpha]$ the integral part of a complex number α, defined as $[\alpha] = [\operatorname{Re} \alpha]$.

Since $\mathbf{\Gamma}$ is defined on $\mathbb{C} \sim \{-k : k \in \mathbb{N}_0\}$, the normalization factor μ_α from (2.4) is defined for all $\alpha \in \mathbb{C}$ except for the elements of the set

$$\{-n-1, -n-3, \dots\} \cup \{1, 3, \dots\}$$

and the normalization factor ν_α from (2.6) is defined for all $\alpha \in \mathbb{C}$ except for the elements of the set

$$\{-n, -n-2, \dots\} \cup \{0, 2, \dots\}.$$

Let us put

$$\mathbb{E}_\mu := \{-n-1, -n-3, \dots\}, \quad \mathbb{E}_\nu := \{-n, -n-2, \dots\}.$$

4.1 Definitions (Fractional gradient and divergence) The fractional gradient ∇^α of order $\alpha \in \mathbb{R} \sim \mathbb{E}_\mu$ is a vector-valued operator on $\mathscr{D}(\mathbb{R}^n)$ defined by

$$\nabla^\alpha f(x) = \begin{cases} \mu_\alpha \int\limits_{\mathbb{R}^n} \dfrac{hf(x+h)}{|h|^{n+\alpha+1}}\,dh, \\ \mu_\alpha \operatorname{Pv} \int\limits_{\mathbb{R}^n} \dfrac{h\Big(f(x+h) - \sum\limits_{i=0}^{2[(\alpha-1)/2]+1} \dfrac{\nabla^i f(x)\cdot h^i}{i!}\Big)}{|h|^{n+\alpha+1}}\,dh, \\ (-\Delta)^{(\alpha-1)/2}\nabla f(x) \end{cases} \tag{4.2}$$

for every $f \in \mathscr{D}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$; the three regimes in (4.2) correspond to the following three conditions on $\alpha \in \mathbb{R} \sim \mathbb{E}_\mu$:

$$\text{(i)}: \ \alpha < 0, \qquad \text{(ii)}: \ \alpha \ge 0 \ \text{ not odd}, \qquad \text{(iii)}: \ \alpha \ge 0 \ \text{ odd}, \tag{4.3}$$

respectively. The fractional divergence $\operatorname{div}^\alpha$ of order $\alpha \in \mathbb{R} \sim \mathbb{E}_\mu$ is a scalar-valued operator on $\mathscr{D}(\mathbb{R}^n, \mathbb{C}^n)$ defined by

$$\operatorname{div}^\alpha v(x) = \begin{cases} \mu_\alpha \int\limits_{\mathbb{R}^n} \dfrac{h\cdot v(x+h)}{|h|^{n+\alpha+1}}\,dh, \\ \mu_\alpha \operatorname{Pv} \int\limits_{\mathbb{R}^n} \dfrac{h\cdot\Big(v(x+h) - \sum\limits_{i=0}^{2[(\alpha-1)/2]+1} \dfrac{\nabla^i v(x)\cdot h^i}{i!}\Big)}{|h|^{n+\alpha+1}}\,dh, \\ (-\Delta)^{(\alpha-1)/2}\operatorname{div} v(x), \end{cases} \tag{4.4}$$

for every $v \in \mathscr{D}(\mathbb{R}^n, \mathbb{C}^n)$ and $x \in \mathbb{R}^n$; the three regimes in (4.4) correspond to Conditions (i), (ii), and (iii) in (4.3).

We shall see in Section 7 that the integrals in (4.2), (4.4) and in (4.5), below, converge and the principal values exist. Furthermore, recall that the restriction $\alpha \notin \mathbb{E}_\mu$ comes from the fact that μ_α is undefined on $\mathbb{E}_\mu$; if we omit that factor, then the right-hand sides of (4.2) and (4.4) are meaningful for all $\alpha \in \mathbb{R}$. We use the symbols $\nabla_\circ^\alpha f$, and $\operatorname{div}_\circ^\alpha v$ for the so simplified expressions.

We have the following two generalizations of Theorem 2.2.

4.2 Theorem *If $\alpha \in \mathbb{R}$, then $\nabla_\circ^\alpha$ is a vector-valued, translationally and rotationally invariant, α-homogeneous, and $\mathscr{D}$-continuous operator on $\mathscr{D}(\mathbb{R}^n)$; conversely, any operator with these properties is a multiple of $\nabla_\circ^\alpha$.*

The rotational invariance in Theorems 4.2 and 4.3 is interpreted in the sense of (2.2). The proofs of all results in this section are deferred to Sections 7 and 8. The following theorem strengthens Theorem 4.2 if $\alpha \ge -n$.

4.3 Theorem *If $\alpha \ge -n$, then Equation* (4.2) *is meaningful for all $f \in \mathscr{T}(\mathbb{R}^n)$, which defines ∇^α as a translationally and rotationally invariant, α-homogeneous, and $\mathscr{T}$-continuous vector-valued operator on $\mathscr{T}(\mathbb{R}^n)$; conversely, any operator with these properties is a multiple of ∇^α.*

4.4 Remark *If $f \in \mathscr{T}(\mathbb{R}^n)$ and $\alpha \in (1,2) \cup (3,4) \cup (5,6) \cup \dots$, then the principal value symbol in* $(4.2)_2$ *can be omitted.*

With this assertion we see that the fractional gradient introduced in Definition 4.1 reduces to that from Definition 2.1 if $0 \le \alpha \le 2$.

For completeness, we note that if $\alpha < -n$, there is no nontrivial translationally and rotationally invariant, α-homogeneous, and $\mathscr{T}$-continuous vector-valued operator on $\mathscr{T}(\mathbb{R}^n)$. Briefly and roughly, the reason is that the $\mathscr{T}$-continuity enforces that the operator must be given by $(4.2)_1$ for all $f \in \mathscr{T}(\mathbb{R}^n)$. However, if $\alpha < -n$, then $\mathscr{T}(\mathbb{R}^n)$ contains many elements f for which the integral in $(4.2)_1$ diverges. The same applies to scalar-valued operators on $\mathscr{T}(\mathbb{R}^n)$ and on $\mathscr{T}(\mathbb{R}^n, \mathbb{C}^n)$.

There are analogs of Theorems 4.2 and 4.3 and of Remark 4.4 for the fractional divergence. These analogs generalize Theorem 2.4 and show that the fractional divergence introduced in Definition 4.1 reduces to that in Definition 2.3 if $0 \le \alpha \le 2$. Since the modifications are obvious, explicit statements are omitted.

4.5 Definition (Fractional laplacean) The fractional laplacean of order $\alpha/2$, where $\alpha \in \mathbb{R} \sim \mathbb{E}_v$, is a scalar-valued operator on $\mathscr{D}(\mathbb{R}^n)$ defined by

$$(-\Delta)^{\alpha/2} f(x) = \begin{cases} v_\alpha \int\limits_{\mathbb{R}^n} \dfrac{f(x+h)}{|h|^{n+\alpha}}\, dh, \\[2ex] v_\alpha \operatorname{Pv} \int\limits_{\mathbb{R}^n} \dfrac{f(x+h) - \sum\limits_{i=0}^{2[\alpha/2]} \dfrac{\nabla^i f(x) \cdot h^i}{i!}}{|h|^{n+\alpha}}\, dh, \\[2ex] (-\Delta)^{\alpha/2} f(x) \end{cases} \tag{4.5}$$

for every $f \in \mathscr{D}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$; the three regimes in (4.5) correspond to the following three conditions on $\alpha \in \mathbb{R} \sim \mathbb{E}_v$:

$$\text{(i)}: \ \alpha < 0, \qquad \text{(ii)}: \ \alpha \ge 0 \ \text{ not even}, \qquad \text{(iii)}: \ \alpha \ge 0 \ \text{ even},$$

respectively.

Recall that the restriction $\alpha \notin \mathbb{E}_v$ comes from the fact that v_α is undefined on $\mathbb{E}_v$; if we omit that factor, then the right-hand side of (4.5) is meaningful for all $\alpha \in \mathbb{R}$. We use the symbol $(-\Delta)_\circ^{\alpha/2} f$ for the so simplified expression.

The following two theorems generalize Theorem 2.6.

4.6 Theorem *If $\alpha \in \mathbb{R}$, then $(-\Delta)_\circ^{\alpha/2}$ is a scalar-valued, translationally and rotationally invariant, α-homogeneous, and $\mathscr{D}$-continuous operator on $\mathscr{D}(\mathbb{R}^n)$; conversely, any operator with these properties is a multiple of $(-\Delta)_\circ^{\alpha/2}$.*

In Theorems 4.6 and 4.7, the rotational invariance is interpreted in the sense of (2.7).

4.7 Theorem *If $\alpha > -n$, then Equation* (4.5) *is meaningful for all $f \in \mathscr{T}(\mathbb{R}^n)$, which defines $(-\Delta)^{\alpha/2}$ as a translationally and rotationally invariant, α-homogeneous, and $\mathscr{T}$-continuous scalar-valued operator on $\mathscr{T}(\mathbb{R}^n)$; conversely, any operator with these properties is a multiple of $(-\Delta)^{\alpha/2}$.*

The following remark shows that the fractional laplacean introduced in Definition 4.5 reduces to that in Definition 2.5 if $0 \le \alpha \le 2$.

4.8 Remark *If* $f \in \mathscr{T}(\mathbb{R}^n)$ *and* $\alpha \in (0,1) \cup (2,3) \cup (4,5) \cup \dots$, *then the principal value symbol in* $(4.5)_2$ *can be omitted.*

We conclude this section with a discussion of the definitions in this section.

Formulas (4.5) are standard; if $-n < \alpha < 0$, then $(-\Delta)^{\alpha/2} f$ is called M. Riesz's fractional integral of f of order $\beta = -\alpha$ while if $\alpha \geq 0$, then $(-\Delta)^{\alpha/2} f$ is called Riesz-Bochner-Feller's fractional power of the laplacean or M. Riesz's fractional derivative of f of order α. The term 'fractional power' is in harmony with the Balakrishnan theory of fractional powers of nonnegative operators, see, e.g., [22]. Equations $(4.5)_{2,3}$ are particular cases of the formulas for the power A^γ of a general operator A with $A = -\Delta$ and $\gamma = \alpha/2$. There is a large literature on Riesz's fractional integro-differentiation and on fractional powers of operators, see [36] and [22] for detailed expositions of the theory, history and bibliography of these subjects. A recent paper [19] reviews possible definitions of the fractional laplacean and describes their relationships in various function spaces.

The fractional gradient ∇^0 of order $\alpha = 0$ is the classical Riesz's transform, which has numerous applications in analysis. The generalization to all α, i.e., the object denoted here by $\nabla^\alpha f$, is due to J. Horváth [16–17], although he did not interpret his construction as the fractional gradient. Horváth defines ∇^α equivalently as the distributional gradient of Riesz's fractional derivative. The interpretation of Horváth's distribution as the fractional gradient appears under the name *Riesz's fractional gradient* in the recent papers [39, 37, 40]. The notion of fractional divergence seems to be new, but it is analogous to the fractional gradient.

There are several equivalent formulas for the fractional laplacean, which include those containing iterated finite differences [36; Chapter 5, Equation (25.61)], and the formulas splitting the integration over $\mathbb{R}^n$ into those near the origin and over the rest of $\mathbb{R}^n$, with a simultaneous introduction of regularizing terms [36; Chapter 5, Equation (26.66)]. We also refer to [16, 31] and the references therein. Similar alternatives can be constructed for the fractional gradient and divergence.

5 Analyticity, ranges, and fractional vector identities

As mentioned in Section 2, the normalization factors μ_α and ν_α are introduced to get suitable analyticity properties of ∇^α, $\operatorname{div}^\alpha$ and $(-\Delta)^{\alpha/2}$ and simple forms of fractional vector identities for them. The present section briefly discusses these questions.

5.1 Theorem (Analyticity)

(i) *For each* $f \in \mathscr{D}(\mathbb{R}^n)$, $v \in \mathscr{D}(\mathbb{R}^n, \mathbb{C}^n)$ *and* $x \in \mathbb{R}^n$ *the functions which associate with any* $\alpha \in \mathbb{R} \sim \mathbb{E}_\mu$ *the values*

$$\nabla^\alpha f(x), \quad \operatorname{div}^\alpha v(x), \tag{5.1}$$

and with any $\alpha \in \mathbb{R} \sim \mathbb{E}_\nu$ *the number*

$$(-\Delta)^{\alpha/2} f(x) \tag{5.2}$$

have analytic extensions to the set $\mathbb{C} \sim \mathbb{E}_\mu$ *and* $\mathbb{C} \sim \mathbb{E}_\nu$, *respectively.*

(ii) *If* $v \in \mathscr{T}(\mathbb{R}^n, \mathbb{C}^n)$ *and* $f \in \mathscr{T}(\mathbb{R}^n)$*, the expressions in* (5.1) *and* (5.2) *have analytic extensions from the set* $\{\alpha \in \mathbb{R} : \alpha > -n\}$ *to the set* $\{\alpha \in \mathbb{C} : \operatorname{Re}\alpha > -n\}$.

The extensions of $f \in \mathscr{D}(\mathbb{R}^n)$ are given by the right-hand sides of (4.2) and (4.4) without any modification for all $\alpha \in \mathbb{C} \sim \mathbb{E}_\mu$ except for the "punctuated vertical lines" in $\mathbb{C}$ constituting the set

$$\mathbb{F}_\mu := \{\alpha \in \mathbb{C} : \operatorname{Re}\alpha = 1, 3, \dots, \operatorname{Im}\alpha \neq 0\}.$$

The extension of (5.2) is given by the right-hand side of (4.5) for all $\alpha \in \mathbb{C} \sim \mathbb{E}_\nu$ except for the punctuated vertical lines in $\mathbb{C}$ constituting the set

$$\mathbb{F}_\nu := \{\alpha \in \mathbb{C} : \operatorname{Re}\alpha = 0, 2, \dots, \operatorname{Im}\alpha \neq 0\}.$$

The formulas for the extensions on the exceptional lines are omitted.

The analyticity is a standard tool for classes of problems like the present one, dating back to the foundational works of M. Riesz [34], L. Schwartz [38], I. M. Gel'fand and G. E. Shilov [13–14], N. S. Landkof [20] and others. The reader is referred to any of these works for the proof the analyticity of $(-\Delta)^{\alpha/2} f(x)$ for any $f \in \mathscr{D}(\mathbb{R}^n)$. The proof of the remaining assertions of Theorem 5.1 are analogous.

5.2 Proposition (Ranges) *If* $\operatorname{Re}\alpha > 0$, *then the operators* ∇^α *and* $(-\Delta)^{\alpha/2}$ *map the space* $\mathscr{T}(\mathbb{R}^n)$ *into the spaces* $\mathscr{T}(\mathbb{R}^n, \mathbb{C}^n)$ *and* $\mathscr{T}(\mathbb{R}^n)$*, respectively, while the operator* $\operatorname{div}^\alpha$ *maps* $\mathscr{T}(\mathbb{R}^n, \mathbb{C}^n)$ *into* $\mathscr{T}(\mathbb{R}^n)$.

This invariance of the spaces of the type $\mathscr{T}$ opens the way to the fractional vector identities and to the weak definitions of the fractional operators in Section 6. This is the main motivation for the spaces of type $\mathscr{T}$. The classical test function spaces $\mathscr{D}(\mathbb{R}^n)$ of compactly supported functions or the space $\mathscr{S}(\mathbb{R}^n)$ of rapidly decreasing functions *do not* enjoy similar invariances. Also note that Proposition 5.2 does not hold for $\operatorname{Re}\alpha < 0$. We refer to [23] for the proof of Proposition 5.2 in the case of $(-\Delta)^{\alpha/2}$; the cases of ∇^α and $\operatorname{div}^\alpha$ follow by easy modifications.

5.3 Theorem (Fractional vector identities) *If* $\alpha, \beta \in \mathbb{C}$ *satisfy* $\operatorname{Re}\alpha > -n$ *and* $\operatorname{Re}\beta \ge 0$, *we have the following identities*:

$$\nabla^\alpha(-\Delta)^{\beta/2} = \nabla^{\alpha+\beta}, \tag{5.3}$$

$$\operatorname{div}^\alpha(-\Delta)^{\beta/2} = \operatorname{div}^{\alpha+\beta}, \tag{5.4}$$

$$(-\Delta)^{\alpha/2}(-\Delta)^{\beta/2} = (-\Delta)^{(\alpha+\beta)/2}, \tag{5.5}$$

$$\operatorname{div}^\alpha \nabla^\beta = -(-\Delta)^{(\alpha+\beta)/2} \tag{5.6}$$

for arguments from the spaces $\mathscr{T}(\mathbb{R}^n)$ *and* $\mathscr{T}(\mathbb{R}^n, \mathbb{C}^n)$, *or, if possible or if needed, to the spaces of tensorial extensions of the fractional operators described in Subsection* 2.7. *Formulas* (5.5) *and* (5.6) *imply the following inversion formulas if* $0 \le \operatorname{Re}\alpha < n$:

$$(-\Delta)^{-\alpha/2}(-\Delta)^{\alpha/2} = \operatorname{id}_{\mathscr{T}(\mathbb{R}^n)},$$

$$-\operatorname{div}^{-\alpha}\nabla^\alpha = \operatorname{id}_{\mathscr{T}(\mathbb{R}^n)}$$

where $\operatorname{id}_{\mathscr{T}(\mathbb{R}^n)}$ *is the identity map on* $\mathscr{T}(\mathbb{R}^n)$.

Our restrictions on α and β come from our choice of the test functions spaces $\mathscr{T}$. Different choices lead to less restrictive conditions, see [16–17, 31] and the references therein. Equation (5.5) is standard while (5.3) is due to Horváth (in a different notation). We conclude the discussion with the following particular cases of (5.3) and (5.4): *If* $-n+1 \le \alpha < \infty$ *then*

$$\nabla^\alpha f(x) = \nabla(-\Delta)^{(\alpha-1)/2} f(x),$$
$$\operatorname{div}^\alpha v(x) = \operatorname{div}(-\Delta)^{(\alpha-1)/2} v(x).$$

These formulas can be used to define ∇^α and $\operatorname{div}^\alpha$ in terms of the fractional laplacean.

6 Weak definitions of ∇^α, $\operatorname{div}^\alpha$, and $(-\Delta)^{\alpha/2}$

In this section we briefly discuss the extension of the fractional operators to larger classes of objects than functions from $\mathscr{D}(\mathbb{R}^n)$ or $\mathscr{T}(\mathbb{R}^n)$.

The definitions are based on the duality between the fractional gradient and divergence and on the formal self-adjointness of the fractional laplacean, i.e., on the easily verifiable relations

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha v\, dx = -\int_{\mathbb{R}^n} v \cdot \nabla^\alpha f\, dx$$

and

$$\int_{\mathbb{R}^n} f(-\Delta)^{\alpha/2} g\, dx = \int_{\mathbb{R}^n} g(-\Delta)^{\alpha/2} f\, dx$$

for every $f, g \in \mathscr{T}(\mathbb{R}^n)$, $v \in \mathscr{T}(\mathbb{R}^n, \mathbb{C}^n)$, and $\alpha \ge 0$.

We denote by $\mathscr{T}'(\mathbb{R}^n, Z)$ and $\mathscr{D}'(\mathbb{R}^n, Z)$ the spaces of continuous functionals on $\mathscr{T}(\mathbb{R}^n, Z)$ and $\mathscr{D}(\mathbb{R}^n, Z)$. The space $\mathscr{T}'(\mathbb{R}^n, Z)$ can be interpreted as the set of all distributions from $\mathscr{D}'(\mathbb{R}^n, Z)$ that have a $\mathscr{T}$-continuous extension to $\mathscr{T}(\mathbb{R}^n, Z)$. In this interpretation, for example, Dirac's δ-function belongs to $\mathscr{T}'(\mathbb{R}^n)$ since the formula $\delta(f) = f(0)$ gives a continuous functional on $\mathscr{T}(\mathbb{R}^n)$. The difference between $\mathscr{D}'(\mathbb{R}^n, Z)$ and $\mathscr{T}'(\mathbb{R}^n, Z)$ can be seen on distributions represented by functions. It is well-known that the distribution $H \in \mathscr{D}'(\mathbb{R}^n, Z)$, given by

$$H(g) = \int_{\mathbb{R}^n} f(x) \cdot g(x)\, dx, \tag{6.1}$$

for every $g \in \mathscr{D}(\mathbb{R}^n)$ is well defined if f is a locally integrable function on $\mathbb{R}^n$. On the other hand, the Banach space duality theory shows that the distribution H can be extended to a continuous functional on $\mathscr{T}(\mathbb{R}^n)$ if and only if f can be written as $f = f_1 + f_\infty$ where $f_1 \in L^1(\mathbb{R}^n)$ and $f_\infty \in L^\infty(\mathbb{R}^n)$. The latter fact is written symbolically as

$$f \in L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n).$$

Thus, e.g., the functional (6.1) with, say, $f(x) = |x|^\beta$, belongs to $\mathscr{D}'(\mathbb{R}^n)$ for all β satisfying $-n < \beta < \infty$ but to $\mathscr{T}'(\mathbb{R}^n)$ only if $-n < \beta \le 0$.

6.1 Definition Let $\operatorname{Re}\alpha \geq 0$. If $\mathfrak{f} \in \mathscr{T}'(\mathbb{R}^n)$ and $\mathfrak{v} \in \mathscr{T}'(\mathbb{R}^n, \mathbb{C}^n)$, we define $\nabla^\alpha \mathfrak{f}$ as an element of $\mathscr{T}'(\mathbb{R}^n, \mathbb{C}^n)$ and $(-\Delta)^{\alpha/2}\mathfrak{f}$ and $\operatorname{div}^\alpha \mathfrak{v}$ as elements of $\mathscr{T}'(\mathbb{R}^n)$ by the formulas

$$(\nabla^\alpha \mathfrak{f})(w) = -\mathfrak{f}(\operatorname{div}^\alpha w),$$

$$((-\Delta)^{\alpha/2}\mathfrak{f})(g) = \mathfrak{f}((-\Delta)^{\alpha/2}g),$$

$$(\operatorname{div}^\alpha \mathfrak{v})(g) = -\mathfrak{v}(\nabla^\alpha g),$$

for every $g \in \mathscr{T}(\mathbb{R}^n)$ and $w \in \mathscr{T}(\mathbb{R}^n, \mathbb{C}^n)$.

Ordinary functions are covered by these definitions by associating with any $f \in L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ and any $v \in L^1(\mathbb{R}^n, \mathbb{C}^n) + L^\infty(\mathbb{R}^n, \mathbb{C}^n)$ the functionals $\mathfrak{f} \in \mathscr{T}'(\mathbb{R}^n)$ and $\mathfrak{v} \in \mathscr{T}'(\mathbb{R}^n, \mathbb{C}^n)$ given by

$$\mathfrak{f}(g) = \int_{\mathbb{R}^n} gf\,dx, \quad \mathfrak{v}(w) = \int_{\mathbb{R}^n} w \cdot v\,dx$$

for any $g \in \mathscr{T}(\mathbb{R}^n)$ and $w \in \mathscr{T}(\mathbb{R}^n, \mathbb{C}^n)$.

The widened scope of the fractional operators allows us to present the following examples.

6.2 Example Let us calculate $\nabla^\alpha \delta$ for $0 \leq \alpha < 1$. The Definitions 6.1 and 2.1 give

$$(\nabla^\alpha \delta)(v) = -(\operatorname{div}^\alpha v)(0) = -\mu_\alpha \operatorname{Pv} \int_{\mathbb{R}^n} \frac{x \cdot v(x)}{|x|^{n+\alpha+1}}\,dx$$

for every $v \in \mathscr{T}(\mathbb{R}^n)$; this can be written symbolically as

$$\nabla^\alpha \delta(x) = -\mu_\alpha \operatorname{Pv} \frac{x}{|x|^{n+\alpha+1}}.$$

6.3 Example Let $n = 1$ and let $\theta : \mathbb{R} \to \mathbb{R}$ be the Heaviside function,

$$\theta(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$$

We shall show that the fractional gradient (derivative) $\mathrm{D}^\alpha := \nabla^\alpha$ of θ for $0 < \alpha < 1$ is given by

$$\mathrm{D}^\alpha \theta(x) = \frac{\nu_{\alpha-1}}{|x|^\alpha} \tag{6.2}$$

where $\nu_{\alpha-1}$ is given by (2.6) with $n = 1$, i.e.,

$$\nu_{\alpha-1} = \frac{2^{\alpha-1}\Gamma(\alpha/2)}{\pi^{1/2}\Gamma((1-\alpha)/2)}.$$

Note that $\mathrm{D}^\alpha \theta(x)$ approaches $\mathrm{D}\theta(x) = \delta(x)$ as α approaches 1, i.e.,

$$\frac{\nu_{\alpha-1}}{|x|^\alpha} \to \delta(x) \quad \text{in} \quad \mathscr{T}'(\mathbb{R}) \quad \text{as} \quad \alpha \to 1.$$

Thus the family of functions $g_\alpha(x) := \nu_{\alpha-1}/|x|^\alpha$, $0 < \alpha < 1$, is an approximation of the delta function in the sense that

$$\int_{\mathbb{R}} g_\alpha(x) f(x)\, dx \to f(0) \quad \text{as} \quad \alpha \to 1 \tag{6.3}$$

for every $f \in \mathscr{T}(\mathbb{R})$. Unlike the usual family of Friedrich's mollifiers

$$g_\rho(x) = \rho^{-1}\varphi(x/\rho), \quad 0 < \rho < 1,$$

with the support of $\varphi \subset (-1, 1)$ and $\int_{\mathbb{R}} \varphi(x)\, dx = 1$, here the support of $g_\alpha = \mathbb{R}$ and $\int_{\mathbb{R}} g_\alpha(x)\, dx = \infty$. We shall see that to establish (6.3), it suffices to use the localization properties saying that for any $\varepsilon > 0$ and any $f \in \mathscr{T}(\mathbb{R})$,

$$\int_{-\varepsilon}^{\varepsilon} g_\alpha(x)\, dx \to 1 \quad \text{and} \quad \int_{|x|>\varepsilon} g_\alpha(x) f(x)\, dx \to 0 \quad \text{as} \quad \alpha \to 1. \tag{6.4}$$

These, in turn, follow from the asymptotics

$$\nu_{\alpha-1} \sim (\alpha - 1)/2 \quad \text{near} \quad \alpha = 1, \tag{6.5}$$

which is a consequence of $\mathbf{\Gamma}(z) \sim 1/z$ near $z = 0$. Of course, alternatively, (6.3) follows from the analyticity of $\mathrm{D}^\alpha\theta$ asserted by Theorem 5.1.

Proof We have

$$\mathrm{D}^\alpha\theta(v) = -\theta(\mathrm{div}^\alpha v) = -\theta\big((-\Delta)^{(\alpha-1)/2}\,\mathrm{div}\, v\big)$$

for each $v \in \mathscr{T}(\mathbb{R})$, where the first equality is the definition and the second equality follows from the identity $\mathrm{div}^\alpha v = (-\Delta)^{(\alpha-1)/2}\,\mathrm{div}\, v$, a particular case of (5.4). Using

$$(-\Delta)^{(\alpha-1)/2}\,\mathrm{div}\, v(x) = \nu_{\alpha-1} \int_{\mathbb{R}} \frac{v'(x+h)}{|h|^\alpha}\, dh$$

we find

$$\mathrm{D}^\alpha\theta(v) = -\nu_{\alpha-1} \int_{\mathbb{R}} \theta(x) \int_{\mathbb{R}} \frac{v'(x+h)}{|h|^\alpha}\, dh\, dx.$$

Exchanging the orders of integration and evaluating the inner integral according to $\int_{\mathbb{R}} \theta(x) v'(x+h)\, dx = -v(h)$ we find, finally,

$$\mathrm{D}^\alpha\theta(v) = \nu_{\alpha-1} \int_{\mathbb{R}} v(h)\, |h|^{-\alpha}\, dh,$$

which is (6.2).

Next, we prove (6.4). Assertion $(6.4)_1$ follows by combining the equality $\int_{-\varepsilon}^{\varepsilon} |x|^{-\alpha}\, dx = 2\varepsilon^{1-\alpha}/(1-\alpha)$ with (6.5). Further, we observe that

$$\Big| \int_{|x|>\varepsilon} g_\alpha(x) f(x)\, dx \Big| \le \nu_{\alpha-1}\varepsilon^{-\alpha} \int_{|x|>\varepsilon} |f(x)|\, dx;$$

since the integral on the right-hand side is finite, the limit $\alpha \to 1$ using (6.5) provides $(6.4)_2$. We now employ (6.4) to prove (6.3). To this end, we choose $\eta > 0$ and use the continuity of f at 0 to find $\varepsilon > 0$ such that $|f(x) - f(0)| < \eta$ for all $|x| < \varepsilon$. Then $|\int_{-\varepsilon}^{\varepsilon} g_\alpha(x)\big(f(x) - f(0)\big)\, dx| \le \eta \int_{-\varepsilon}^{\varepsilon} g_\alpha(x)\, dx$. We now use the estimate

$$\begin{aligned}\Big|\int_{\mathbb{R}} g_\alpha(x) f(x)\, dx - f(0)\Big| &\le \Big| \int_{-\varepsilon}^{\varepsilon} g_\alpha(x)\big(f(x) - f(0)\big)\, dx\Big| \\ &\quad + \Big|f(0) - f(0) \int_{-\varepsilon}^{\varepsilon} g_\alpha(x)\Big| + \Big| \int_{|x|>\varepsilon} g_\alpha(x) f(x)\, dx\Big|\end{aligned} \tag{6.6}$$

in the following way. By the above, the first term on the right-hand side of (6.6) is estimated by $\eta \int_{-\varepsilon}^{\varepsilon} g_\alpha(x)\,dx$. By $(6.4)_1$ this term converges to η as $\alpha \to 1$. The remaining two terms on the right-hand side of (6.6) converge to 0 by $(6.4)_{1,2}$. Thus

$$\limsup_{\alpha\to 0} \Big| \int_{\mathbb{R}} g_\alpha(x) f(x)\,dx - f(0) \Big| \le \eta$$

and the arbitrariness of $\eta > 0$ gives (6.3). □

7 Consistency of the definitions of ∇^α, $\operatorname{div}^\alpha$, and $(-\Delta)^{\alpha/2}$

In this section we shall show that the right-hand sides of Formulas (4.2), (4.4), and (4.5) in the definitions of the fractional operators deliver finite numbers and well-defined vectors. In view of the translational invariance, it suffices to consider only the special case $x = 0$ in these formulas. This reduces the proofs in this section and in Section 8 to the theory of homogeneous distributions, see Schwartz [38], Gel'fand & Shapiro [12], Gel'fand & Shilov [13; Sections I.3 and III.3], Lemoine [21], Hörmander [15; Section 3.2] and Estrada & Kanwal [11; Section 2.6]. Even though the results can be reconstructed from these sources, we present complete proofs for the reader's convenience.

In addition to setting $x = 0$, we omit the unessential normalization factors μ_α and ν_α and we also change the notation by writing x for the variable previously denoted by h.

If f is a k-times continuously differentiable function on $\mathbb{R}^n$ with values in Z, we denote by $\mathbf{T}^k f$ the Taylor expansion of order k of f at 0, i.e., a function defined on $\mathbb{R}^n$ with values in Z given by

$$\mathbf{T}^k f(x) = \sum_{i=0}^{k} \frac{\nabla^i f(0)\cdot x^i}{i!}$$

for any $x \in \mathbb{R}^n$, where we use the notation (4.1).

7.1 Proposition

(i) *If $\alpha \in \mathbb{R}$ then $(-\Delta)_\circ^{\alpha/2}$ is $\mathscr{D}$-continuous, α-homogeneous, and translationally and rotationally invariant scalar operator on $\mathscr{D}(\mathbb{R}^n)$.*

(ii) *If $\alpha > -n$ then an analogous assertion holds for the operator $(-\Delta)^{\alpha/2}$ on $\mathscr{T}(\mathbb{R}^n)$.*

(iii) *If $\alpha \in (0,1) \cup (2,3) \cup (4,5) \cup \dots$, the principal value symbol in the second regime in* (4.5) *can be omitted for any $f \in \mathscr{T}(\mathbb{R}^n)$.*

Here the rotational invariance is interpreted in the sense of (2.7).

Proof It suffices to prove only Assertions (ii) and (iii); Assertion (i) is proved by obvious simplifications of the proof of (ii).

Proof of (ii): Let $\alpha > -n$ and $f \in \mathscr{T}(\mathbb{R}^n)$ and prove that Equation (4.5) gives a well-defined element in $\mathscr{T}'(\mathbb{R}^n)$. Since the regime $(4.5)_3$ is clear, we consider only the regimes $(4.5)_{1,2}$.

Prove that *if $\alpha < 0$, the integral in $(4.5)_1$ absolutely converges, i.e., that*

$$\int_{\mathbb{R}^n} |x|^{-n-\alpha}|f(x)|\,dx < \infty. \tag{7.1}$$

We split the integral in (7.1) into the sum

$$\int_{|x|\le 1} |x|^{-n-\alpha}|f(x)|\,dx + \int_{|x|>1} |x|^{-n-\alpha}|f(x)|\,dx.$$

The first integral converges since f is bounded and $x \mapsto |x|^{-n-\alpha}$ locally integrable. To prove that the second integral converges also, we note that $0 < n+\alpha < n$ so that $n/(n+\alpha) > 1$. We choose any $p > n/(n+\alpha)$ and use Hölder's inequality

$$\int_{|x|>1} |x|^{-n-\alpha}|f(x)|\,dx \le \Big(\int_{|x|>1} |x|^{-p(n+\alpha)}\,dx\Big)^{1/p}\Big(\int_{|x|>1} |f(x)|^q\,dx\Big)^{1/q} \tag{7.2}$$

where $q = p/(p-1)$. The first integral on the right-hand side of (7.2) converges by our choice of p; the second integral converges also since $\mathscr{T}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ for any $q \ge 1$. Thus the integral in $(4.5)_1$ absolutely converges.

Prove that *if* $\alpha \ge 0$ *is not even, the principal value in* $(4.5)_2$ *exists and is finite.* We shall prove the following formula for the principal value:

$$\lim_{\varepsilon\downarrow 0}\int_{|x|>\varepsilon} \frac{f(x) - \mathbf{T}^{2[\alpha/2]}f(x)}{|x|^{n+\alpha}}\,dx = \int_{|x|\le 1} \frac{f(x) - \mathbf{T}^{2[\alpha/2]+1}f(x)}{|x|^{n+\alpha}}\,dx + \int_{|x|>1} \frac{f(x) - \mathbf{T}^{2[\alpha/2]}f(x)}{|x|^{n+\alpha}}\,dx \tag{7.3}$$

where the two integrals on the right-hand side absolutely converge.

To establish the convergence of the first integral on the right-hand side of (7.3), we observe that the order of the singularity at 0 is $\mathrm{O}(|x|^{2[\alpha/2]+2-n-\alpha})$. The singularity is integrable since $2[\alpha/2]+2-n-\alpha > -n$ as a consequence of the definition of the integral part of a number. To establish the convergence of the second integral on the right-hand side of (7.3), we note that the order of the integrand at ∞ is $\mathrm{O}(|x|^{2[\alpha/2]-n-\alpha})$. Since $\alpha \ge 0$ is not even, we have $[\alpha/2] < \alpha/2$ and thus the exponent $2[\alpha/2]-n-\alpha$ satisfies $2[\alpha/2]-n-\alpha < -n$. This proves the convergence of the second integral.

To prove the equality in (7.3), we note that the convergence of the first integral on the right-hand side of (7.3) allows us to replace that integral by

$$\lim_{\varepsilon\downarrow 0}\int_{\varepsilon<|x|\le 1} \frac{f(x) - \mathbf{T}^{2[\alpha/2]+1}f(x)}{|x|^{n+\alpha}}\,dx. \tag{7.4}$$

Since the last term in the Taylor expansion $\mathbf{T}^{2[\alpha/2]+1}f(x)$, i.e., the term

$$\nabla^{2[\alpha/2]+1}f(0)\cdot x^{2[\alpha/2]+1}/(2[\alpha/2]+1)!$$

is an odd function of the integration variable x, it is annihilated by the integration over the symmetric domain in (7.4). So we can replace $\mathbf{T}^{2[\alpha/2]+1}f(x)$ in (7.4) by $\mathbf{T}^{2[\alpha/2]}f(x)$. Then the right-hand side of (7.3) becomes

$$\lim_{\varepsilon\downarrow 0}\int_{\varepsilon<|x|\le 1} \frac{f(x) - \mathbf{T}^{2[\alpha/2]}f(x)}{|x|^{n+\alpha}}\,dx + \int_{|x|>1} \frac{f(x) - \mathbf{T}^{2[\alpha/2]}f(x)}{|x|^{n+\alpha}}\,dx$$

which is exactly the left-hand side of that equation. Thus, the principal value in $(4.5)_2$ exists and is finite.

To complete the proof of (ii), we already know that the right-hand side of (4.5) provides a well-defined scalar-valued operator L on $\mathscr{T}(\mathbb{R}^n)$. Elementary substitutions in the integrals in (4.5) show that L is translationally and rotationally invariant and α-homogeneous. Also, it is a matter of a routine use of Lebesgue's dominated convergence theorem to establish the $\mathscr{T}$-continuity of L. These steps are omitted and the proof of (ii) is complete.

Proof of (iii): We have to establish the absolute convergence of the integral

$$\int_{\mathbb{R}^n} \frac{f(x) - \mathbf{T}^{2[\alpha/2]} f(x)}{|x|^{n+\alpha}}\, dx.$$

In view of the above discussion it suffices to estimate the integrand near the origin. By the mean value theorem, the integrand is of order $\mathrm{O}(|x|^{2[\alpha/2]+1-n-\alpha})$. The integrability near the origin then follows from $2[\alpha/2] + 1 - n - \alpha > -n$ which in turn follows from the hypothesis $0 < \alpha - 2[\alpha/2] < 1$. □

7.2 Proposition

(i) *If* $\alpha \in \mathbb{R}$, *then* $\nabla^\alpha_\circ$ *and* $\operatorname{div}^\alpha_\circ$ *are* $\mathscr{D}$*-continuous,* α*-homogeneous, and translationally and rotationally invariant vector and scalar operators on* $\mathscr{D}(\mathbb{R}^n)$ *and* $\mathscr{D}(\mathbb{R}^n, \mathbb{C}^n)$.

(ii) *If* $\alpha > -n$, *then analogous assertions hold for the operators* ∇^α *and* $\operatorname{div}^\alpha$ *on* $\mathscr{T}(\mathbb{R}^n)$ *and* $\mathscr{T}(\mathbb{R}^n, \mathbb{C}^n)$.

(iii) *If* $\alpha \in (1,2) \cup (3,4) \cup (5,6) \cup \dots$, *the principal value symbols in* $(4.2)_2$ *and* $(4.4)_2$ *can be omitted.*

Here the rotational invariance is interpreted in the sense of (2.2) and (2.5), respectively.

Proof The assertions can be proved in several ways. One possibility is to adapt the proof of Proposition 7.1 to the present situation. Perhaps a more conceptual and definitely shorter way is to reduce the present proposition to Proposition 7.1. This is the way we shall go. The proof will explain, among other things, the shift by -1 in the regimes of α in (4.2) and (4.4) with respect to those in (4.5).

In the case of ∇^α the reduction to Proposition 7.1 is accomplished by the following formula:

$$\nabla^\alpha f(0) = -\frac{(-\Delta)^{(\alpha+1)/2} v(0)}{\alpha + 1} \tag{7.5}$$

for any $f \in \mathscr{T}(\mathbb{R}^n)$, where $v(x) = x f(x)$, $x \in \mathbb{R}^n$. We note that the case $\alpha = -1$ must be excluded, but this gap is easily overcome since the value $\alpha = -1$ is in no way special for the fractional gradient.

To establish (7.5), in Equation (4.5) we replace the function f by the function v and the exponent α by $\alpha + 1$. This reduces the right-hand side of (4.5) to the right-hand side of (4.2) by noting that the Leibniz rule for derivatives of order k gives

$$\nabla^k v(0) \cdot x^k = k x \nabla^{k-1} f(0) \cdot x^{k-1}$$

and consequently,

$$\mathbf{T}^{2[(\alpha+1)/2]}v(x) = x\mathbf{T}^{2[(\alpha-1)/2]+1}f(x),$$

where we use that $2[(\alpha+1)/2]-1 = 2[(\alpha-1)/2]+1$. The proof is completed by the identity

$$\nu_{\alpha+1} = -(\alpha+1)\mu_\alpha, \tag{7.6}$$

which follows from (2.4) and (2.6).

The assertion about $\operatorname{div}^\alpha$ is reduced to Proposition 7.1 by noting that

$$\operatorname{div}^\alpha v(0) = -\frac{(-\Delta)^{(\alpha+1)/2}f(0)}{\alpha+1}$$

for any $v \in \mathscr{T}(\mathbb{R}^n, \mathbb{C}^n)$, where $f(x) = x \cdot v(x)$, $x \in \mathbb{R}^n$. This is proved by employing the formula

$$\nabla^k f(0) \cdot x^k = kx \cdot \nabla^{k-1} v(0) \cdot x^{k-1},$$

its consequence

$$\mathbf{T}^{2[(\alpha+1)/2]}f(x) = x \cdot \mathbf{T}^{2[(\alpha-1)/2]+1}v(x),$$

and (7.6). □

8 Invariant α-homogeneous operators

In this section we complete the proofs of the results in Sections 2 and 4 by proving that any operator meeting our requirements of invariance, homogeneity and continuity is a multiple of one of the fractional operators of Section 4. The most systematic and conceptual proof seems to be the one which would combine the description of a general α-homogeneous distribution by Lemoine [21; Theorem 3.1.1]★ with the rotational invariance. Nevertheless, a direct proof is given below without the reference to Lemoine's result.

We say that a scalar-valued function $\tilde{f} : \mathbb{R}^n \to \mathbb{C}$ is rotationally invariant if $\tilde{f}(qx) = \tilde{f}(x)$ for every $x \in \mathbb{R}^n$ and $q \in \mathrm{O}(n)$. We say that a vector-valued function $\tilde{v} : \mathbb{R}^n \to \mathbb{C}^n$ is rotationally invariant if $\tilde{v}(qx) = q\tilde{v}(x)$ for every $x \in \mathbb{R}^n$ and $q \in \mathrm{O}(n)$. It is well-known that $\tilde{f}$ is rotationally invariant if and only if there exists a function $\varphi : [0,\infty) \to \mathbb{C}$ such that

$$\tilde{f}(x) = \varphi(|x|) \tag{8.1}$$

for every $x \in \mathbb{R}^n$ and $\tilde{v}$ is rotationally invariant if and only if there exists a rotationally invariant scalar-valued function $\tilde{f}$ such that

$$\tilde{v}(x) = x\tilde{f}(x) \tag{8.2}$$

for every $x \in \mathbb{R}^n$.

We denote by $\mathscr{D}(\mathbb{R}^n_\mp)$ the Schwartz space of all complex-valued functions f on $\mathbb{R}^n$ with compact support that is contained in $\mathbb{R}^n_\mp := \mathbb{R}^n \sim \{0\}$ and by $\mathscr{D}'(\mathbb{R}^n_\mp)$ the dual of $\mathscr{D}(\mathbb{R}^n_\mp)$.

★ Incidentally, an essential hypothesis, stated elsewhere in Lemoine's paper, viz. Equation (1.3.1), is missing in Part b) of [21; Theorem 3.1.1]. See [11; Theorem 18, Section 2.6] for a complete statement (without proof, though).

8.1 Lemma *Let $\alpha \in \mathbb{R}$ and let $F \in \mathscr{D}'(\mathbb{R}^n)$ be a rotationally invariant and α-homogeneous functional,★ i.e., let F satisfy*

$$F(f \circ \rho_q) = F(f), \quad F(f \circ \eta_\lambda) = \lambda^\alpha F(f) \tag{8.3}$$

for every $f \in \mathscr{D}(\mathbb{R}^n)$, every $q \in O(n)$, and every $\lambda > 0$. Then there exists a $c \in \mathbb{C}$ such that

$$F(f) = c \int_{\mathbb{R}^n} \frac{f(x)}{|x|^{n+\alpha}}\, dx \tag{8.4}$$

for every $f \in \mathscr{D}(\mathbb{R}^n_{\ddagger})$.

Proof Let us first show that there exists a $c \in \mathbb{C}$ such that

$$F(\tilde{f}) = c \int_0^\infty \frac{\varphi(s)}{s^{\alpha+1}}\, ds \tag{8.5}$$

for every rotationally invariant function $\tilde{f} \in \mathscr{D}(\mathbb{R}^n_{\ddagger})$ where φ is as in (8.1). We denote by $\mathscr{C}$ the set of all functions $\varphi : \mathbb{R} \to \mathbb{C}$ with compact support that is contained in $(0, \infty)$. Let $A : \mathscr{C} \to \mathbb{C}$ be defined by

$$A(\varphi) = F(\tilde{f})$$

for every $\varphi \in \mathscr{C}$ where $\tilde{f}$ is given by (8.1). Clearly, A is α-homogeneous, i.e.,

$$A(\psi_\lambda) = \lambda^\alpha A(\psi) \tag{8.6}$$

for any $\lambda > 0$ and $\psi \in \mathscr{C}$, where ψ_λ is given by $\psi_\lambda(t) = \psi(\lambda t)$, for $t \geq 0$. We differentiate (8.6) with respect to λ at $\lambda = 1$ to obtain

$$\alpha A(\psi) = \frac{A(\psi_\lambda)}{d\lambda}\Big|_{\lambda=1} = A\Big(\frac{d\psi_\lambda}{d\lambda}\Big|_{\lambda=1}\Big). \tag{8.7}$$

That the differentiation can be absorbed into the argument of the functional is justified by the continuity of A inherited from the $\mathscr{D}$-continuity of F. It is easily found that

$$\frac{d\psi_\lambda}{d\lambda}\Big|_{\lambda=1}(t) = t\psi'(t),$$

where the prime denotes the standard differentiation with respect to t. Denoting by $\mathscr{m}$ the operation of multiplication by the independent variable, i.e., $(\mathscr{m}\psi)(t) = t\psi(t)$, we rewrite (8.7) as

$$A\big(\mathscr{m}^{\alpha+1}(\mathscr{m}^{-\alpha}\psi)'\big) = 0. \tag{8.8}$$

Let us fix any $\omega \in \mathscr{C}$ such that $\int_0^\infty \omega(s)/s^{\alpha+1}\, ds = 1$. Let now $\tilde{f} \in \mathscr{D}(\mathbb{R}^n_{\ddagger})$ be rotationally invariant with φ as in (8.1). Define $\psi : \mathbb{R} \to \mathbb{C}$ by

$$\psi(t) = t^\alpha \int_0^t \frac{\varphi(s) - \bar{c}\omega(s)}{s^{\alpha+1}}\, ds \tag{8.9}$$

for any $t \geq 0$, where

★ This terminology differs from the standard one, where F is said to be a-homogeneous if it is $(-a-n)$-homogeneous in the present terminology. A distribution corresponding to an a-homogeneous function is a-homogeneous in the standard terminology. The definition adopted here is more convenient for our purpose.

$$\bar{c} = \int_0^\infty \frac{\varphi(s)}{s^{\alpha+1}}\, ds.$$

The choice of $\bar{c}$ and the fact that both φ and ω have bounded supports imply that $\psi(t)$ vanishes for all sufficiently large t. Thus, $\psi \in \mathscr{C}$. Further, a differentiation of (8.9) yields that

$$\varphi = m^{\alpha+1}(m^{-\alpha}\psi)' + \bar{c}\omega$$

and hence

$$A(\varphi) = A\big(m^{\alpha+1}(m^{-\alpha}\psi)'\big) + \bar{c}A(\omega) = \bar{c}A(\omega)$$

by (8.8). Invoking the definition of $\bar{c}$, we obtain (8.5) with $c = A(\omega)$.

Let now $f \in \mathscr{D}(\mathbb{R}^n_\ddagger)$. The integration of $(8.3)_1$ with respect to the Haar measure $\mathfrak{h}$ on $\mathrm{O}(n)$ (see, e.g., [35; Theorem 5.14]) provides

$$F(f) = \int_{\mathrm{O}(n)} F(f \circ \rho_q)\, d\,\mathfrak{h}(q) = F\Big(\int_{\mathrm{O}(n)} f \circ \rho_q\, d\,\mathfrak{h}(q)\Big),$$

where we have used the $\mathscr{D}$-continuity and linearity of F. The function

$$\tilde{f} := \int_{\mathrm{O}(n)} f \circ \rho_q\, d\,\mathfrak{h}(q)$$

is rotationally invariant and thus it admits a representation as (8.1), where

$$\varphi(r) = \frac{1}{\kappa_{n-1} r^{n-1}} \int_{S_r} f(y)\, d\mathscr{H}^{n-1}(y) \tag{8.10}$$

for $r > 0$, where κ_{n-1} is the area of the unit sphere in $\mathbb{R}^n$. Thus $F(\tilde{f})$ is given by (8.5) and hence

$$F(f) = c\int_0^\infty \frac{\varphi(s)}{s^{\alpha+1}}\, ds.$$

The formula (8.10) and the spherical Fubini's theorem gives (8.4). □

8.2 Lemma *Let $\alpha \in \mathbb{R}$ and let $F \in \mathscr{D}'(\mathbb{R}^n)$ be a rotationally invariant and α-homogeneous functional in the sense of Lemma* 8.1. *Then there exists a $c \in \mathbb{C}$ such that*

$$F(f) = c(-\Delta)_\circ^{\alpha/2} f(0)$$

for every $f \in \mathscr{D}(\mathbb{R}^n)$.

Proof The objective is to prove that F is a multiple of the functional given by the right-hand side of (4.5) with $x = 0$, where we omit the normalization factor ν_α.

If $\alpha < 0$, then the right-hand side of (8.4) is meaningful for every $f \in \mathscr{D}(\mathbb{R}^n)$. Hence, if $F(f)$ is given by (8.4) for every $f \in \mathscr{D}(\mathbb{R}^n)$, the proof is complete.

Next we consider the case $\alpha \geq 0$ and not even. By Lemma 8.1 there exists a $c \in \mathbb{C}$ such that (8.4) holds for every $f \in \mathscr{D}(\mathbb{R}^n_\ddagger)$. Let $\tilde{F} : \mathscr{D}(\mathbb{R}^n) \to \mathbb{C}$ be given by

$$\tilde{F}(f) = c\,\mathrm{Pv}\int_{\mathbb{R}^n} \frac{f(x) - \mathbf{T}^{2[\alpha/2]} f(x)}{|x|^{n+\alpha}}\, dx \tag{8.11}$$

for every $f \in \mathscr{D}(\mathbb{R}^n)$. We note that for $f \in \mathscr{D}(\mathbb{R}^n_\ddagger)$ we have $\tilde{F}(f) = F(f)$ by (8.4). The proof of the proposition in the present case will be complete if we show that $F = \tilde{F}$ on $\mathscr{D}(\mathbb{R}^n)$. Since $F = \tilde{F}$ on $\mathscr{D}(\mathbb{R}^n_{\neq})$, the difference $F - \tilde{F}$ is a distribution with the support is contained in $\{0\}$. It is well-known that such a distribution is a linear combination of finitely many gradients of the Dirac distribution δ at $0 \in \mathbb{R}^n$, i.e.,

$$F - \tilde{F} = \sum_{l=0}^{k} c_l \cdot \nabla^l \delta \tag{8.12}$$

for some nonnegative integer k and some (tensorial) constants c_l. However, the distribution $F - \tilde{F}$ must be α-homogeneous. Since $\alpha \geq 0$ is not even, the right-hand side of (8.12) is inconsistent with α-homogeneity of $F - \tilde{F}$ unless all c_l vanish. Thus $F = \tilde{F}$ on $\mathscr{D}(\mathbb{R}^n)$. Thus, F is given by the right-hand side of (8.11), which is also a multiple of the right-hand side of $(4.5)_2$ with $x = 0$. The proof of the proposition in the present case is complete.

Finally, let $\alpha \geq 0$ be even. Lemma 8.1 says that there exists a $c \in \mathbb{C}$ such that $F(f)$ is given by (8.4) for every $f \in \mathscr{D}(\mathbb{R}^n_{\neq})$. We shall now show that there is no way to extend the right-hand side of (8.4) to a rotationally invariant, α-homogeneous and $\mathscr{D}$-continuous functional on $\mathscr{D}(\mathbb{R}^n)$ unless $c = 0$. To prove the claim, let $\omega \in \mathscr{D}(\mathbb{R}^n)$ be any function satisfying $\omega = 1$ in some neighborhood of $0 \in \mathbb{R}^n$ and let $\tilde{F} : \mathscr{D}(\mathbb{R}^n) \to \mathbb{C}$ be defined by

$$\tilde{F}(f) = c \int_{\mathbb{R}^n} |x|^{-n-\alpha} \big(f(x) - \omega(x) \mathbf{T}^\alpha f(x)\big)\, dx$$

for every $f \in \mathscr{D}(\mathbb{R}^n)$. It is not hard to see that $\tilde{F}$ is a well-defined and $\mathscr{D}$-continuous functional, not necessarily α-homogeneous and rotationally invariant. Moreover, $F(f) = \tilde{F}(f)$ for every $f \in \mathscr{D}(\mathbb{R}^n_{\neq})$. Thus, by an argument similar to that given above, the difference $F - \tilde{F}$ takes the form (8.12). If f is such that

$$\nabla^l f(0) = 0, \quad 0 \leq l \leq k, \quad l \neq \alpha,$$

then (8.12) takes the form

$$F(f) = (-1)^\alpha c_\alpha \cdot \nabla^\alpha f(0) + c \int_{\mathbb{R}^n} |x|^{-n-\alpha} \big(f(x) - \omega(x) \cdot \nabla^\alpha f(0) \cdot x^\alpha\big)\, dx.$$

The condition of α-homogeneity reads

$$\begin{aligned}\lambda^\alpha F(f) = {} & (-1)^\alpha \lambda^\alpha c_\alpha \cdot \nabla^\alpha f(0) \\ & + c \int_{\mathbb{R}^n} |x|^{-n-\alpha} \big(f(\lambda x) - \lambda^\alpha \omega(x) \cdot \nabla^\alpha f(0) \cdot x^\alpha\big)\, dx.\end{aligned}$$

The substitution $x \mapsto y = \lambda x$ and the division by λ^α transforms the last equation into

$$F(f) = (-1)^\alpha c_\alpha \cdot \nabla^\alpha f(0) + c \int_{\mathbb{R}^n} |y|^{-n-\alpha} \big(f(y) - \omega(y/\lambda) \cdot \nabla^\alpha f(0) \cdot y^\alpha\big)\, dy.$$

A differentiation of the last relation with respect to λ at $\lambda = 1$ yields

$$c \int_{\mathbb{R}^n} |y|^{-n-\alpha} \big(y \cdot \nabla \omega(y)\big) \cdot \nabla^\alpha f(0) \cdot y^\alpha\, dy = 0$$

and consequently, taking the trace, noting that $\operatorname{tr} y^\alpha = |y|^\alpha$, and using the arbitrariness of $\nabla^\alpha f(0)$ we obtain

$$c \int_{\mathbb{R}^n} |y|^{-n} y \cdot \nabla \omega(y)\, dy = 0. \tag{8.13}$$

We recall that the function ω is completely at our disposal provided $\omega = 1$ in some neighborhood of $0 \in \mathbb{R}^n$. In particular we can assume that $\omega(x) = \theta(|x|)$ for every $x \in \mathbb{R}^n$ where θ is smooth, $\theta = 1$ in some right neighborhood of $0 \in \mathbb{R}$, and $\theta(r) = 0$ for all sufficiently large $r > 0$. Equation (8.13) is then equivalent to

$$c \int_0^\infty \theta'(r)\,dr = -c\theta(0) = -c = 0.$$

Thus $\tilde{F} = 0$ and (8.12) yields

$$F(f) = \sum_{l=0}^{k} (-1)^l c_l \cdot \nabla^l f(0)$$

for every $f \in \mathscr{D}(\mathbb{R}^n)$. The differential operator on the right-hand side is rotationally invariant and α-homogeneous only if it reduces, in a standard way, to a multiple of $\Delta^{\alpha/2}$. Thus, we can write

$$F(f) = a(-\Delta)^{\alpha/2} f(0)$$

for every $f \in \mathscr{D}(\mathbb{R}^n)$ and some $a \in \mathbb{C}$. □

8.3 Lemma *Let $\alpha \in \mathbb{R}$ and let $F \in \mathscr{D}'(\mathbb{R}^n, \mathbb{C}^n)$ be a rotationally invariant and α-homogeneous functional, i.e., let F satisfy*

$$F\big(q^{\mathrm{T}}(v \circ \rho_q)\big) = F(v), \quad F(v \circ \eta_\lambda) = \lambda^\alpha F(v) \tag{8.14}$$

for every $v \in \mathscr{D}(\mathbb{R}^n, \mathbb{C}^n)$, every $q \in \mathrm{O}(n)$, and every $\lambda > 0$. Then there exists a $c \in \mathbb{C}$ such that

$$F(v) = c \operatorname{div}_\circ^\alpha v(0)$$

for every $v \in \mathscr{D}(\mathbb{R}^n, \mathbb{C}^n)$.

Proof Define a $G \in \mathscr{D}'(\mathbb{R}^n)$ by

$$G(f) = F(v) \tag{8.15}$$

for every $f \in \mathscr{D}(\mathbb{R}^n)$, where v is given by

$$v(x) = x f(x),$$

$x \in \mathbb{R}^n$. Then G is rotationally invariant in the sense of $(8.3)_1$ and positively homogeneous of degree $\alpha - 1$. By Lemma 8.2 there exists a $c \in \mathbb{C}$ such that

$$G(f) = c(-\Delta)_\circ^{(\alpha-1)/2} f(0) \tag{8.16}$$

for every $f \in \mathscr{D}(\mathbb{R}^n)$. The integration of the identity $(8.14)_1$ over $\mathrm{O}(n)$ with respect to Haar's measure yields, in the same way as in the proof of Lemma 8.1, the identity

$$F(v) = F(\tilde{v})$$

for every $v \in \mathscr{D}(\mathbb{R}^n, \mathbb{C}^n)$ where

$$\tilde{v}(x) = \int_{\mathrm{O}(n)} q^{\mathrm{T}} v(qx)\, d\mathfrak{h}(q)$$

for every $x \in \mathbb{R}^n$. Clearly, $\tilde{v}$ is rotationally invariant and hence it has the representation (8.2) through a rotationally invariant scalar valued function $\tilde{f}$. Equations (8.15) and (8.16) then give

$$F(\tilde{v}) = G(\tilde{f}) = c(-\Delta)_\circ^{(\alpha-1)/2} \tilde{f}(0).$$

The proof is completed by showing that $(-\Delta)_\circ^{(\alpha-1)/2} \tilde{f}(0) = \operatorname{div}_\circ^\alpha v(0)$. This follows from the definitions of $(-\Delta)_\circ^{\alpha/2}$ and $\operatorname{div}_\circ^\alpha$ in (4.5) and (4.4) by a straightforward computation. □

8.4 Proposition

(i) *If* $\alpha > -n$, *then any translationally and rotationally invariant,* α*-homogeneous, and* $\mathscr{T}$*-continuous scalar-valued operator on* $\mathscr{T}(\mathbb{R}^n)$ *is a multiple of* $(-\Delta)^{\alpha/2}$. *Analogous statements hold for vector-valued operators on* $\mathscr{T}(\mathbb{R}^n)$ *and for scalar-valued operators on* $\mathscr{T}(\mathbb{R}^n, \mathbb{C}^n)$, *which are multiples of* ∇^α *and of* $\operatorname{div}^\alpha$, *respectively.*

(ii) *If* $\alpha \in \mathbb{R}$, *then the operators on* $\mathscr{D}(\mathbb{R}^n)$ *and* $\mathscr{D}(\mathbb{R}^n, \mathbb{C}^n)$ *with similar properties are multiples of* $(-\Delta)_\circ^{\alpha/2}$, $\nabla_\circ^\alpha$ *and* $\operatorname{div}_\circ^\alpha$.

Proof The assertions about scalar-valued operators on the space $\mathscr{D}(\mathbb{R}^n)$ and $\mathscr{D}(\mathbb{R}^n, \mathbb{C}^n)$ follow from Lemmas 8.2 and 8.3 by translation invariance. The assertion about a vector-valued operator on the space $\mathscr{D}(\mathbb{R}^n)$ is reduced to the preceding case by duality: if G is a translationally and rotationally invariant, α-homogeneous, and $\mathscr{D}$-continuous vector-valued operator on $\mathscr{D}(\mathbb{R}^n)$ then its dual (adjoint) S is a translationally and rotationally invariant, α-homogeneous, and $\mathscr{D}$-continuous scalar-valued operator on $\mathscr{D}(\mathbb{R}^n, \mathbb{C}^n)$. The application of the result on S gives eventually the result on G.

The results for $\alpha > -n$ on the spaces of the type $\mathscr{T}$ are derived from those on the spaces the type $\mathscr{D}$ by density: the restrictions of the operators on $\mathscr{T}$ to $\mathscr{D}$ are multiples of $(-\Delta)_\circ^{\alpha/2}$ and $\nabla_\circ^\alpha$ on $\mathscr{D}$. Since the latter space is dense in $\mathscr{T}$ the result extends to $\mathscr{T}$. □

Finally, we note that a combination of the results of Section 7 *and of those of the present section yields the proof of all results of Sections* 2 *and* 4.

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