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A variational approach to nonlinear electro-magneto-elasticity: convexity conditions and existence theorems

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Abstract Electro- or magneto-sensitive elastomers are smart materials whose mechanical properties change instantly by the application of an electric or magnetic fields. The paper analyses the convexity conditions (quasiconvexity, polyconvexity, ellipticity) of the free energy of such materials. These conditions are treated within the framework of the general $\mathscr{A}$-quasiconvexity theory for the constraints

$$\operatorname{curl} F = 0, \quad \operatorname{div} d = 0, \quad \operatorname{div} b = 0, \qquad (*)$$

where F is deformation gradient, d is the electric displacement and b is the magnetic induction. If the energy depends separately only on F, or on d, or on b, the $\mathscr{A}$-quasiconvexity reduces, respectively, to Morrey's quasiconvexity, polyconvexity and ellipticity conditions or to convexity in d or in b. In the present case, the simultaneous occurrence of F, d, and b leads to the cross-phenomena: mechanic-electric, mechanic-magnetic, and electro-magnetic.

The main results of the paper are:

- In dimension 3 there are 32 linearly independent scalar $\mathscr{A}$-affine functions (and 15 in dimension 2) corresponding to the constraints $(*)$.
- Therefore, an energy function $\psi(F, d, b)$ is $\mathscr{A}$-polyconvex if and only if it is of the form

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$$\psi(F, d, b) = \Phi(F, \operatorname{cof} F, \det F, d, b, Fd, Fb)$$

where Φ is a convex function (of 31 scalar variables). Apart from the expected terms F, $\operatorname{cof} F$, $\det F$, d, and b, we have the cross-effect terms Fd, Fb (and in dimension 2 also $d \times b$).
- An existence theorem is proved for a state of minimum energy for a system consisting of an $\mathscr{A}$-polyconvex electro-magneto-elastic solid plus the vacuum electromagnetic field outside the body.

Keywords Electromechanical and magnetomechanical interactions, finite strain, constitutive equations, energy methods, variational principles, instabilities, $\mathscr{A}$-quasiconvexity, $\mathscr{A}$-polyconvexity

Contents

1 Introduction

Electro- or magneto-sensitive elastomers are smart materials whose mechanical properties change instantly by the application of an electric or magnetic field. The sensitivity to the electromagnetic fields is due to the manufacturing process in which some metallic electro- or magneto- sensitive inclusions (such as alumina particles or iron powder) are deposited in an elastomeric (usually rubber) matrix. If the fabrication process is conducted under the external electric or magnetic fields, it produces an alignment of the inclusions and consequently an anisotropy; the latter is combined with large deformations of the matrix. One is thus faced with full nonlinear couplings of the mechanical response with the electric and magnetic fields and also with an indirect magneto-electric coupling.

As is well-known, for large deformations the well-posedness questions play an important role.

For nonlinear elastostatics Ball [1] showed that Morrey's quasiconvexity condition [30–31] has a direct relevance for the behavior of the body; moreover, recognized the importance of Morrey's sufficient condition for quasiconvexity [31; Theorem 4.4.10], for which he introduced the term polyconvexity. He showed that the polyconvexity is compatible with the realistic constraint for the energy function ψ, viz.,

$$\psi(F) \to \infty \quad \text{as} \quad \det F \to 0$$

and leads to a satisfactory existence theorems in nonlinear elasticity under realistic assumptions. These convexity conditions are known to be one of the main guiding principles for the formation of the nonlinear constitutive equations, e.g., [44, 48] and [22].

In this paper I extend the quasiconvexity and polyconvexity notions to the completely coupled problem in electro-magneto-rheological elastomers. There are new issues beyond the purely mechanical case, as we shall see below.

Our choice of the basic variables in the constitutive equations are the deformation gradient F, the (lagrangean) electric displacement d and the (lagrangean) magnetic induction b. Among the new issues that we encounter is that the electromagnetic variables satisfy

$$\operatorname{div} d = 0, \quad \operatorname{div} b = 0 \tag{1.1}$$

identically as a counterpart of

$$\operatorname{curl} F = 0 \tag{1.2}$$

for the deformation gradient F. We make a full use of (1.1) and (1.2) by adopting the convexity theory under differential constraints known as the $\mathscr{A}$-quasiconvexity theory [6, 15, 35–36, 4, 26, 28]. Central to the theory are the notions of $\mathscr{A}$-quasiconvex function, $\mathscr{A}$-quasiaffine function, and $\mathscr{A}$-polyconvex function, see Definitions 5.2 and 5.5.

The novel feature of the present paper is that it adopts these "$\mathscr{A}$-notions" to the full combined electro-magneto-elastic interactions, i.e., to the combinations of the constrains (1.1) and (1.2) (Definitions 6.1 and 6.4). For brevity, in this special case we omit the modifier "$\mathscr{A}$-" and use the terms quasiconvex function, quasiaffine function, and polyconvex function. The main results of the paper are as follows.

- In dimension 3 there are 32 linearly independent scalar quasiaffine functions; dimension 2 there are 15 linearly independent scalar quasiaffine functions, see the lists in (6.4) in Theorem 6.3, below. Here 'linear independence' means linear independence in the linear space of (unrestricted) functions $f = f(F, d, b)$ defined on the space $\mathbb{M}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^n$, $n = 2$ or 3, under standardly defined addition and multiplication by scalars.⋆ The nontrivial proof is given in Section 7.⋆⋆
- In dimension 3, an energy function $\psi = \psi(F, d, b)$ of an electro-magneto-elastic material is polyconvex if and only if it is of the form

$$\psi(F, d, b) = \Phi(F, \operatorname{cof} F, \det F, d, b, Fd, Fb) \tag{1.3}$$

where Φ is a convex function (of 31 scalar variables). In dimension 2, ψ is polyconvex if and only if it is of the form

$$\psi(F, d, b) = \Phi(F, \det F, d, b, Fd, Fb, d \times b)$$

⋆ See Section 2 for the notation.

⋆⋆ The paper [46] proves a general result which yields the electro-magneto-elastic quasiaffine functions as a special case.

where Φ is a convex function (of 14 scalar variables), see Theorem 6.5, below. Apart from the expected terms F, $\operatorname{cof} F$, $\det F$, d, and b, we have the cross-effect terms Fd, Fb (and in dimension 2 also $d \times b$). This description applies to materials of arbitrary symmetry. ⋆

- An existence theorem is proved for a state of minimum energy for a system consisting of a polyconvex electro-magneto-elastic solid plus the vacuum electromagnetic field outside the body, see Theorem 8.3, below.

To mention restricted cases (as opposed to the full case described above), we start with noting that in the absence of electromagnetic phenomema the $\mathscr{A}$-quasiconvexity under the constraint (1.2) reduces to the aforementioned Morrey's quasiconvexity. The convexity conditions for the electric or magnetic phenomena in rigid bodies (no deformation) have been studied in [49, 6, 39, 15, 7]. These works show that the $\mathscr{A}$-quasiconvexity under $(1.1)_1$ or under $(1.1)_2$ reduces to the ordinary convexity.

The quasiconvexity for combinations of mechanical and magnetic phenomena has been discussed in [24] and [23], but ignoring the constraint $(1.1)_2$, which substantially reduces the class of quasiconvex and polyconvex energies. The paper [16] briefly mentions, as an example, a combination of mechanical and magnetic phenomena in 2 dimensions within a different framework, but without any further development.

Note After the research presented in this paper had been completed, the author became aware of the recent papers by Gil & Ortigosa [19, 37–38],⋆⋆ which postulate (1.3) also. The main motivation for (1.3) in [19, 37–38] comes from the electro-magneto-elastic ellipticity condition, which is implied by condition (1.3). Accordingly, the developments and motivations of [19, 37–38] are different from the present work since here I *derive* the polyconvexity (1.3) from the general concepts of $\mathscr{A}$-quasiconvexity theory (Theorem 6.5) and prove a result specific to polyconvexity: the existence theorem (Theorem 8.3). The electro-magneto-elastic ellipticity condition figures as a consequence of the $\mathscr{A}$-quasiconvexity (Proposition 6.2, below) and the non-trivial proof of the form of the polyconvexity presented in Section 7 has no counterpart in [19, 37–38].

The paper is organized as follows. Section 2 describes the notation and presents the basic definitions from the ordinary convexity. Section 3 gives a survey of the equilibrium and constitutive equations for the static electro-magneto-elasticity. Formal aspects of the variational principle of the electro-magneto-elasticity (the total energy, its first and second variations and the variational derivation of the equilibrium equations) are treated in Section 4. The optional Section 5 introduces the $\mathscr{A}$-quasiconvexity in the general case. A specialization of the $\mathscr{A}$-quasiconvexity to electro-magneto-elasticity is provided in Section 6. This central section can be read independently of Section 5 since independent definitions are given therein. Section 7 provides the proof of Theorem 6.3. Section 8 establishes an existence theorem. The remaining sections are appendices. Section 9 summarizes some results on the classical rank 1 convexity needed in our proofs. Section 10 collects the results on the weak convergence necessary for the existence theorem.

⋆ A companion paper [47] treats isotropic polyconvex electro-magneto-elastic bodies.

⋆⋆ I thank M. Itskov for drawing my attention to these papers.

2 Preliminaries: notation, brief convexity

We use the direct notation with the same conventions as in [50, 45]. The following sets are used throughout:

$$
\begin{aligned}
\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\} &= \text{ the extended real line,}\\
\mathbb{R}^n &= \text{ the } n\text{-dimensional euclidean space,}\\
\mathbb{Z}^n &= \text{ the set of all } n\text{-tuples of integers,}\\
\mathbb{M}^{n\times n} &= \text{ the space of all real } n \times n \text{ matrices,}\\
\mathbb{M}^{n\times n}_+ &= \{F \in \mathbb{M}^{n\times n} : \det F > 0\}.
\end{aligned}
$$

We interpret the matrices from $\mathbb{M}^{n\times n}$ as second–order tensors on $\mathbb{R}^n$. We denote by $1 \in \mathbb{M}^{n\times n}$ is the unit matrix, by $a \cdot b$ the usual scalar product of two vectors in $\mathbb{R}^n$ and by $A \cdot B := \operatorname{tr}(A^{\mathrm{T}}B)$ the scalar product of tensors. We recall that the tensor of cofactors of $F \in \mathbb{M}^{n\times n}_+$ is given by $\operatorname{cof} F = (\det F)F^{-\mathrm{T}}$.

If $n = 3$, we define the vector product and the curl in the usual way as vectors in $\mathbb{R}^3$ while if $n = 2$ then both the vector product and the curl are the numbers $a \times b = a_1b_2 - a_2b_1$, $\operatorname{curl} a = a_{2,1} - a_{1,2}$.

Although the main theme of the paper are various weakened notions of convexity, an essential use is made of the classical convexity. We refer to [43] and [10] for systematic expositions of the convexity theory; here we only outline basic notions. A function $f : \mathrm{X} \to \bar{\mathbb{R}}$ on a vector space X is said to be convex if

$$
f\big((1-t)\xi_1 + t\xi_2\big) \le (1-t)f(\xi_1) + tf(\xi_2) \tag{2.1}
$$

for every $\xi_1, \xi_2 \in \mathrm{X}$ and every $t \in (0,1)$. The function f is said to be affine if we have the equality sign in (2.1) holding identically. f is affine if and only if there is a linear functional φ on X and a constant $c \in \mathbb{R}$ such that

$$
f(\xi) = \langle \varphi, \xi \rangle + c \tag{2.2}
$$

for every $\xi \in \mathrm{X}$ where $\langle \varphi, \xi \rangle$ is the value of φ on ξ. If $\mathrm{X} = \mathbb{R}^m$ then (2.2) reads $f(\xi) = \varphi \cdot \xi + c$ where $\varphi \in \mathbb{R}^m$.

We conclude this section by recording Jensen's inequality [14; Theorem 4.80], which underlies the notion of polyconvexity. *If $\Phi : \mathbb{R}^m \to \bar{\mathbb{R}}$ is a convex lowersemicontinuous function and $Q = (0,1)^n$ then*

$$
\int_Q \Phi(z(x))\,dx \ge \Phi\Big(\int_Q z(x)\,dx\Big) \tag{2.3}
$$

for any measurable map $z : Q \to \mathbb{R}^m$.

3 Equilibrium and constitutive equations for electro-magneto-elasticity

With the exception of Section 5, we work in the space dimensions $n = 2$ or 3. Recall from the introduction that the variables in the constitutive equations are the deformation gradient $F \in \mathbb{M}^{n\times n}_+$, the referential electric displacement $d \in \mathbb{R}^n$ and the magnetic induction $b \in \mathbb{R}^n$. Throughout the section, $\psi : \mathbb{M}_+^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes the energy function of an electro-magneto-elastic body and (F, d, b) is an element of $\mathbb{M}_+^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^n$.

The coupling between electricity, magnetism and nonlinear elasticity is well studied since the sixties of the last century, as illustrated by the book expositions [51, 5, 21, 29, 12–13, 25] and others. Our situation is purely static, so that only the static form of Maxwell's equations and the mechanical equilibrium of forces govern the behavior of the body.

3.1 Equilibrium equations

3.1.1 ***Actual ("eulerian") configuration*** The basic electromagnetic variables are the electric and magnetic fields, the electric displacement and the magnetic induction, denoted, respectively, by E, H, D, B. The mechanical variables are the Cauchy stress tensor T, the density of the body force g, and the actual density of mass ρ. The equilibrium equations are

$$\operatorname{Div} D = 0, \quad \operatorname{Div} B = 0, \quad \operatorname{Curl} E = 0, \quad \operatorname{Curl} H = 0 \text{ on } \mathbb{R}^n, \tag{3.1}$$

$$\operatorname{Div} T + \rho g = 0 \text{ on } \omega \tag{3.2}$$

where Curl and Div denote the curl and divergence with respect to the actual position and ω is the actual configuration of the body. In Section 4, the equilibrium equations will be derived from a variational principle. The equations (3.1) and (3.2) are assumed to hold in the weak sense, which then includes the well-known jump conditions for the electromagnetic variables on the boundary of the body. This is not repeated here. Furthermore, below we shall consider only the Dirichlet boundary conditions for the deformation; thus there is no equation for the surface traction on the boundary. Outside ω we have the ether relations

$$E = D, \quad H = B; \tag{3.3}$$

on ω, we have the constitutive relations for E and H to be discussed below.

3.1.2 ***Referential ("lagrangian") configuration*** We denote by $\Omega \subset \mathbb{R}^n$ the reference configuration of the body and by $y : \Omega \to \mathbb{R}^n$ the deformation. We prescribe the Dirichlet boundary conditions on $\partial\Omega$, i.e.,

$$y = \tilde{y} \quad \text{on } \partial\Omega \tag{3.4}$$

where $\tilde{y} : \partial\Omega \to \mathbb{R}^n$ is a given function. We assume that $\tilde{y}$ can be extended to an equally denoted injective function on $\mathbb{R}^n \sim \operatorname{cl}\Omega$ such that $\det \nabla\tilde{y} > 0$ on $\mathbb{R}^n \sim \Omega$. For notational convenience we define the deformation gradient $F : \mathbb{R}^n \to \mathbb{M}_+^{n\times n}$ by

$$F = \begin{cases} \nabla y & \text{on } \Omega, \\ \nabla \tilde{y} & \text{on } \mathbb{R}^n \sim \operatorname{cl}\Omega. \end{cases} \tag{3.5}$$

We now use the classical Piola transformation [20], [27; Chapter I, §§7.18–7.20] to introduce the referential (lagrangean) quantities by

$$\begin{aligned} &e = F^{\mathrm{T}} E, \quad h = F^{\mathrm{T}} H, \quad d = (\operatorname{cof} F)^{\mathrm{T}} D, \quad b = (\operatorname{cof} F)^{\mathrm{T}} B && \text{on } \mathbb{R}^n, \\ &S = T \operatorname{cof} F && \text{on } \Omega, \end{aligned} \tag{3.6}$$

where, of course the spatial variables $E, \dots, T$ are now expressed as functions of the referential variable. The referential forms of the equilibrium equations read

$$\operatorname{div} d = 0, \quad \operatorname{div} b = 0, \quad \operatorname{curl} e = 0, \quad \operatorname{curl} h = 0, \ \text{on} \ \mathbb{R}^n, \tag{3.7}$$

$$\operatorname{div} S + g = 0 \ \text{on} \ \Omega, \tag{3.8}$$

where curl and div denote the referential forms of the curl and divergence, i.e., the same differential operators as Curl and Div, but with the derivatives with respect to the actual position replaced by the derivatives with respect to referential position. The ether relations (3.3) read in terms of the referential variables as

$$e = F^{\mathrm{T}} F d / \det F, \quad h = F^{\mathrm{T}} F b / \det F \tag{3.9}$$

outside Ω.

3.2 Constitutive relations The density of the free energy is

$$\psi = \psi(F, d, b)$$

which is a twice continuously differentiable function with the domain

$$\mathbb{D}^n_+ := \mathbb{M}^{n \times n}_+ \times \mathbb{R}^n \times \mathbb{R}^n \tag{3.10}$$

which is a subset of

$$\mathbb{D}^n := \mathbb{M}^{n \times n} \times \mathbb{R}^n \times \mathbb{R}^n. \tag{3.11}$$

We have the potential relations

$$S = \mathrm{D}_F \psi, \quad e = \mathrm{D}_d \psi, \quad h = \mathrm{D}_b \psi. \tag{3.12}$$

We assume that ψ satisfies the principle of material frame indifference

$$\psi(QF, d, b) = \psi(F, d, b) \tag{3.13}$$

for all $(F, d, b) \in \mathbb{D}^n_+$ and all proper orthogonal tensors Q. A standard argument shows that (3.13) implies the symmetry of the stress,

$$SF^{\mathrm{T}} = FS^{\mathrm{T}}, \quad T^{\mathrm{T}} = T.$$

3.3 Example (Non-interacting matter) Consider an elastic body Ω in the external electromagnetic field but suppose that there is no field-matter interaction. Therefore, the energy splits into the sum

$$\psi(F, d, b) = \psi_1(F) + \psi_2(F, d, b)$$

of the elastic energy $\psi_1(F)$ and of the energy of the vacuum electromagnetic field $\psi_2(F, d, b)$. In the reference configuration Ω, ψ_2 is given by

$$\psi_2(F, d, b) = \frac{1}{2} (\det F)^{-1} \big(|Fd|^2 + |Fb|^2 \big). \tag{3.14}$$

Indeed, passing from the reference variable x to the spatial variable $y = y(x)$ and employing the transformation rules (3.6) we obtain the vacuum energy of the electromagentic field, i.e.,

$$\frac{1}{2} \int_{\Omega} (\det F)^{-1} \big(|Fd|^2 + |Fb|^2 \big) \, dx = \frac{1}{2} \int_{\omega} \big(|D|^2 + |B|^2 \big) \, dy$$

where dx and dy are the referential and actual elements of volume if $n = 3$ or those of area if $n = 2$, D and B are the spatial electric displacement and magnetic induction and $\omega = y(\Omega)$ is the actual configuration of the body. The potential relations $(3.12)_{2,3}$ yield the ether relations (3.9); the stress relation $(3.12)_1$ yields

$$S = S_1 + S_2 \quad \text{where} \quad S_1(F) = \mathrm{D}_F\psi_1(F), \quad S_2(F,d,b) = \mathrm{D}_F\psi_2(F,d,b)$$

where S_1 is the elastic stress while a calculation shows that S_2 is given by

$$S_2(F,d,b) = (\det F)^{-1}\big(Fd \otimes d + Fb \otimes b - \frac{1}{2}F^{-\mathrm{T}}(|Fd|^2 + |Fb|^2)\big).$$

Let us show that

$$\operatorname{div} S_2 = 0 \tag{3.15}$$

for any deformation y of Ω and any vector fields d and b that satisfy $(3.7)_{1,2}$ on Ω. Indeed, passing to the spatial stress $T_2 = S_2 \operatorname{cof} F^{-1}$, we obtain the vacuum Maxwell tensor

$$T_2 = D \otimes D + B \otimes B - \frac{1}{2}\,(|D|^2 + |D|^2)1$$

whose spatial divergence is known to vanish as a consequence of $(3.1)_{1,2}$:

$$\operatorname{Div} T_2 = 0.$$

The referential form (3.15) then follows by Piola's transformation. The equilibrium equation (3.8) with the total stress S then reduces to the equilibrium for the elastic stress

$$\operatorname{div} S_1 + g = 0 \quad \text{on} \quad \Omega.$$

We thus summarize that the total stress S is different form zero even in the (idealized) absence of matter as a consequence of the geometric factors in (3.14); however, its divergence identically vanishes.

4 Variational principle

This section presents a preliminary analysis of a variational principle for an electro-magneto-elastic body. We consider a state of minimum energy of the system consisting of an elastic body Ω interacting with the electromagnetic field inside Ω and the vacuum electromagnetic field in its exterior. Section 8 treats the same minimum principle under natural, weakened assumptions on y, d, b which ensure the existence of a minimizer. As already mentioned, the proof is currently available only for the Dirichlet data for the deformation. Even though the considerations to be presented in this section can be carried out for the general boundary conditions, we assume the Dirichlet data for notational simplicity also here.

4.1 The system and its states We assume that the reference configuration Ω is bounded and has class C^2 boundary $\partial\Omega$. We denote by $\Omega^{\mathrm{c}} = \mathbb{R}^n \sim (\Omega \cup \partial\Omega)$ the complement of the body and by n the outer normal to $\partial\Omega$.

By a state we mean any triplet $\sigma = (y,d,b)$ of maps

$$y : \Omega \to \mathbb{R}^n, \quad d : \mathbb{R}^n \to \mathbb{R}^n, \quad b : \mathbb{R}^n \to \mathbb{R}^n;$$

these represent the deformation of the body and the referential electric displacement and magnetic induction, respectively. We assume that

(i) y is twice continuously differentiable with y and its derivatives up to the order 2 having continuous extensions to the closure $\operatorname{cl}\Omega$ of Ω, and with $\det \nabla y > 0$ on $\operatorname{cl}\Omega$;

(ii) y satisfies Dirichlet's boundary condition (3.4);

(iii) d and b are continuously differentiable in Ω and in Ω^c with d and b and their derivatives having continuous extensions from Ω to $\operatorname{cl}\Omega$ and from Ω^c to $\operatorname{cl}\Omega^c$.

(iv) d and b satisfy

$$\begin{aligned} &\operatorname{div} d = 0, \qquad \operatorname{div} b = 0 \qquad \text{on } \Omega \cup \Omega^c, \\ &[\![d]\!] \cdot n = 0, \qquad [\![b]\!] \cdot n = 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where $[\![\cdot]\!]$ the jump across $\partial\Omega$.

We denote by $\mathfrak{S}$ the set of all states.

4.2 The total energy The total energy of a state $\sigma = (y, d, b) \in \mathfrak{S}$ is defined by

$$\mathsf{E}(\sigma) = \int_\Omega \psi(\nabla y, d, b)\, dx - \int_\Omega g \cdot y\, dx + \frac{1}{2} \int_{\Omega^c} J^{-1}\big(|Fd|^2 + |Fb|^2\big)\, dx \tag{4.1}$$

where the deformation gradient outside Ω is defined by (3.5) using the extension $\tilde{y}$ on Ω^c which is fixed, and $J = \det F$. Following [8; Chapter 8], we note that the last term in (4.1) is independent of the choice of the fictitious 'deformation' $\tilde{y}$ since it can be transformed into the vacuum energy

$$\frac{1}{2} \int_{\omega^c} \big(|D|^2 + |B|^2\big)\, dy$$

as in Example 3.3, where $\omega^c = \mathbb{R}^n \sim y(\Omega)$ is the exterior of the actual configuration $y(\Omega)$.

The set $\delta\,\mathfrak{S}$ of admissible variations of state is the set of triplets (κ, δ, β) of infinitely differentiable functions on $\mathbb{R}^n$ with values in $\mathbb{R}^n$ such that

$$\kappa = 0 \quad \text{on} \quad \Omega^c, \quad \operatorname{div} \delta = 0, \quad \operatorname{div} \beta = 0 \quad \text{in} \quad \mathbb{R}^n \tag{4.2}$$

and δ, β vanish outside some (varying) bounded subset of $\mathbb{R}^n$. If $\sigma = (y, d, b)$ is a state, we define the first and second variations $\delta\,\mathsf{E}(\sigma)[\cdot]$ and $\delta^2\,\mathsf{E}(\sigma)[\cdot]$ of energy at σ as linear and quadratic functionals on $\delta\,\mathfrak{S}$ by

$$\left.\begin{aligned} \delta\,\mathsf{E}(\sigma)[\kappa, \delta, \beta] &= \int_\Omega \mathrm{D}\psi(\nabla y, d, b)[\nabla\kappa, \delta, \beta]\, dx - \int_\Omega g \cdot \kappa\, dx \\ &\quad + \frac{1}{2} \int_{\Omega^c} J^{-1}\big((Fd \cdot F\delta) + (Fb \cdot F\beta)\big)\, dx, \\ \delta^2\,\mathsf{E}(\sigma)[\kappa, \delta, \beta] &= \int_\Omega \mathrm{D}^2\psi(\nabla y, d, b)[(\nabla\kappa, \delta, \beta), (\nabla\kappa, \delta, \beta)]\, dx \\ &\quad + \int_{\Omega^c} J^{-1}\big(|F\delta|^2 + |F\beta|^2\big)\, dx. \end{aligned}\right\} \tag{4.3}$$

4.3 Equilibrium states A state $\sigma = (y, d, b) \in \mathfrak{S}$ is said to be an equilibrium state if $\mathsf{E}(\sigma) < \infty$ and

$$\mathsf{E}(\sigma) \le \mathsf{E}(\bar\sigma)$$

for all $\bar\sigma \in \mathfrak{S}$. Necessary conditions for the minimum are, standardly,

$$\delta\,\mathsf{E}(\sigma)[\kappa, \delta, \beta] = 0, \quad \delta^2\,\mathsf{E}(\sigma)[\kappa, \delta, \beta] \ge 0 \tag{4.4}$$

for each $(\kappa, \delta, \beta) \in \delta\,\mathfrak{S}$. Moreover, $(4.4)_1$ is equivalent to the equilibrium conditions

$$\left.\begin{array}{llll} \operatorname{div} S + g = 0 & & \text{in} & \Omega, \\ \operatorname{curl} e = 0, & \operatorname{curl} h = 0 & \text{in} & \Omega \cup \Omega^{\mathrm{c}}, \\ [\![e]\!] \times n = 0, & [\![h]\!] \times n = 0 & \text{on} & \partial\Omega, \end{array}\right\} \tag{4.5}$$

where the associated stress S on Ω and the electric and magnetic fields e and h on the entire space are given by

$$\left\{\begin{array}{lll} S = \mathrm{D}_F\psi(F, d, b) & & \text{on} \quad \Omega \\ e = \mathrm{D}_d\psi(F, d, b), & h = \mathrm{D}_b\psi(F, d, b) & \text{on} \quad \Omega \\ e = J^{-1}F^{\mathrm{T}}Fd, & h = J^{-1}F^{\mathrm{T}}Fb & \text{on} \quad \Omega^{\mathrm{c}}. \end{array}\right.$$

To derive (4.5), note that if $(\kappa, \delta, \beta) \in \delta\,\mathfrak{S}$, then

$$\delta\,\mathsf{E}(\sigma)[\kappa, \delta, \beta] = \int_\Omega (S \cdot \nabla\kappa - g \cdot \kappa)\,dx + \int_{\mathbb{R}^n} (e \cdot \delta + h \cdot \beta)\,dx = 0 \tag{4.6}$$

by $(4.4)_1$. By $(4.2)_{2,3}$ we may write $\delta = \operatorname{curl}\pi$, $\beta = \operatorname{curl}\rho$; inserting this in to (4.6) and integrating by parts we obtain

$$\int_\Omega (-\operatorname{div} S - g) \cdot \kappa\,dx + \int_{\mathbb{R}^n} (\pi \cdot \operatorname{curl} e + \rho \cdot \operatorname{curl} h)\,dx = 0.$$

The arbitrariness of κ, π, and ρ then gives (4.5). □

5 $\mathscr{A}$-quasiconvexity: the general case

Our treatment of the convexity properties for the electro-magneto-elasticity is based on the $\mathscr{A}$-quasiconvexity theory, which includes the associated notions $\mathscr{A}$-quasiaffinity, $\mathscr{A}$-polyconvexity, Λ-convexity and Λ-ellipticity. The $\mathscr{A}$-quasiconvexity theory has been introduced in [6] and further developed in [15]. Closely related is the compensated compactness theory [35–36]. The reader is referred to [4, 26, 28] for more recent developments and additional literature. This section discusses these notions from a general point of view; the specialization to electro-magneto-elastic materials is the subject of the succeeding sections.

5.1 The differential operator $\mathscr{A}$ and the characteristic cone Λ The following dimensions will be needed in the subsequent discussion:

$$\begin{aligned} n &= \text{the number of independent variables, } x = (x_1, \dots, x_n), \\ d &= \text{the number of dependent variables, } u = (u_1, \dots, u_d), \\ l &= \text{the number of differential constrains.} \end{aligned}$$

Let $Q = (0,1)^n$ be the unit cube, let $C^\infty_{\text{per}}(\mathbb{R}^n, \mathbb{R}^d)$ denote the set of all infinitely differentiable Q-periodic maps $u : \mathbb{R}^n \to \mathbb{R}^d$. We shall consider the first–order differential constraint $\mathscr{A}v = 0$ on a map $v \in C^\infty(\mathbb{R}^n, \mathbb{R}^d)$ where

$$\mathscr{A}v = \sum_{i=1}^n A^{(i)} v_{,i}$$

with $A^{(i)} \in \operatorname{Lin}(\mathbb{R}^d, \mathbb{R}^l)$. For each $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$ define

$$\mathbb{A}(\eta) = \sum_{i=1}^n \eta_i A^{(i)},$$

which is an element of $\operatorname{Lin}(\mathbb{R}^d, \mathbb{R}^l)$, and make the standing assumption that the rank of $\mathbb{A}(\eta)$ is the same for all $\eta \neq 0$. We define the characteristic cone

$$\Lambda = \{u \in \mathbb{R}^d : \mathbb{A}(\eta)u = 0 \quad \text{for some } \eta \in \mathbb{R}^n,\ \eta \neq 0\}.$$

5.2 Definition A continuous function $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ is said to be
(i) $\mathscr{A}$-quasiconvex if

$$\int_Q f(u + v(x))\, dx \ge f(u) \tag{5.1}$$

for all $u \in \mathbb{R}^d$ and all $v \in C^\infty_{\text{per}}(\mathbb{R}^n, \mathbb{R}^d)$ such that $\mathscr{A}v = 0$ on $\mathbb{R}^n$ and $\int_Q v\, dx = 0$;
(ii) $\mathscr{A}$-quasiaffine if it takes only finite values and both f and $-f$ are $\mathscr{A}$-quasiconvex;
(iii) Λ-convex if

$$f(tu_1 + (1-t)u_2) \le tf(u_1) + (1-t)f(u_2)$$

for every $t \in (0,1)$ and $u_1, u_2 \in \mathbb{R}^d$ such that $u_2 - u_1 \in \Lambda$;
(iv) Λ-affine if it takes only finite values and both f and $-f$ are Λ-convex.

If f is continuously differentiable then the Λ-convexity is equivalent to the Λ-ellipticity

$$\mathrm{D}^2 f(u)(l,l) \ge 0$$

for every $u \in \mathbb{R}^n$ and $l \in \Lambda$.

5.3 Theorem ([15; Proposition 3.4]) *If $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ is a continuous $\mathscr{A}$-quasiconvex function then f is Λ-convex; consequently, if f is $\mathscr{A}$-quasiaffine then f is Λ-affine.*

The following weak sequential lower semicontinuity theorem is the main motivation for the $\mathscr{A}$-quasiconvexity. We refer to Section 10 (below) for our conventions about the weak convergence. The weak sequential lower semicontinuity is the basic ingredient of the direct method of the calculus of variations. It should be also noted that for the proof of the existence of the minimizer in electro-magneto-elasticity in Theorem 8.3 (below) the sequential lower semicontinuity theorem cannot be used as the hypothesis (5.2) is inconsistent with the requirement $\psi(F,d,b) \to \infty$ for $\det F \to 0$. One has to use the $\mathscr{A}$-polyconvexity defined below.

5.4 Theorem ([15; Theorem 3.7]) *Let $1 \le p < \infty$ and suppose that $f : \Omega \times \mathbb{R}^d \to [0,\infty)$ is a Carathéodory integrand such that $f(x,\cdot)$ is $\mathscr{A}$-quasiconvex and*

$$0 \le f(x,u) \le a(x)(1 + |u|^p) \tag{5.2}$$

for all $x \in \Omega$ and $u \in \mathbb{D}^d$ where $a : \Omega \to [0,\infty)$ is a bounded function. If u and u_k belong to $L^p(\Omega,\mathbb{R}^d)$ and satisfy

$$u_k \rightharpoonup u \quad \text{in } L^p(\Omega,\mathbb{R}^d) \quad \text{and} \quad \mathscr{A}u_k \to 0 \quad \text{in} \quad W^{-1,p}(\Omega,\mathbb{R}^d)$$

then

$$\liminf_{k\to\infty} \int_\Omega f(x,u_k(x))\,dx \ge \int_\Omega f(x,u(x))\,dx.$$

5.5 Definition A continuous function $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ is said to be $\mathscr{A}$-polyconvex if there exists a finite number of $\mathscr{A}$-quasiaffine functions $f_1,\dots,f_m$ and a convex lowersemicontinuous function $\Phi : \mathbb{R}^m \to \bar{\mathbb{R}}$ such that

$$f(u) = \Phi(f_1(u),\dots,f_m(u)) \tag{5.3}$$

for each $u \in \mathbb{R}^d$.

5.6 Theorem ([6; Corollary 2.5]) *Any $\mathscr{A}$-polyconvex function is $\mathscr{A}$-quasiconvex.*

Proof Since the polyconvexity is central for the present paper, let us outline the proof. Thus let $f = f(u)$ be as in (5.3) and prove (5.1) for all u, v as in Definition 5.2(i). The quasiaffinity of $f_1,\dots,f_m$ means that

$$\int_Q f_i(u + v(x))\,dx = f_i(u), \quad 1 \le i \le m,$$

and hence the application of Jensen's inequality (2.3) with z given by

$$z(x) = \big(f_1(u + v(x)),\dots,f_m(u + v(x))\big)$$

gives

$$\begin{aligned}\int_Q f(u + v(x))\,dx &= \int_Q \Phi\big(f_1(u + v(x)),\dots,f_m(u + v(x))\big)\,dx\\ &\ge \Phi\Big(\int_Q f_1(u + v(x))\,dx,\dots,\int_Q f_m(u + v(x))\,dx\Big)\\ &= \Phi(f_1(u),\dots,f_m(u)) = f(u). \quad \square\end{aligned}$$

6 A specialization to electro-magneto-elasticity

We apply the formalism of the preceding section with $n = 2$ or 3, and with the identifications

$$v = (F,d,b)$$

where F is the deformation gradient, d the electric displacement and b the magnetic induction. In view of the constraint $\det F > 0$, we apply the $\mathscr{A}$-quasiconvexity notions to functions f defined on the domain $\mathbb{D}^n_+$, see (3.10). To obtain an agreement with the general theory of the preceding section, where only functions f defined on the entire $\mathbb{R}^d$ have been considered, we tacitly extend $f : \mathbb{D}^n_+ \to \bar{\mathbb{R}}$ to the entire $\mathbb{D}^n$ from (3.11) by setting f equal to ∞ on $\mathbb{D}^n \sim \mathbb{D}^n_+$.

The functions v of Section 5 will be identified with the triples $(\varphi, \delta, \beta) : \mathbb{R}^n \to \mathbb{D}^n$ and the operator $\mathscr{A}$ with

$$\mathscr{A}(\varphi, \delta, \beta) = (\operatorname{curl} \varphi, \operatorname{div} \delta, \operatorname{div} \beta). \tag{6.1}$$

Here curl of $\varphi = [\varphi_{ij}]^n_{i,j=1}$ is defined by

$$(\operatorname{curl} \varphi)_{il} = \sum_{j,k=1}^{3} \varepsilon_{ljk} \varphi_{ij,k}, \qquad (\operatorname{curl} \varphi)_i = \sum_{j,k=1}^{2} \varepsilon_{jk} \varphi_{ij,k},$$

in dimensions $n = 3$ and $n = 2$, respectively, where $i,\ l = 1, 2, 3$ or $i = 1, 2$ and ε_{ijk} and ε_{ij} are the three– and two– dimensional permutation symbols. Analogously, if $\eta \in \mathbb{R}^n$, we define $\varphi \times \eta$ by

$$(\varphi \times \eta)_{il} = \sum_{j,k=1}^{3} \varepsilon_{ljk} \varphi_{ij} \eta_k, \qquad (\varphi \times \eta)_i = \sum_{j,k=1}^{2} \varepsilon_{jk} \varphi_{ij} \eta_k$$

in dimensions $n = 3$ and $n = 2$, respectively.

To determine the characteristic cone $\Lambda \equiv \Lambda_{\mathrm{E}}$ corresponding to the system (6.1), we replace the partial derivatives $\nabla \varphi$, $\nabla \delta$, $\nabla \beta$ in (6.1) by the tensor products $\varphi \otimes \eta$, $\delta \otimes \eta$, $\beta \otimes \eta$ where $\eta \in \mathbb{R}^n$ is an arbitrary nonzero vector. This transforms (6.1) into

$$\varphi \times \eta = 0, \quad \delta \cdot \eta = 0, \quad \beta \cdot \eta = 0; \tag{6.2}$$

noting that $(6.2)_1$ is satisfied in and only if $\varphi = \xi \otimes \eta$ for some $\xi \in \mathbb{R}^n$, one obtains

$$\Lambda_{\mathrm{E}} = \{(\xi \otimes \eta, \delta, \beta) \in \mathbb{D}^n : \xi,\ \delta,\ \beta,\ \eta \in \mathbb{R}^n,\ \delta \cdot \eta = \beta \cdot \eta = 0,\ \eta \neq 0\}.$$

We now specialize the general definitions of Section 5 to the present case.

6.1 Definition A continuous function $f : \mathbb{D}^n \to \bar{\mathbb{R}}$ is said to be

(i) quasiconvex at $(F, d, b) \in \mathbb{D}^n$ if

$$\int_Q f(F + \varphi(x), d + \delta(x), b + \beta(x))\, dx \ge f(F, d, b)$$

for each triplet $(\varphi, \delta, \beta) \in C^\infty_{\mathrm{per}}(\mathbb{R}^n, \mathbb{D}^n)$ satisfying

$$\operatorname{curl} \varphi = 0, \quad \operatorname{div} \delta = \operatorname{div} \beta = 0 \quad \text{on} \quad \mathbb{R}^n \quad \text{and} \quad \int_Q (\varphi, \delta, \beta)\, dx = 0;$$

(ii) quasiconvex if it is quasiconvex at every point of $\mathbb{D}^n$;

(iii) quasiaffine if f takes only finite values, and both f and $-f$ are quasiconvex;

(iv) Λ_{E}-convex if

$$f(F + t\xi \otimes \eta, d + t\delta, b + t\beta) \le (1-t) f(F, d, b) + t f(F + \xi \otimes \eta, d + \delta, b + \beta)$$

for every $t \in (0,1)$ and $(F, d, b) \in \mathbb{D}^n$ and every $\xi,\ \delta,\ \beta,\ \eta \in \mathbb{R}^n$ such that

$$\delta \cdot \eta = \beta \cdot \eta = 0 \quad \text{and} \quad \eta \neq 0; \tag{6.3}$$

(v) Λ_{E}-affine if f takes only finite values, and both f and $-f$ are Λ_{E}-convex.

6.2 Proposition *Let $f : \mathbb{D}^n_+ \to \mathbb{R}$ be twice continuously differentiable.*
(i) *If f is quasiconvex at $(F, d, b) \in \mathbb{D}^n_+$ then f is elliptic at (F, d, b), i.e.,*

$$\mathrm{D}^2\psi(F, d, b)[(\xi \otimes \eta, \delta, \beta), (\xi \otimes \eta, \delta, \beta)] \ge 0$$

for every $\xi,\ \delta,\ \beta,\ \eta \in \mathbb{R}^n$ satisfying (6.3);
(ii) *if f is quadratic, i.e., if*

$$f(F, d, b) = C[(F, d, b), (F, d, b)]$$

for every $(F, d, b) \in \mathbb{D}^n$ and some symmetric bilinear form C then f is quasiconvex at some point $\Leftrightarrow$ f is quasiconvex $\Leftrightarrow$ f is elliptic at some point $\Leftrightarrow$ f is elliptic at every point of $\mathbb{D}^n$.

Here the second derivative $\mathrm{D}^2\psi(F, d, b)[\cdot, \cdot]$ of ψ at (F, d, b) is interpreted as a bilinear form in the incremental variable $(\xi \otimes \eta, \delta, \beta)$. The proof of Proposition 6.2 is only a minor variation of van Hove's original proof [52] in the gradient case; it is therefore omitted.

Proposition 6.2 can be restated equivalently in terms of the second variation $\delta^2\, \mathsf{E}(\sigma)$ of the total energy [see (4.1) and (4.3)]. Namely, the ellipticity of ψ at (F, d, b) is equivalent to the nonnegativity of the second variation $\delta^2\, \mathsf{E}(\sigma)$ at the homogeneous state with data (F, d, b) under the Dirichlet boundary conditions, i.e.,

$$\delta^2\, \mathsf{E}(\sigma)[\kappa, \delta, \beta] \ge 0$$

for every triplet of infinitely differentiable functions $\kappa, \delta, \beta : \mathbb{R}^n \to \mathbb{R}^n$ which vanish outside Ω.

The main results of this paper are the following theorem and Theorem 6.5, below.

6.3 Theorem *A continuous function $f : \mathbb{D}^n \to \mathbb{R}$ is quasiaffine $\Leftrightarrow$ f is Λ_{E}-affine $\Leftrightarrow$ f is a linear combination, with constant coefficients, of the following functions*:

$$\left.\begin{array}{llllllllll} 1, & F, & & \det F, & d, & b, & Fd, & Fb, & d \times b & \text{if} \quad n = 2, \\ 1, & F, & \operatorname{cof} F, & \det F, & d, & b, & Fd, & Fb & & \text{if} \quad n = 3. \end{array}\right\} \quad (6.4)$$

Here the expressions involving the variables F, d, b in (6.4) stand for the functions of (F, d, b) defined on the domain $\mathbb{D}^n$; linear independence is understood in the linear space of functions $f = f(F, d, b)$ on $\mathbb{D}^n$ under standardly defined addition and multiplication by scalars.

Thus there are 15 linearly independent quasiaffine functions if $n = 2$ and 32 linearly independent quasiaffine functions if $n = 3$, including constants. The proof of Theorem 6.3 is deferred to Section 7.

6.4 Definition A continuous function $f : \mathbb{D}^n \to \bar{\mathbb{R}}$ is said to be polyconvex if there exists a finite number of quasiaffine functions $f_1, \dots, f_m$ and a convex lowersemicontinuous function $\Phi : \mathbb{R}^m \to \bar{\mathbb{R}}$ such that

$$f(F, d, b) = \Phi(f_1(F, d, b), \dots, f_m(F, d, b))$$

for each $(F, d, b) \in \mathbb{D}^n$.

Theorem 6.3 has the following corollary.

6.5 Theorem *A continuous function $f : \mathbb{D}^n \to \bar{\mathbb{R}}$ is polyconvex if and only is f is of the following form:*

$$f(F,d,b) = \Phi(\wedge(F,d,b))$$

for every $(F,d,b) \in \mathbb{D}^n_+$, where we abbreviate

$$\wedge(F,d,b) = \begin{cases} (F, \det F, d, b, Fd, Fb, d \times b) & \text{if} \quad n = 2, \\ (F, \operatorname{cof} F, \det F, d, b, Fd, Fb) & \text{if} \quad n = 3 \end{cases}$$

and where Φ is a convex lowersemicontinuous function on

$$\tilde{\mathbb{D}}^n = \begin{cases} \mathbb{M}^{2\times 2} \times \mathbb{R} \times (\mathbb{R}^2)^4 \times \mathbb{R} & \text{if} \quad n = 2, \\ \mathbb{M}^{3\times 3} \times \mathbb{M}^{3\times 3} \times \mathbb{R} \times (\mathbb{R}^3)^4 & \text{if} \quad n = 3. \end{cases}$$

Thus Φ is a function of 14 and 31 scalar variables, respectively.

7 Proof of Theorem 6.3

Recall from Definition 6.1 that a continuous function $f : \mathbb{D}^n \to \mathbb{R}$ is quasiaffine if

$$\int_Q f(F + \varphi, d + \delta, b + \beta)\, dx = f(F,d,b) \tag{7.1}$$

for each triplet $(\varphi, \delta, \beta) \in C^\infty_{\text{per}}(\mathbb{R}^n, \mathbb{D}^n)$ satisfying

$$\operatorname{curl} \varphi = 0, \quad \operatorname{div} \delta = \operatorname{div} \beta = 0 \quad \text{on} \quad \mathbb{R}^n \quad \text{and} \quad \int_Q (\varphi, \delta, \beta)\, dx = 0, \tag{7.2}$$

and that f is Λ_{E}-affine if

$$f(F + t\xi \otimes \eta, d + t\delta, b + t\beta) = (1-t) f(F,d,b) + t f(F + \xi \otimes \eta, d + \delta, b + \beta) \tag{7.3}$$

for every $t \in (0,1)$ and $(F,d,b) \in \mathbb{D}^n$ and every ξ, δ, β, $\eta \in \mathbb{R}^n$ such that

$$\delta \cdot \eta = \beta \cdot \eta = 0 \quad \text{and} \quad \eta \neq 0. \tag{7.4}$$

The proof of Theorem 6.3 is divided into several lemmas. We start with the analysis of the *separate* Λ_{E}-affinity with respect to the variables F, d, and b. The cross effects will be analyzed subsequently. We refer to Section 9 for the rank 1 affinity which underlies Item (i) of the following result and many points in the subsequent treatment.

7.1 Lemma *Let $f : \mathbb{D}^n \to \mathbb{R}$ be a Λ_{E}-affine function. Then*

(i) *for each d, $b \in \mathbb{R}^n$ the function $f(\cdot, d, b)$ is rank 1 affine, i.e., it is a linear combination, with coefficients depending on d, b, of the functions occurring in* (9.2);

(ii) *for each $F \in \mathbb{M}^{n\times n}$ the function $f(F, \cdot, \cdot)$ is a linear combination, with coefficients depending on F, of the functions*

$$\begin{array}{ll} 1,\ d,\ b,\ d \times b & \text{if} \quad n = 2, \\ 1,\ d,\ b & \text{if} \quad n = 3. \end{array}$$

Proof (i): Fixing d, $b \in \mathbb{R}^n$, taking $\delta = \beta = 0$ in (7.3) and denoting $g(\cdot) = f(\cdot, d, b)$ we obtain Inequality (9.1) with the equality sign for every $t \in (0,1)$, every and $F \in \mathbb{M}^{n\times n}$ and every ξ, $\eta \in \mathbb{R}^n$. Thus $f(\cdot, d, b)$ is rank 1 affine and Lemma 9.2 yields the assertion.

(ii): Employing (7.3) with $\xi = \beta = 0$ and noting that there always exists an $\eta \in \mathbb{R}^n$, $\eta \neq 0$, such that $\delta \cdot \eta = \beta \cdot \eta = 0$, we obtain

$$f(F, d + t\delta, b) = (1 - t) f(F, d, b) + t f(F, d + \delta, b)$$

for every F, d, b and δ. Thus $f(F, \cdot, b)$ is affine and hence

$$f(F, d, b) = \Delta(F, b) + \varepsilon(F, b) \cdot d,$$

for each $(F, d, b) \in \mathbb{D}^n$, where $\Delta(F, b) \in \mathbb{R}$ and $\varepsilon(F, b) \in \mathbb{R}^n$. Repeating the same argument for $\delta = 0$, β arbitrary, we obtain

$$f(F, d, b) = \Gamma(F, d) + \zeta(F, d) \cdot b,$$

$(F, d, b) \in \mathbb{D}^n$, where $\Gamma(F, d) \in \mathbb{R}$ and $\zeta(F, d) \in \mathbb{R}^n$. Thus

$$\Gamma(F, d) + \zeta(F, d) \cdot b = \Delta(F, b) + \varepsilon(F, b) \cdot d.$$

Since the left–hand side is affine in b at any fixed d, we see that the functions Δ and ε must be affine functions of b as well, i.e.,

$$\Delta(F, b) = c_2(F) \cdot b + c_4(F), \quad \varepsilon(F, b) = c_1(F) + A(F) b,$$

$b \in \mathbb{R}^n$, where $c_2(F)$, $c_1(F) \in \mathbb{R}^n$, $c_4(F) \in \mathbb{R}$ and $A(F) \in \mathbb{M}^{n\times n}$. Hence

$$f(F, d, b) = c_1(F) \cdot d + c_2(F) \cdot b + A(F) b \cdot d + c_4(F). \tag{7.5}$$

To complete the proof, we return to (7.3), this time with $\xi = 0$, so that we have

$$f(F, d + t\delta, b + t\beta) = (1 - t) f(F, d, b) + t f(F, d + \delta, b + \beta) \tag{7.6}$$

for every $t \in (0,1)$, every $(F, d, b) \in \mathbb{D}^n$ and every δ, β, $\eta \in \mathbb{R}^n$ such that (7.4) holds. This gives

$$A(F)(b + t\beta) \cdot (d + t\delta) = (1 - t) A(F) b \cdot d + t A(F)(b + \beta) \cdot (d + \delta).$$

The left–hand side contains a quadratic term (i.e., the coefficient of t^2) which is equal to $A(F)\beta \cdot \delta$ and hence we have to have

$$A(F)\beta \cdot \delta = 0 \tag{7.7}$$

for every δ, β such that $\delta \cdot \eta = \beta \cdot \eta = 0$ for some $\eta \neq 0$.

If $n = 3$, then for a given pair (δ, β) there always exists a $\eta \neq 0$ such that $\delta \cdot \eta = \beta \cdot \eta = 0$. Hence (7.6) asserts that $f(F, \cdot, \cdot)$ is affine. Thus the bilinear term $A(F) b \cdot d$ in (7.5) must vanish and hence $f(F, \cdot, \cdot)$ is of the form asserted in (ii).

If $n = 2$ then for a given pair (δ, β) there exists a $\eta \neq 0$ such that $\delta \cdot \eta = \beta \cdot \eta = 0$ if and only if δ and β are parallel, i.e., $\delta \times \beta = 0$. Thus (7.7) requires

$$A(F) b \cdot d = c_3(F)(d \times b)$$

for all d, $b \in \mathbb{R}^2$ and some $c_3(F) \in \mathbb{R}$. Then (7.5) gives the asserted form. □

We are about to pass to the cross effects. In view of the results of Lemma 7.1 it suffices to consider functions of very special forms considered in Lemmas 7.2–7.5, as explained in the proof of Lemma 7.6.

7.2 Lemma *Let* $f : \mathbb{D}^n \to \mathbb{R}$ *be given by*

$$f(F,d,b) = \Omega(F) \cdot d$$

$(F,d,b) \in \mathbb{D}^n$ *where* Ω *is a linear transformation from* $\mathbb{M}^{n\times n}$ *into* $\mathbb{R}^n$, *written* $F \mapsto \Omega(F)$. *Then* f *is* Λ_{E}*-affine if and only if* f *is of the form*

$$f(F,d,b) = Fd \cdot c$$

for all $(F,d,b) \in \mathbb{D}^n$ *and some* $c \in \mathbb{R}^n$.

Proof The choice $F = 0$, $d = b = 0$ in (7.3) yields

$$t^2 \Omega(\xi \otimes \eta) \cdot \delta = t\Omega(\xi \otimes \eta) \cdot \delta \tag{7.8}$$

for every $t \in (0,1)$ and every $\xi,\ \delta,\ \eta \in \mathbb{R}^n$ such that

$$\delta \cdot \eta = 0, \quad \eta \neq 0. \tag{7.9}$$

Thus $\Omega(\xi \otimes \eta) \cdot \delta = 0$ for every $\xi,\ \delta,\ \eta \in \mathbb{R}^n$ such that (7.9) holds. Consequently,

$$\Omega(\xi \otimes \eta) = m(\xi)\eta$$

$\xi,\ \eta \in \mathbb{R}^n$ where $m(\xi) \in \mathbb{R}$. The linearity in ξ requires $m(\xi) = c \cdot \xi$ for some $c \in \mathbb{R}^n$ and all $\xi \in \mathbb{R}^n$; thus $\Omega(\xi \otimes \eta) = (c \cdot \xi)\eta = (\xi \otimes \eta)^{\mathrm{T}} c$ for all $\xi, \eta \in \mathbb{R}^n$. Since every $A \in \mathbb{M}^{n\times n}$ is a sum of tensor products $\xi \otimes \eta$, the linearity of $\Omega(\cdot)$ yields $\Omega(A) = A^{\mathrm{T}} c$ for all $A \in \mathbb{M}^{n\times n}$. Hence

$$f(F,d,b) = \Omega(A) \cdot d = F^{\mathrm{T}} c \cdot d \equiv Fd \cdot c.$$

This completes the proof of the direct implication; the proof of the converse implication is straightforward and the details are omitted. □

7.3 Lemma *Let* $n = 3$ *and let* $f : \mathbb{D}_3 \to \mathbb{R}$ *be given by*

$$f(F,d,b) = \Psi(\operatorname{cof} F) \cdot d$$

$(F,d,b) \in \mathbb{D}^n$ *where* Ψ *is a linear transformation from* $\mathbb{M}^{3\times 3}$ *into* $\mathbb{R}^3$, *written* $A \mapsto \Psi(A)$. *Then* f *is* Λ_{E}*-affine if and only if* $f = 0$ *identically.*

Proof We apply (7.8) with $F = 1$, $d = 0$ and $\xi,\ \delta,\ \eta \in \mathbb{R}^3$ as in (7.9). Using the formula

$$\operatorname{cof}(1 + \xi \otimes \eta) = \big((1 + \xi \cdot \eta)1 - \eta \otimes \xi\big)$$

one finds that (7.3) is equivalent to

$$t^2 \Psi\big((\xi \cdot \eta)1 - t\eta \otimes \xi\big) \cdot \delta = t\Psi\big((\xi \cdot \eta)1 - \eta \otimes \xi\big) \cdot \delta$$

This requires

$$\Psi\big((\xi \cdot \eta)1 - \eta \otimes \xi\big) \cdot \delta = 0$$

for every $\xi,\ \delta,\ \eta \in \mathbb{R}^3$ as above. Hence

$$\Psi((\eta \cdot \xi)1 - \eta \otimes \xi) = m(\xi)\eta$$

$\xi,\ \eta \in \mathbb{R}^3$ where $m(\xi) \in \mathbb{R}$. The linearity in ξ provides $m(\xi) = -c \cdot \xi$ for some $c \in \mathbb{R}^3$ and all ξ; hence

$$\Psi((\eta \cdot \xi)1 - \eta \otimes \xi) = -(c \cdot \xi)\eta = -(\eta \otimes \xi)c.$$

Setting $a := \Psi(1)$, we obtain

$$\Psi(\eta \otimes \xi) = (\eta \cdot \xi)a + (\eta \otimes \xi)c.$$

Since every $A \in \mathbb{M}^{3\times 3}$ is a sum of tensor products $\eta \otimes \xi$, the linearity of $\Psi(\cdot)$ yields

$$\Psi(A) = (\operatorname{tr} A)a + Ac$$

for each $A \in \mathbb{M}^{3\times 3}$. The consistency requires $a = \Psi(1) = 3a + c$ and hence

$$\Psi(A) = -\frac{1}{2}(\operatorname{tr} A)c + Ac.$$

To complete the proof, let us show that $c = 0$. Let $\eta \in \mathbb{R}^3$ be any unit vector, and apply (7.3) with $F = 1 - \eta \otimes \eta$, $d = 0$, $\delta \in \mathbb{R}^3$ satisfying $\delta \cdot \eta = 0$. Using

$$\operatorname{cof}(F + t\eta \otimes \eta) = tF + \eta \otimes \eta$$

one finds that Equation (7.3) reads

$$t\big(tFc - (1 + t/2)c\big) \cdot \delta = -tc \cdot \delta/2;$$

the arbitrariness of t then leads to the unique consequence

$$c \cdot \delta = 0$$

for any δ such that $\delta \cdot \eta = 0$ for some unit vector η. Taking η such that $\eta \cdot c = 0$, we can take $\delta = c$ to obtain $c = 0$. □

7.4 Lemma *Let $f : \mathbb{D}^n \to \mathbb{R}$ be given by*

$$f(F, d, b) = (\det F)c \cdot d$$

for every $(F, d, b) \in \mathbb{D}^n$ where $c \in \mathbb{R}^n$ is a constant. If f is Λ_{E}-affine then $f = 0$ identically.

Proof We apply (7.3) with $F = 1$, $d = 0$ and ξ, δ, $\eta \in \mathbb{R}^n$ as in (7.9). This gives

$$t(1 + t\xi \cdot \eta)(c \cdot d) = t(c \cdot d) + (1 - t)(1 + \xi \cdot \eta)(c \cdot d)$$

and clearly this can hold only if $c = 0$. □

7.5 Lemma *Let $n = 2$ and let $f : \mathbb{D}_2 \to \mathbb{R}$ be given by*

$$f(F, d, b) = m(F)(d \times b)$$

for every $(F, d, b) \in \mathbb{D}^n$ where $m : \mathbb{M}^{2\times 2} \to \mathbb{R}$ is a rank 1 affine function. Then f is Λ_{E}-affine if and only if

$$f(F, d, b) = c(d \times b) \tag{7.10}$$

for all $(F, d, b) \in \mathbb{D}_2$ and some $c \in \mathbb{R}$.

Proof By Lemma 9.2 we have $m(F) = A \cdot F + b \det F + c$ for each $F \in \mathbb{M}^{2\times 2}$ where $A \in \mathbb{M}^{2\times 2}$ and $b, c \in \mathbb{R}$ are constants. Hence

$$f(F, d, b) = (A \cdot F + b \det F + c)(d \times b).$$

Let $\eta \in \mathbb{R}^2$ be any unit vector, let $\beta = \eta^{\perp}$ and $\lambda > 0$. Let us write the equality (7.3) with $F = \lambda 1$, $d = \eta$, $b = 0$, $\xi \in \mathbb{R}^2$ arbitrary, $\delta = 0$. This gives

$$t\left[A\cdot(\lambda 1+t\xi\otimes\eta)+b\lambda^2\left(1+t\lambda^{-1}(\xi\cdot\eta)\right)+c\right](\eta\times\eta^{\perp})$$
$$=(1-t)f(F,d,b)+tf(F+\xi\otimes\eta,d,b+\beta).$$

The quadratic term (i.e., the coefficient of t^2) on the left–hand side is

$$A\cdot(\xi\otimes\eta)+b\lambda(\xi\cdot\eta).$$

This term must vanish. The arbitrariness of λ, ξ, η then gives $A=0$, $b=0$. Thus we have (7.10). Conversely, if f is given by (7.10), then clearly, f is Λ_{E}-affine. □

7.6 Lemma *A continuous function $f:\mathbb{D}^n\to\mathbb{R}$ is Λ_{E}-affine if and only if f is a linear combination, with constant coefficients, of the functions occurring in* (6.4).

Proof Let f be Λ_{E}-affine. By Lemma 7.1(ii) then

$$f(F,d,b)=c_0(F)+c_1(F)\cdot d+c_2(F)\cdot b+c_3(F)(d\times b)$$

for each $(F,d,b)\in\mathbb{D}^n$ where the last term must be omitted if $n=3$. By Item (i) of the same lemma then $f(\cdot,d,b)$ is a rank 1 affine function for each d, $b\in\mathbb{R}^n$. The independence of d, b and $d\times b$ then implies that each of the coefficients c_0 to c_3 are rank 1 affine functions. Lemma 9.2 then asserts that c_0 and c_3 are exactly of the form described in (9.2). Since c_1 and c_2 are vector valued functions, a componentwise application of Lemma 9.2 gives

$$c_1(F)=c+\Omega(F)+\Psi(\operatorname{cof}F)+d\det F$$

for every $F\in\mathbb{M}^{n\times n}$, where $c\in\mathbb{R}^n$ and Ω, Ψ are linear transformations from $\mathbb{M}^{n\times n}$ to $\mathbb{R}^n$ with $\Psi=0$ if $n=2$. A similar form applies to c_2. Using the just described forms of $c_0(F)$ to $c_3(F)$ and collecting some of the terms of the same type into one, it is found that

$$\begin{aligned}f(F,d,b)&=m_0(F,d,b)+M_1\cdot\operatorname{cof}F\\&\quad+\Omega_1(F)\cdot d+\Omega_2(F)\cdot b+\Psi_1(\operatorname{cof}F)\cdot d+\Psi_2(\operatorname{cof}F)\cdot b\\&\quad+(m_2\cdot d+m_3\cdot b)\det F+m_4(F)(d\times b)\end{aligned}\tag{7.11}$$

for every $(F,d,b)\in\mathbb{D}^n$ where m_0 is an affine function of F, d, b, the tensor M_1 is in $\mathbb{M}^{n\times n}$ with $M_1=0$ if $n=2$, the objects Ω_i, Ψ_i $(i=1,2)$ are linear transformations from $\mathbb{M}^{n\times n}$ into $\mathbb{R}^n$ with $\Psi_i=0$ if $n=2$, the vectors $m_i\in\mathbb{R}^n$ $(i=1,2)$ are constants and m_4 is a rank 1 affine function.

We shall now make use of the full power of the Λ_{E}-affinity equality (7.3) (so far only various particular cases have been used). Inserting the form of f from (7.11) into (7.3) and noting that the affine function m_0 and the term $M_1\cdot\operatorname{cof}F$ trivially satisfy that equality, we see that we have to require that the function f_1 given by

$$\begin{aligned}f_1(F,d,b)&=\Omega_1(F)\cdot d+\Omega_2(F)\cdot b+\Psi_1(\operatorname{cof}F)\cdot d+\Psi_2(\operatorname{cof}F)\cdot b\\&\quad+(m_2\cdot d+m_3\cdot b)\det F+m_4(F)(d\times b)\end{aligned}$$

has to satisfy (7.3). Then of course the terms of different order in F have to satisfy the equality individually as well as the terms with d and b. Thus each of the functions

$$\Omega_1(F)\cdot d,\quad \Omega_2(F)\cdot b,$$
$$\Psi_1(\operatorname{cof}F)\cdot d,\quad \Psi_2(\operatorname{cof}F)\cdot b,\quad m_i\cdot d\det F,\quad m_4(F)(d\times b)$$

must be $\Lambda_{\rm E}$-affine. By Lemma 7.2 then $\Omega_1(F)\cdot d = Fd\cdot c_1$, $\Omega_2(F)\cdot b = Fb\cdot c_2$ where $c_i \in \mathbb{R}^n$ are constants, by Lemma 7.3 then $\Psi_i = 0$, $i = 1,2$, by Lemma 7.4 $m_i = 0$ and by Lemma 7.5 m_4 is constant. The asserted form of f follows. The converse implication is immediate. □

We conclude this section with the following converse statement.

7.7 Lemma *Each function from the list* (6.4) *is quasiaffine.*

Proof We have to prove that any function f from the list (6.4) satisfies the equality (7.1) for each triplet $(\varphi,\delta,\beta)\in C^\infty_{\rm per}(\mathbb{R}^n,\mathbb{D}^n)$ satisfying (7.2).

Consider first the case $n = 3$. The system (7.2) implies that there are functions $\omega,\pi,\rho\in C^\infty(\mathbb{R}^3,\mathbb{R}^3)$, such that $\varphi=\nabla\omega$, $\delta=\operatorname{curl}\pi$, $\beta=\operatorname{curl}\rho$; Equation (7.1) then reads

$$\int_Q f(F+\nabla\omega, d+\operatorname{curl}\pi, b+\operatorname{curl}\rho)\,dx = f(F,d,b). \tag{7.12}$$

An argument described in [2; Remark, p. 141] shows that it suffices to verify (7.12) only for $\omega,\pi,\rho\in C_0^\infty(Q,\mathbb{R}^3)$. The verification of (7.12) for the functions $(F,d,b)\mapsto F$, $\operatorname{cof}F$, $\det F$ is standard, see, e.g., [1]. Consider now the functions $(F,d,b)\mapsto d$, b. Then (7.12) reads

$$\int_Q (d+\operatorname{curl}\pi)\,dx = d$$

and a similar equation for b, which is true since

$$\int_Q \operatorname{curl}\pi\,dx = \int_{\partial Q} \pi\times n\,dA(x) = 0$$

by Gauss theorem (where n is the normal to ∂Q) and π vanishes on ∂Q. Finally consider the functions $(F,d,b)\mapsto Fd$, Fb. We have

$$\begin{aligned}\int_Q \big((F+\nabla\omega)(d+\delta)\big)\,dx &= \int_Q \operatorname{div}\big((Fx+\omega)\otimes(d+\delta)\big)\,dx\\ &= \int_{\partial Q}(Fx+\omega)\big((d+\delta)\cdot n\big)\,dA(x)\\ &= \int_{\partial Q} Fx\,(d\cdot n)\,dA(x) \quad (\text{since } \delta=0 \text{ on } \partial Q)\\ &= \int_Q \operatorname{div}\big((Fx)\otimes d\big)\,dx = Fd.\end{aligned}$$

This completes the proof for $n = 3$. The case $n = 2$ is similar; the details are omitted. □

8 Existence theorem

The present section deals with the existence of minimum energy states for the energy E of a polyconvex solid and the surrounding vacuum electromagnetic field. As is usual, the state space $\mathfrak{S}$ from Section 4 has to be enlarged as described in Definition 8.1 (below). Let us recall the definitions of $\mathbb{D}^n_+$ and $\mathbb{D}^n$ in (3.10) and (3.11).

The existence theory for a purely elastic material with a polyconvex energy is well understood [1, 17, 34, 18]. The corresponding part of the proof is based on the sequential weak continuity of the cofactor and determinant. The additional electromagnetic variables d and b interact with the mechanical variable in the nonlinear terms Fd and Fb and in dimension 2 we have also electrical–magnetic interactions $d \times b$. These terms are sequentially weakly continuous as well, but this time one has to use the div–curl lemma. We summarize the results on the weak convergence and weak continuity in Section 10, below.

8.1 Definition Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. We denote by $\mathfrak{S}$ the set of all triplets $(y, d, b) \in W^{1,1}(\Omega, \mathbb{R}^n) \times L^1(\Omega, \mathbb{R}^n) \times L^1(\Omega, \mathbb{R}^n)$ such that

$$\left.\begin{aligned} \operatorname{div} d = 0, \quad \operatorname{div} b = 0 \quad \text{in} \ \mathbb{R}^n, \\ y = \tilde{y} \quad \text{on} \quad \partial\Omega \end{aligned}\right\} \tag{8.1}$$

in the sense of distributions and in the sense of traces, respectively. Here $\tilde{y} : \partial\Omega \to \mathbb{R}^n$ is a prescribed function. As in § 3.1.2, we assume that $\tilde{y}$ can be extended to an equally denoted injective function on $\partial\Omega \cup \Omega^{\mathrm{c}}$; in the present section we assume that

$$c \cdot 1 \geq J^{-1} F^{\mathrm{T}} F \geq c^{-1} \cdot 1 \tag{8.2}$$

on Ω^{c} where F is as in (3.5) and c is a positive constant.

8.2 Definition The total energy of a state $\sigma = (y, d, b) \in \mathfrak{S}$ is defined by the original formula (4.1) where $\psi : \mathbb{D}^n_+ \to \mathbb{R}$ is the energy, which is assumed to be continuous and bounded from below and where we assume that the body force g is in $L^\infty(\Omega, \mathbb{R}^n)$ for notational simplicity.

8.3 Theorem *Let* (8.2) *hold and let* p, r, *and* s *be numbers satisfying*

$$2 \leq p < \infty, \quad 1/r + 1/p \leq 1, \quad 1/s + 1/p \leq 1,$$

and additionally

$$\begin{cases} \textit{let} \ 1/r + 1/s \leq 1 & \text{if} \quad n = 2, \\ \textit{let} \ q \ \textit{be a number satisfying} \ 3/2 \leq q < \infty & \text{if} \quad n = 3. \end{cases}$$

Extend the energy function $\psi : \mathbb{D}^n_+ \to \mathbb{R}$ *to* $\tilde{\psi} : \mathbb{D}^n \to \mathbb{R}$ *by setting* $\tilde{\psi}(F, d, b) = \infty$ *if* $\det F \leq 0$ *and assume that the following conditions hold*:

(i) *there exists a continuous convex and bounded from below function* $\Phi : \tilde{\mathbb{D}}^n \to \mathbb{R} \cup \{\infty\}$ *such that*

$$\tilde{\psi}(F, d, b) = \Phi(\wedge(F, d, b)) \tag{8.3}$$

for every $(F, d, b) \in \mathbb{D}^n$;

(ii) *we have*

$$\psi(F, d, b) \geq \begin{cases} c(|F|^p + |d|^r + |b|^s) + d & \text{if} \quad n = 2, \\ c(|F|^p + |\operatorname{cof} F|^q + |d|^r + |b|^s) + d & \text{if} \quad n = 3 \end{cases}$$

for some $c > 0$, $d \in \mathbb{R}$ *and all* $(F, d, b) \in \mathbb{D}^n_+$.

If $\mathfrak{S}$ *contains an element of finite total energy then there exists a* $\sigma = (y, d, b) \in \mathfrak{S}$ *such that*

$$\mathsf{E}(\sigma) \le \mathsf{E}(\bar\sigma)$$

for all $\bar\sigma \in \mathfrak{S}$; each such a σ satisfies

$$\det \nabla y > 0 \ \textit{for almost every point of } \Omega.$$

Note that the continuity of Φ, the definition of $\tilde\psi$, and (8.3) imply that $\psi(F, d, b) \to \infty$ if $\det F \to 0$.

Proof Let $n = 3$ and assume that $q > 3/2$ rather than $q \ge 3/2$ to simplify the matters; the case $q = 3/2$ is similar but slightly more complicated (cf. [32; Proof of Theorem 5. 1, Case 2] for purely elastic bodies).

In the proof, we are going to apply Propositions 10.1 and 10.2, below. These propositions involve hypotheses on the exponents; we leave to the reader to verify that the hypotheses of the present theorem on p, q, r, and s are chosen exactly to satisfy the hypotheses of Propositions 10.1 and 10.2.

Let $\sigma_k = (y_k, d_k, b_k) \in \mathfrak{S}$ be a minimizing sequence and write $F_k = \nabla y_k$ for brevity. The coercivity condition (ii) implies that the sequence y_k is bounded in $W^{1,p}(\Omega, \mathbb{R}^3)$, the sequence $\operatorname{cof} F_k$ is bounded in $L^q(\Omega, \mathbb{M}^{3\times 3})$, the restrictions of d_k and b_k to Ω are bounded in $L^r(\Omega, \mathbb{R}^3)$ and $L^s(\Omega, \mathbb{R}^3)$, respectively, and the restrictions of d_k and b_k to Ω^{c} are bounded in $L^2(\Omega^{\mathrm{c}}, \mathbb{R}^3)$. The reflexivity of these spaces implies that it is possible to extract a subsequence of the sequence $\sigma_k = (y_k, d_k, b_k)$, again denoted by σ_k, such that

$$\begin{aligned} y_k &\rightharpoonup y \quad \text{in} \quad W^{1,p}(\Omega, \mathbb{R}^3), \\ d_k &\rightharpoonup d \quad \text{in} \quad \begin{cases} L^r(\Omega, \mathbb{R}^3), \\ L^2(\Omega^{\mathrm{c}}, \mathbb{R}^3), \end{cases} \\ b_k &\rightharpoonup b \quad \text{in} \quad \begin{cases} L^s(\Omega, \mathbb{R}^3), \\ L^2(\Omega^{\mathrm{c}}, \mathbb{R}^3) \end{cases} \end{aligned}$$

for some (y, d, b) the indicated spaces. Proposition 10.1 then implies that

$$\operatorname{cof} F_k \rightharpoonup \operatorname{cof} F \quad \text{in } L^q(\Omega, \mathbb{M}^{3\times 3}), \tag{8.4}$$

$$\det F_k \rightharpoonup \det F \quad \text{in } L^{2q/3}(\Omega). \tag{8.5}$$

Furthermore, the components of the vector Fd with a general $F = \nabla y$ and a general d are $F_i \cdot d$ where $F_i = (F_{i1}, F_{i2}, F_{i3})$ and since

$$\operatorname{curl} F_i = 0, \quad \operatorname{div} d = 0,$$

the div–curl lemma Proposition 10.2 implies

$$F_k d_k \rightharpoonup Fd \ \text{ in } L^1(\Omega, \mathbb{R}^3) \quad \text{and similarly} \quad F_k b_k \rightharpoonup Fb \ \text{ in } L^1(\Omega, \mathbb{R}^3).$$

To summarize, we have

$$\wedge(F_k, d_k, b_k) \rightharpoonup \wedge(F, d, b) \quad \text{in } L^1(\Omega) \tag{8.6}$$

and hence Proposition 10.3 gives

$$\liminf_{k\to\infty} \int_\Omega \Phi(\wedge(F_k, d_k, b_k))\, dx \ge \int_\Omega \Phi(\wedge(F, d, b))\, dx.$$

This can be rewritten as

$$\liminf_{k\to\infty}\int_{\Omega}\tilde{\psi}(F_k,d_k,b_k)\,dx \geq \int_{\Omega}\tilde{\psi}(F,d,b)\,dx. \tag{8.7}$$

The integral on the right–hand side is finite and the fact that $\tilde{\psi}=\infty$ if $\det F \leq 0$, we see that $\det\nabla y > 0$ for almost every $x \in \Omega$. Furthermore, the weak form of $(8.1)_1$ reads

$$\int_{\mathbb{R}^3} d_k\cdot\nabla\phi\,dx = 0, \qquad \int_{\mathbb{R}^3} b_k\cdot\nabla\phi\,dx = 0$$

for each indefinitely integrable function $\phi : \mathbb{R}^3 \to \mathbb{R}$ with compact support. The convergence indicated in $(8.6)_{4,5}$ then yields

$$\int_{\mathbb{R}^3} d\cdot\nabla\phi\,dx = 0, \qquad \int_{\mathbb{R}^3} b\cdot\nabla\phi\,dx = 0,$$

i.e.,

$$\operatorname{div} d = 0, \quad \operatorname{div} b = 0 \quad \text{in } \mathbb{R}^3 \text{ in the sense of distributions.}$$

As one also finds that $y_k = \tilde{y}$ on $\partial\Omega$ implies $y = \tilde{y}$ on $\partial\Omega$, we see that the triplet $\sigma = (y,d,b)$ belongs to $\mathfrak{S}$. Also, trivially in view of (8.2),

$$\liminf_{k\to\infty}\int_{\Omega^c} J^{-1}(|Fd_k|^2+|Fb_k|^2)\,dx \geq \int_{\Omega^c} J^{-1}(|Fd|^2+|Fb|^2)\,dx, \tag{8.8}$$

$$\lim_{k\to\infty}\int_{\Omega} y_k\cdot g\,dx = \int_{\Omega} y\cdot g\,dx. \tag{8.9}$$

Inequalities (8.7)–(8.9) can be collected to show that

$$\liminf_{k\to\infty}\mathsf{E}(\sigma_k) \geq \mathsf{E}(\sigma)$$

which shows that σ is the required minimizer. This completes the proof in the case $n = 3$.

The proof is similar if $n = 2$. Instead of Proposition 10.1 one has to use the simpler result of Reshetnyak [40, 42] and Ball [1] that $y_k \rightharpoonup y$ in $W^{1,p}(\Omega,\mathbb{R}^2)$ with $p > 2$ implies $\det F_k \rightharpoonup F$ in $L^{p/2}(\Omega)$; moreover, one more use of the div–curl lemma Proposition 10.2 is needed to show that $d_k \rightharpoonup d$ in $L^r(\Omega,\mathbb{R}^2)$ and $b_k \rightharpoonup b$ in $L^s(\Omega,\mathbb{R}^2)$ implies that $d_k\times b_k \rightharpoonup d\times b$ in $L^1(\Omega)$. For this, one has to identify the sequence g_k of Proposition 10.2 with $(d_{2,k},-d_{1,k})$ so that $\operatorname{div} d_k = 0$ reads $\operatorname{curl} g_k = 0$. □

9 Appendix A: rank 1 convex and rank 1 affine functions

The reader is referred to [1, 6] and [33] for the following notions.

9.1 Definitions Let $g : \mathbb{M}^{n\times n} \to \bar{\mathbb{R}}$.

(i) g is said to be rank 1 convex if

$$g(F+t\xi\otimes\eta) \leq (1-t)g(F) + tg(F+\xi\otimes\eta) \tag{9.1}$$

for every $t \in (0,1)$, every $F \in \mathbb{M}^{n\times n}$ and every $\xi,\eta \in \mathbb{R}^n$.

(ii) g is said to be rank 1 affine if it taken only finite values and (9.1) holds with the equality sign for every t, F, ξ, and η as in (i).

(iii) The rank 1 convex envelope $\mathrm{R}g : \mathbb{M}^{n\times n} \to \bar{\mathbb{R}}$ of g is defined by

$$\mathrm{R}g(F) = \sup\{h(F) : h \quad \text{is rank 1 convex and} \quad h \le g \quad \text{on} \quad \mathbb{M}^{n\times n}\},$$

$F \in \mathbb{M}^{n\times n}$.

The following result is standard.

9.2 Lemma ([11], [9], [1; Theorem 4.1]) *A continuous function $g : \mathbb{M}^{n\times n} \to \mathbb{R}$ is rank 1 affine if and only if g is a linear combination, with constant coefficients, of the functions*

$$\left.\begin{array}{llll} 1, \ F, & & \det F, & \text{if} \quad n = 2, \\ 1, \ F, & \operatorname{cof} F, & \det F & \text{if} \quad n = 3. \end{array}\right\} \tag{9.2}$$

10 Appendix B: weak convergence

We here gather some basic facts about maps that are continuous under the weak convergence.

Let $1 \le p \le \infty$ and let θ_k and θ be measurable functions on open subset Ω of $\mathbb{R}^n$. In this situation, we define the following three types of weak convergence:

- $\theta_k \rightharpoonup \theta \quad \text{in} \quad L^p(\Omega, \mathbb{R}^m),$
- $\theta_k \overset{*}{\rightharpoonup} \theta \quad \text{in} \quad \mathscr{M}(\Omega, \mathbb{R}^m),$
- $\theta_k \rightharpoonup \theta \quad \text{in} \quad W^{1,p}(\Omega, \mathbb{R}^m)$

which mean, respectively,

- that $\theta, \theta_k \in L^p(\Omega, \mathbb{R}^m)$ and

$$\int_\Omega \theta_k \cdot \phi \, dx \to \int_\Omega \theta \cdot \phi \, dx \tag{10.4}$$

  for each $\phi \in L^q(\Omega, \mathbb{R}^m)$ where $1/p + 1/q = 1$; we then say that the sequence θ_k converges weakly to θ in $L^p(\Omega, \mathbb{R}^m)$;
- that $\theta, \theta_k \in L^1(\Omega, \mathbb{R}^m)$ and (10.4) holds for each continuous function $\phi : \mathbb{R}^n \to \mathbb{R}^m$ which vanishes outside Ω; we then say that the sequence θ_k converges weak* to θ in the sense of measures;
- that $\theta, \theta_k \in W^{1,p}(\Omega, \mathbb{R}^m)$ and $\theta_k \rightharpoonup \theta$ in $L^p(\Omega, \mathbb{R}^m)$ and $\nabla\theta_k \rightharpoonup \nabla\theta$ in $L^p(\Omega, \mathbb{M}^{m\times n})$; we then say that θ_k converges weakly to θ in $W^{1,p}(\Omega, \mathbb{R}^m)$.

If $p = \infty$, we should actually write $\overset{*}{\rightharpoonup}$ instead of $\rightharpoonup$ and speak about the weak* convergence; however, this is consistently ignored here.

10.1 Proposition (Müller, Tang & Yan [34]) *Let Ω be a bounded open subset of $\mathbb{R}^n$ where n is arbitrary, let*

$$p \ge n - 1, \quad q > n/(n-1)$$

and let $y, y_k \in W^{1,p}(\Omega, \mathbb{R}^n)$ satisfy

$$y_k \rightharpoonup y \quad \text{in } W^{1,p}(\Omega, \mathbb{R}^n), \tag{10.5}$$

$$\operatorname{cof} F_k \quad \text{is bounded in } L^q(\Omega, \mathbb{M}^{n\times n}) \tag{10.6}$$

where $F = \nabla y$, $F_k = \nabla y_k$. *Then*

$$\operatorname{cof} F_k \rightharpoonup \operatorname{cof} F \quad \text{in } L^q(\Omega, \mathbb{M}^{n\times n}), \tag{10.7}$$

$$\det F_k \rightharpoonup \det F \quad \text{in } L^r(\Omega), \quad r = q(n-1)/n. \tag{10.8}$$

If $q = n/(n-1)$ *and* $\det F_k \geq 0$ *then instead of* (10.8) *we have*

$$\det F_k \rightharpoonup \det F \quad \text{in } L^1(K),$$

for all compact subsets $K \subset \Omega$.

10.2 Proposition (Murat [35], Tartar [49]) *Let* $\Omega \subset \mathbb{R}^n$ *be open bounded and let* $1 < p, q < \infty$ *satisfy* $1/p + 1/q = 1$. *Suppose* $d, d_k \in L^p(\Omega; \mathbb{R}^n)$, $g, g_k \in L^q(\Omega, \mathbb{R}^n)$ *are sequences such that*

$$\begin{aligned} d_k &\rightharpoonup d && \text{in} && L^p(\Omega, \mathbb{R}^n) \\ g_k &\rightharpoonup g && \text{in} && L^q(\Omega, \mathbb{R}^n), \\ \operatorname{div} d_k &\to \operatorname{div} d && \text{in} && W^{-1,1}(\Omega), \\ \operatorname{curl} g_k &\to \operatorname{curl} g && \text{in} && W^{-1,1}(\Omega). \end{aligned}$$

Then

$$d_k \cdot g_k \rightharpoonup d \cdot g \quad \text{in } \mathscr{M}(\Omega).$$

Here $W^{-1,1}(\Omega)$ is the dual of $W_0^{1,\infty}(\Omega)$.

10.3 Proposition (Reshetnyak [41], Ball & Murat [3]) *Let* $\Phi : \mathbb{R}^m \to \bar{\mathbb{R}}$ *be convex, lower semicontinuous and bounded below. Let* θ, $\theta_k \in L^1(\Omega, \mathbb{R}^m)$ *with* $\theta_k \overset{*}{\rightharpoonup} \theta$ *in the sense of measures. Then*

$$\liminf_{j\to\infty} \int_\Omega \Phi(\theta_k)\, dx \geq \int_\Omega \Phi(\theta)\, dx.$$

11 References

1 Ball, J. M.: *Convexity conditions and existence theorems in nonlinear elasticity* Arch. Rational Mech. Anal. **63** (1977) 337–403

2 Ball, J. M., Currie, J. C.; Olver, P. J.: *Null lagrangians, weak continuity and variational problems of any order* J. Funct. Anal. **41** (1981) 135–174

3 Ball, J. M.; Murat, F.: *$W^{1,p}$-quasiconvexity and variational problems for multiple integrals* J. Funct. Anal. **58** (1984) 225–253

4 Braides, A.; Fonseca, I.; Leoni, G.: *A-quasiconvexity: relaxation and homogenization* ESAIM: Control, Optimisation and Calculus of Variations **5** (2000) 539–577

5 Brown, W. F.: *Magnetoelastic interactions* Springer, Berlin 1966

6 Dacorogna, B.: *Weak continuity and weak lower semicontinuity of non-linear functionals* Springer, Berlin 1982

7 Dacorogna, B.; Fonseca, I.: *A-B quasiconvexity and implicit partial differential equations* Calc. of Var. PDE **14** (2002) 115–149 `https://10.1007/s005260100092`

8 Dorfmann, L.; Ogden, R. W.: *Nonlinear theory of electroelastic and magnetoelastic interactions* Springer, New York 2014

9 Edelen, D. G. B.: *The null set of the Euler–Lagrange operator* Arch. Rational Mech. Anal. **11** (1962) 117–121

10 Ekeland, I.; Temam, R.: *Convex analysis and variational problems* SIAM, Philadelphia 1999

11 Ericksen, J. L.: *Nilpotent energies in liquid crystal theory* Arch. Rational Mech. Anal. **10** (1962) 189–196

12 Eringen, A. C.; Maugin, G. A.: *Electrodynamics of continua I* Springer, New York 1990

13 Eringen, A. C.; Maugin, G. A.: *Electrodynamics of continua II* Springer, New York 1990

14 Fonseca, I.; Leoni, G.: *Modern Methods in the Calculus of Variations: L^p Spaces* Springer, New York 2007

15 Fonseca, I.; Müller, S.: *A-quasiconvexity, lower semicontinuity, and Young measures* SIAM J. Math. Anal. **30** (1999) 1355–1390

16 Foss, M.; Randriampiry, N.: *Some two-dimensional A-quasiaffine functions* Contemporary Mathematics, vol. 514 **514** (2010) 133–141 (A co-publication of the AMS and Bar-Ilan University.)

17 Giaquinta, M.; Modica, G.; Souček, J.: *Cartesian currents, weak diffeomorphisms and existence theorems in non-linear elasticity* Arch. Rational Mech. Anal. **106** (1989) 97–160

18 Giaquinta, M.; Modica, G.; Souček, J.: *Cartesian currents in the calculus of variations I, II* Springer, Berlin 1998

19 Gil, A. J.; Ortigosa, R.: *A new framework for large strain electromechanics based on convex multi-variable strain energies: Variational formulation and material characterisation* Comput. Methods Appl. Mech. Engrg. **302** (2016) 293–328

20 Hutter, K.: *On thermodynamics and thermostatics of viscous thermoelastic solids in the electromagnetic fields. A Lagrangian formulation* Arch. Rational Mech. Anal. **58** (1975) 339–368 `()`

21 Hutter, K.; van de Ven, A. A. F.; Ursescu, A.: *Electromagnetic field matter interactions in thermoelastic solids and viscous fluids.* Lect. Notes Phys. 710 Springer, Berlin 2006

22 Itskov, M.; Aksel, N.: *A class of orthotropic and transversely isotropic hyperelastic constitutive models based on a polyconvex strain energy function* International Journal of Solids and Structures **41** (2004) 3833–3848

23 Itskov, M.; Khiem, V. N.: *A polyconvex anisotropic free energy function for electro- and magneto-rheological elastomers* Mathematics and Mechanics of Solids **21** (2016) 1126–1137 `https://10.1177/1081286514555140`

24 Kankanala, S. V.; Triantafyllidis, N.: *On finitely strained magnetorheological elastomers* J Mech Phys Solid **52** (2004) 2869–2908

25 Kovetz, A.: *Electromagnetic theory* Oxford University Press, Oxford 2000

26 Krämer, J.; Krömer, S.; Kružík, M.; Pathó, G.: *A-quasiconvexity at the boundary and weak lower semicontinuity of integral functionals* Advances in Calculus of Variations **10** (2015) 49–67

27 Marsden, J. E.; Hughes, T. J. R.: *Mathematical foundations of elasticity* Dover Publications, New York 1994

28 Matias, J.; Morandotti, M.; Santos, P. M.: *Homogenization of functionals with linear growth in the context of A-quasiconvexity* Applied Mathematics and Optimization **72** (2015) 523–547

29 Maugin, G.: *Continuum mechanics of electromagnetic solids* North-Holland, Amsterdam 1988

30 Morrey, Jr, C. B.: *Quasi-convexity and the lower semicontinuity of multiple integrals* Pacific J. Math. **2** (1952) 25–53

31 Morrey, Jr, C. B.: *Multiple integrals in the calculus of variations* Springer, New York 1966

32 Müller, S.: *Higher integrability of determinant and weak convergence in* L^1 J. reine angew. math. **412** (1990) 20–34

33 Müller, S.: *Variational Models for Microstructure and Phase Transitions* In *Calculus of variations and geometric evolution problems (Cetraro, 1996) Lecture notes in Math. 1713* S. Hildebrandt, M. Struwe (ed.), pp. 85–210, Springer, Berlin 1999

34 Müller, S.; Tang, Q.; Yan, B. S.: *On a new class of elastic deformations not allowing for cavitation* Ann. Inst. H. Poincaré, Analyse non linéaire **11** (1994) 217–243

35 Murat, F.: *Compacité par compensation* Ann. Scuola Normale Sup. Pisa Cl. Sci. S. IV **5** (1978) 489–507

36 Murat, F.: *Compacité par compensation: condition nécessaire et suffisante de continuité faible sous une hypothése de rang constant* Ann. Scuola Normale Sup. Pisa Cl. Sci. S. IV **8** (1981) 68–102

37 Ortigosa, R.; Gil, A. J.: *A new framework for large strain electromechanics based on convex multi-variable strain energies: Conservation laws and hyperbolicity and extension to electro-magnetomechanics* Comput. Methods Appl. Mech. Engrg. **317** (2017) 792–816 `http://dx.doi.org/10.1016/j.cma.2016.05.019`

38 Ortigosa, R.; Gil, A. J.: *A new framework for large strain electromechanics based on convex multi-variable strain energies: Finite Element discretisation and computational implementation* Comput. Methods Appl. Mech. Engrg. **302** (2016) 329–360 `http://10.1016/j.cma.2015.12.007`

39 Pedregal, P.: *Parametrized measures and variational principles* Birkhäuser, Basel 1997

40 Reshetnyak, Yu. G.: *On the stability of conformal mappings in multidimensional spaces* Siberian. Math. J. **8** (1967) 91–114

41 Reshetnyak, Yu. G.: *General theorems on semicontinuity and on convergence with a functional* Siberian Math. J. **8** (1967) 801–816

42 Reshetnyak, Yu. G.: *Stability theorems for mappings with bounded excursion* Siberian Math. J. **9** (1968) 499–512

43 Rockafellar, R. T.: *Convex analysis* Princeton University Press, Princeton 1970
44 Schröder, J.; Neff, P.: *Invariant formulation of hyperelastic transverse isotropy* Int. J. Solid Struct. **40** (2003) 401–445
45 Šilhavý, M.: *The mechanics and thermodynamics of continuous media* Springer, Berlin 1997
46 Šilhavý, M.: *Polyconvexity for functions of a system of closed differential forms* Calc. Var. **57:26** (2018) `https://doi.org/10.1007/s00526-017-1298-2`
47 Šilhavý, M.: *Isotropic polyconvex electromagnetoelastic bodies* Mathematics and Mechanics of Solids **24** (2019) 736–747 `https://doi.org/10.1177/1081286518754567`
48 Steigmann, D. J.: *Frame-invariant polyconvex strain-energy functions for some anisotropic solids* Math. and Mech. Solids **8** (2003) 497–506
49 Tartar, L.: *Compensated compactness and applications to partial differential equations* In *Nonlinear Analysis and Mechanics: Heriot-Watt Symposium* R. Knops (ed.), pp. 136–212, Longman, Harlow, 1979
50 Truesdell, C.; Noll, W.: *The non-linear field theories of mechanics* In *Handbuch der Physik III/3* S. Fluegge (ed.), Springer, Berlin 1965
51 Truesdell, C.; Toupin, R. A.: *Classical field theories of mechanics* In *Handbuch der Physik III/1* Springer, Berlin 1960
52 Van Hove, L.: *Sur l'extension de la condition de Legendre du calcul des variations aux intégrales multiples à plusieurs fonctions inconnues* Proc. Koninkl. Ned. Akad. Wetenschap **50** (1947) 18–23