Mathematics and Mechanics of Solids **24** (2019) 736–747
https://doi.org/10.1177/1081286518754567

Isotropic polyconvex electro-magneto-elastic bodies

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Abstract The recent renewal of interest in nonlinear electro-magneto-elastic interactions comes from the technological importance of electro- or magneto-sensitive elastomers, smart materials whose mechanical properties change instantly by the application of an electric or magnetic fields. We consider materials with the free energy functions of the form $\psi = \psi(F, d, b)$, where F is deformation gradient, d is the electric displacement and b is the magnetic induction. In a recent paper [36], it was shown that such an energy function is polyconvex if an only if it is of the form

$$\psi(F, d, b) = \Phi(F, \operatorname{cof} F, \det F, d, b, Fd, Fb) \qquad (*)$$

where Φ is a convex function (of 31 scalar variables). Moreover, an existence theorem was proved for the equilibrium solution for a system consisting of a polyconvex electro-magneto-elastic solid plus the vacuum electromagnetic field outside the body. The condition $(*)$ is not just the combination of Ball's polyconvexity of elastomers $\psi(F) = \Phi(F, \operatorname{cof} F, \det F)$ with the convexity in the electromagnetic variables. The differential constrains

$$\operatorname{div} d = 0, \quad \operatorname{div} b = 0$$

allow for the cross mechanic-electric and mechanic-magnetic terms Fd and Fb, thus substantially enlarging the class of energies covered by the theory. The result $(*)$ applies to a material of any symmetry; the present paper analyses that condition in the case of isotropic materials. A broad sufficient condition for the polyconvexity is given in that case. Further, it is shown that the commonly used isotropic electro-elastic or magneto-elastic invariants are polyconvex except for

Acknowledgment The support of the institutional research plan RVO 67985840 is gratefully acknowledged.

the biquadratic ones; the paper explicitly determines their quasiconvex envelopes and shows that they are polyconvex.

Keywords Electromechanical and magnetomechanical interactions, finite strain, constitutive equations, energy methods, isotropy, invariants

Contents

1 Introduction

Electro- or magneto-sensitive elastomers are smart materials whose manufacturing process results in their ability to instantaneously change their mechanical properties by the application of an electric or magnetic field. The sensitivity to the electromagnetic fields is due to their structure which involves some metallic electro- or magneto-sensitive inclusions deposited in an elastomeric (usually rubber) matrix. Their overall response exhibits a full nonlinear coupling of the mechanical response with the electric and magnetic fields.

In a well-known paper [1] Ball established the (mathematical) consistency of the nonlinear elastostatics by proving satisfactory existence theorems under the polyconvexity condition on the stored energy. In [36], the present author extended the polyconvexity notion to electro-magneto-elastic response. Specifically, [36] treats the stored energies of the form

$$\psi = \psi(F, d, b),$$

where F is deformation gradient, d is the referential electric displacement and b the referential magnetic induction. The triplet (F, d, b) of fields over the reference configuration Ω of the body satisfies the differential constraints ⋆

$$\operatorname{curl} F = 0, \quad \operatorname{div} d = 0, \quad \operatorname{div} b = 0; \tag{1.1}$$

here $(1.1)_1$ is (locally) equivalent to the existence of the displacement y such that $F = \nabla y$ while $(1.1)_{2,3}$ are equivalent to the existence of vector potentials v, w such that $d = \operatorname{curl} v$, $b = \operatorname{curl} w$. The differential constraints (1.1) play decisive role on the form of the quasiconvexity and polyconvexity conditions for electro-magneto-elastic interactions. The analysis of the convexity conditions under differential constraints on the independent variables is the subject of the $\mathscr{A}$-quasiconvexity theory introduced in [4] and further developed in [12].⋆⋆ We refer to Sections 2 and 3 for detailed

⋆ Here curl and div are the referential curl and divergence.

⋆⋆ Related developments are found within the context of compensated compactness [23–24, 38–39]. For more recent works, see [2, 5–6, 19] and the references therein.

presentations of the basic notions of the electro-magneto-elasticity and of the quasiconvexity theory for electro-magneto-elastic interactions.[⋆⋆⋆] We assume that ψ is a continuous function on one of the following two sets [⋆⋆⋆⋆]

$$\mathbb{D}_{+} := \mathbb{M}^{3\times 3}_{+} \times \mathbb{R}^{3} \times \mathbb{R}^{3}, \quad \mathbb{D} := \mathbb{M}^{3\times 3} \times \mathbb{R}^{3} \times \mathbb{R}^{3}. \tag{1.2}$$

Defining the polyconvexity of ψ as the expressibility of ψ as a convex function of the collection of all quasiaffine functions of (F, d, b) (Definitions 3.3 and 3.5, below), the author showed that ψ is polyconvex if an only if it is of the form

$$\psi(F, d, b) = \Phi(F, \operatorname{cof} F, \det F, d, b, Fd, Fb), \tag{1.3}$$

where $\operatorname{cof} F = (\det F)F^{-\mathrm{T}}$ and where Φ is a convex function (of 31 scalar variables), see Theorem 3.6. The condition (1.3) is not just the combination of Ball's polyconvexity of elastomers,

$$\psi(F) = \Phi(F, \operatorname{cof} F, \det F), \tag{1.4}$$

with the convexity in the electromagnetic variables,

$$\psi(F) = \Phi(F, \operatorname{cof} F, \det F, d, b), \tag{1.5}$$

where Φ is a convex function. Indeed, the neglection of the constrains $(1.1)_{2,3}$ would result in the polyconvexity of the form (1.5). The extra cross mechanic-electric and mechanic-magnetic terms Fd and Fb result exactly from the constraints $(1.1)_{2,3}$, in the same way as $(1.1)_1$ results in the occurrence of $\operatorname{cof} F$ and $\det F$ in (1.4) in the purely mechanical case. Of course, the condition (1.3) encompasses a much larger class of energies than (1.5). The "reasonability" of the polyconvexity (1.3) is demonstrated by establishing an existence theorem for the equilibrium state of the system consisting of a polyconvex electro-magneto-elastic body and the induced external electromagnetic field [36]. Only the Dirichlet mechanical boundary conditions are covered presently.

The convexity conditions for the special case of the electric or magnetic phenomena in *rigid bodies* (no deformation) have been studied in a number of papers, see, e.g., [38, 4, 30, 12, 5]. These works show, among other things, that the quasiconvexity under $(1.1)_2$ or under $(1.1)_3$ reduces to the ordinary convexity in d and b, repectively.

The quasiconvexity for combinations of mechanical and magnetic phenomena has been discussed in [17] and [16], but ignoring the constraint $(1.1)_2$, which substantially reduces the class of quasiconvex and polyconvex energies. The paper [13] briefly mentions, as an example, a combination of mechanical and magnetic phenomena in 2 dimensions within a different framework, but without any further development.

After the research of this paper and of [36] had been completed, the author became aware of the recent papers by Gil & Ortigosa [14, 27–28], which contain Condition (1.3) also, but only as a sufficient condition for the electro-magneto-elastic ellipticity condition. Our paper [36] derives (1.3) from the polyconvexity assumption, which is much stronger; only that opens the way to the existence theorem for equilibrium states of electro-magneto-elastic bodies in [36].

⋆⋆⋆ Henceforth we omit the modifier '$\mathscr{A}$' and speak of quasiconvexity, quasiafinity and polyconvexity.

⋆⋆⋆⋆ We denote by $\mathbb{M}^{3\times 3}$ the space of all real 3×3 matrices and by $\mathbb{M}^{3\times 3}_{+}$ the set of $\mathbb{M}^{3\times 3}$ matrices with positive determinant. The paper treats three-dimensional bodies.

The above results apply to materials of any symmetry. Although typically the electro- or magneto-sensitive elastomers display a strongly anisotropic response, the isotropy is not excluded. The goal of the present paper is to combine the polyconvexity (1.3) with the isotropy, to begin with. The isotropy (and the principle of material frame indifference) require that

$$\psi(QFR^{\mathrm{T}}, Rd, Rb) = \psi(F, d, b) \tag{1.6}$$

for each proper orthogonal tensors Q, R, each deformation gradient F with positive determinant and each electric and magnetic vectors d and b.

We summarize the main results of this note.

First, extending a well-known result of Ball [1; Theorem 5.2], in the wording of the subsequent amendments, a sufficient condition is provided for the polyconvexity of isotropic electro-magneto-elastic bodies in Theorem 4.1. That theorem can be used to produce a wide variety of different isotropic polyconvex functions. See Example 4.3.

Second, one popular way to satisfy the symmetry requirement imposed by the isotropy is to express the energy as functions of a minimal family of isotropic invariants. This way already occurred in the basic paper by Toupin [40] (who used a slightly different set of independent variables) and employed numerously in the literature since then. It is easily seen that the following is a complete list of isotropic invariants for electro-magneto-elastic interactions

$$\left.\begin{array}{lll} I_1 = \operatorname{tr} C, & I_2 = \operatorname{tr}\operatorname{cof} C, & I_3 = \det C, \\ K_1^{\mathrm{e}} = |d|^2, & K_2^{\mathrm{e}} = |Fd|^2, & K_3^{\mathrm{e}} = |Cd|^2, \\ K_1^{\mathrm{m}} = |b|^2, & K_2^{\mathrm{m}} = |Fb|^2, & K_3^{\mathrm{m}} = |Cb|^2, \\ & M^{\mathrm{em}} = d \cdot b & \end{array}\right\} \tag{1.7}$$

where

$$C = F^{\mathrm{T}}F.$$

The reader is referred e.g., to [7] and [8] for the isotropic invariants for electro-elastic and magneto-elastic interactions, formed by the relevant subsets of the list (1.7). Theorem 4.4 examines the polyconvexity of the members of (1.7). It turns out that all functions in (1.7) are polyconvex except for K_3^{e}, K_3^{m}, and M^{em}. For the last three, the quasiconvex envelopes are determined explicitly, they are shown to be polyconvex, and functionally dependent on the remaining elements of (1.7).

The paper is organized as follows. Section 2 gives a survey of the equilibrium and constitutive equations for the static electro-magneto-elasticity. Section 3 introduces the specialisation of the $\mathscr{A}$-quasiconvexity notions to the present case. Section 4 presents the main results, Theorems 4.1 and 4.4, together with the illustrations without proofs. Section 5 provides the proofs of Theorems 4.1 and 4.4.

2 Equilibrium and constitutive equations for electro-magneto-elasticity

We briefly recapitulate the static form of Maxwell's equations and the mechanical equilibrium of forces that govern the behavior of the body [41, 3, 15, 21, 10–11, 18].

We work exclusively in the reference configuration $\Omega \subset \mathbb{R}^3$, denote by $y : \Omega \to \mathbb{R}^3$ the deformation and by $F : \mathbb{R}^3 \to \mathbb{M}_+^{3\times 3}$ the deformation gradient

$$F = \nabla y;$$

furthermore, the referential stress tensor is denoted by S and the density of the body force g. The electromagnetic variables are the (referential) electric and magnetic fields, the electric displacement and the magnetic induction, denoted, respectively, by e, h, d, b.⋆ The referential forms of the equilibrium equations read

$$\operatorname{div} d = 0, \quad \operatorname{div} b = 0, \quad \operatorname{curl} e = 0, \quad \operatorname{curl} h = 0, \tag{2.1}$$

$$\operatorname{div} S + g = 0 \tag{2.2}$$

where curl and div denote the referential forms of the curl and divergence. The equations (2.1) and (2.2) are assumed to hold in the weak sense, which then includes the well-known jump conditions for the mechanical and electromagnetic variables on surfaces of discontinuity.

The constitutive equations are

$$S = S(F, d, b), \quad e = e(F, d, b), \quad h = h(F, d, b)$$

where the variables F, d and b satisfy the permanent constraint (1.1). The constitutive functions are derivable from the stored energy $\psi = \psi(F, d, b)$ via the potential relations

$$S = \mathrm{D}_F\psi, \quad e = \mathrm{D}_d\psi, \quad h = \mathrm{D}_b\psi.$$

A standard argument shows that the principle of material frame indifference (a particular case $R = 1$ in (1.6)) yields the symmetry of the stress,

$$SF^{\mathrm{T}} = FS^{\mathrm{T}}.$$

It is well-known that the balance equations $(2.1)_{3,4}$ and (2.2) are the Euler-Lagrange equations corresponding to the variational problem with the energy density $\psi = \psi(F, d, b)$ for the fields F, d, b constrained by (1.1).

3 Electro-magneto-elastic quasiconvexity and polyconvexity

In this section we adapt the basic skeleton of notions of the $\mathscr{A}$-quasiconvexity theory to the differential constraints (1.1). The reader is referred to [4] and [12] (and to the additional references listed in the introduction) for the general notions. The section also presents the results of [36] and [37] on the polyconvexity for electro-magneto-elastic interactions (Theorems 3.4 and 3.6). Throughout the section, ψ is a real-valued function on a domain $\mathscr{D}$ which is equal either to $\mathbb{D}_+$ or to $\mathbb{D}$, as defined in (1.2).

⋆ These are related to the spatial variables T, E, H, D, B by

$$S = T \operatorname{cof} F, \quad e = F^{\mathrm{T}}E, \quad h = F^{\mathrm{T}}H, \quad d = (\operatorname{cof} F)^{\mathrm{T}}D, \quad b = (\operatorname{cof} F)^{\mathrm{T}}B.$$

Definition 3.1 The function ψ is said to be quasiconvex if the inequality

$$\int_Q \psi(F + \varphi(x), d + \delta(x), b + \beta(x))\, dx \geq \psi(F, d, b) \tag{3.1}$$

holds on the unit cube $Q = (0,1)^3$ for each triple of constant values of $(F, d, b) \in \mathscr{D}$ and for each triplet (φ, δ, β) of smooth functions on $\mathbb{R}^3$, periodic with respect to Q, satisfying

$$\operatorname{curl} \varphi = 0, \quad \operatorname{div} \delta = 0, \quad \operatorname{div} \beta = 0 \quad \text{on} \quad \mathbb{R}^3 \tag{3.2}$$

and

$$\int_Q \varphi\, dx = 0, \quad \int_Q \delta\, dx = 0, \quad \int_Q \beta\, dx = 0 \tag{3.3}$$

and such that the argument of ψ in the integral in (3.1) is in $\mathscr{D}$ for every $x \in \mathbb{R}^3$.

The following notion will be applied to the isotropic invariants K_3^{e}, K_3^{m} and M^{em} below.

Definition 3.2 The quasiconvex envelope $\mathbb{Q}\psi : \mathscr{D} \to \mathbb{R}$ of ψ is defined by

$$\mathbb{Q}\psi(F, d, b) = \sup\{\omega(F, d, b) : \omega \text{ is quasiconvex on } \mathscr{D} \text{ and } \omega \leq \psi \text{ on } \mathscr{D}\},$$

$(F, d, b) \in \mathscr{D}$.

Central to the polyconvexity theory is the notion of quasiaffine functions.

Definition 3.3 The function ψ is said to be quasiaffine if it is defined on $\mathbb{D}$ and if the inequality (3.1) holds with the equality sign for all choices of objects occurred there.

The main point about quasiaffine functions is that they are easy to describe. We note that the set of all quasiaffine functions is a linear space under the natural (pointwise) addition and multiplication by a real number.

Theorem 3.4 *In dimension 3, there are exactly 32 linearly independent scalar quasiaffine functions of* (F, d, b)*, viz., the components of*

$$1, \quad F, \quad \operatorname{cof} F, \quad \det F, \quad d, \quad b, \quad Fd, \quad Fb; \tag{3.4}$$

A general quasiaffine function is a linear combination of the functions in (3.4)*.*

This nontrivial result is proved in [36] by elementary means; [37] establishes Theorem 3.4 as a consequence of the polyconvexity of integrands depending on many closed differential forms.

Definition 3.5 The function ψ on $\mathscr{D} = \mathbb{D}_+$ or on $\mathbb{D}$ is said to be polyconvex if there exists a convex lowersemicontinuous function $\Phi : \mathbb{R}^m \to \mathbb{R}$ and quasiaffine functions $\psi_i = \psi_i(F, d, b)$, $i = 1, \ldots, m$, such that

$$\psi(F, d, b) = \Phi(\psi_1(F, d, b), \ldots, \psi_m(F, d, b))$$

for all $(F, d, b) \in \mathscr{D}$.★

★ Recall that a function $f : \mathrm{X} \to \mathbb{R}$ on a vector space X is said to be convex if

$$f((1-t)\xi_1 + t\xi_2) \leq (1-t)f(\xi_1) + tf(\xi_2) \tag{3.5}$$

for every ξ_1, $\xi_2 \in \mathrm{X}$ and every $t \in (0,1)$. Further, f is said to be affine if we have the equality sign in (3.5) holding identically. We refer to [32] and [9] for systematic expositions of the convexity theory.

The polyconvexity is a sufficient condition for quasiconvexity; more importantly, it enables to prove an existence theorem under realistic assumptions on ψ, which is impossible under the mere quasiconvexity.

Theorem 3.4 immediately implies the following.

Theorem 3.6 *The function ψ is polyconvex if and only if it is of the form*

$$\psi(F,d,b) = \Phi(F,\operatorname{cof} F,\det F,d,b,Fd,Fb)$$

where Φ is a convex function (of 31 scalar variables).

Apart from the expected terms F, $\operatorname{cof} F$, $\det F$, d, b, which follow from the separate quasiconvexity with respect to F, d, and b, we have the cross-effect terms Fd, Fb.

4 The results with examples

The first result, Theorem 4.1, is an extension of Ball's sufficient condition [1; Theorem 5.2] for the polyconvexity of isotropic functions. The following terminology is needed. If d is a positive integer and $\phi : [0,\infty)^d \to \mathbb{R}$, we say that ϕ is nondecreasing if for every l satisfying $1 \le l \le d$ and for every $(z_1,\dots,z_d) \in [0,\infty)^d$ the function $t \mapsto \phi(z_1,\dots,z_l+t,\dots,z_d)$ is nondecreasing in t. Furthermore, following [34], we say that ϕ is pairwise nondecreasing if for every pair l, m satisfying $1 \le l < m \le d$ and for every $(z_1,\dots,z_d) \in [0,\infty)^d$ the function $t \mapsto \phi(z_1,\dots,z_l+t,\dots,z_m+t,\dots,z_d)$ is nondecreasing in t.

Theorem 4.1 *Let $\psi : \mathbb{D}_+ \to \mathbb{R}$ be given by*

$$\psi(F,d,b) = \Theta(v_1,v_2,v_3,v_1v_2,v_1v_3,v_2v_3,d^\flat,b^\flat,d^\sharp,b^\sharp,v_1v_2v_3) \tag{4.1}$$

for each $(F,d,b) \in \mathbb{D}_+$ where

v_1, v_2, v_3 are the singular values of F (i.e., the eigenvalues of $\sqrt{F^{\mathrm{T}}F}$),

$$d^\flat = |d|, \quad b^\flat = |b|, \quad d^\sharp = |Fd|, \quad b^\sharp = |Fb|,$$

and where $\Theta = \Theta(z)$, $z = (z_1,\dots,z_{11})$, is a function on $[0,\infty)^{10} \times (0,\infty)$ with the following properties:

- *Θ is convex,*
- *$\Theta(z)$ is pairwise nondecreasing in z_1, z_2, z_3 and in z_4, z_5, z_6,*
- *$\Theta(z)$ is nondecreasing in the variables z_7, …, z_{10},*
- *$\Theta(z)$ is symmetric under the permutations of z_1, z_2, z_3 and of z_4, z_5, z_6.*

Then ψ is a polyconvex isotropic function.

Even in the purely mechanical case, Theorem 4.1 is more general than [1; Theorem 5.2] since the latter requires that $\Theta(z)$ be nondecreasing in $z_1,\dots,z_6$ instead of the more general pairwise nondecreasing character of z_1, z_2, z_3 and z_4, z_5, z_6. In the purely mechanical case, this extension is due to Rosakis [34] (see also [35] for $n = 2$). The following example describes the difference.

Example 4.2 Let $\psi : \mathbb{M}^{3\times 3}_+ \to \mathbb{R}$ be defined by

$$\psi(F) = a(v_1 + v_2 - v_3) + b(v_1 v_2 + v_1 v_3 - v_2 v_3) \tag{4.2}$$

for each $F \in \mathbb{M}^{3\times 3}_+$ where $v_1 \ge v_2 \ge v_3 > 0$ are the singular values of F and where a, b are positive constants. (Let us emphasize that the order $v_1 \ge v_2 \ge v_3 > 0$ is important for the definition (4.2).) We shall prove that ψ is polyconvex by applying Theorem 4.1 (neglecting the electromagnetic variables). Indeed, the right-hand side of (4.2) is pairwise nondecreasing in the variables v_1, v_2, v_3 and in the variables $v_1 v_2, v_1 v_3, v_2 v_3$, but not nondecreasing in each of these variables; hence ψ is out of the scope of [1; Theorem 5.2]. In more detail, let $\Theta : (0,\infty)^6 \to \mathbb{R}$ be defined by

$$\Theta(z_1, \dots, z_6) = a(\tilde z_1 + \tilde z_2 - \tilde z_3) + b(\tilde z_4 + \tilde z_5 - \tilde z_6)$$

for any $(z_1, \dots, z_6) \in (0,\infty)^6$, where $(\tilde z_1, \tilde z_2, \tilde z_3)$ is the unique permutation of (z_1, z_2, z_3) such that $\tilde z_1 \ge \tilde z_2 \ge \tilde z_3$ and $(\tilde z_4, \tilde z_5, \tilde z_6)$ is the unique permutation of (z_4, z_5, z_6) such that $\tilde z_4 \ge \tilde z_5 \ge \tilde z_6$. One has

$$\psi(F) = \Theta(v_1, v_2, v_3, v_1 v_2, v_1 v_3, v_2 v_3)$$

for any F and it is easily verified that Θ is convex, pairwise nondecreasing in z_1, z_2, z_3, pairwise nondecreasing in z_4, z_5, z_6, and symmetric under the permutations of z_1, z_2, z_3 and of z_4, z_5, z_6. Thus the polyconvexity of ψ follows from Theorem 4.1.

Example 4.3 The following are isotropic polyconvex functions of $(F, d, b) \in \mathbb{D}$:

$$\left.\begin{aligned}
&v_1^\alpha + v_2^\alpha + v_3^\alpha \equiv \operatorname{tr}(C^{\alpha/2}),\\
&v_1^\alpha v_2^\alpha + v_2^\alpha v_3^\alpha + v_3^\alpha v_1^\alpha \equiv \operatorname{tr}\operatorname{cof}(C^{\alpha/2}),\\
&|d|^\alpha, \quad |b|^\alpha, \quad |Fd|^\alpha, \quad |Fb|^\alpha,\\
&|d|^\beta/J^\gamma, \quad |b|^\beta/J^\gamma, \quad |Fd|^\beta/J^\gamma, \quad |Fb|^\beta/J^\gamma,\\
&J^{-\gamma},
\end{aligned}\right\} \tag{4.3}$$

where $C = F^{\mathrm{T}} C$ and $J = \det F$, provided

$$\alpha \ge 1, \quad \gamma > 0, \quad \beta - 1 \ge \gamma > 0.$$

Indeed, we shall show that each of the above functions is of the format discussed in Theorem 4.1 with various choices of Θ. The first two members of (4.3) are the well-known forms proposed by Ogden [25–26]; the choice of Θ is obvious for them as well as for the members on the third and fourth lines of (4.3). To prove the polyconvexity of the functions on the last line of (4.3), we use the following elementary fact: if β and γ are positive numbers then the function $(a, b) \mapsto f(a, b) = a^\beta / b^\gamma$ is convex on $(0,\infty)^2$ if and only if $\beta \ge \gamma + 1$. This is verified by a direct check of the positive semidefinite character of the hessian of f. Employing this, we prove the polyconvexity of, say, $\psi(F, d, b) = |Fd|^\beta / J^\gamma$ by observing that ψ is of the format of Theorem 4.1 with $\Theta(z) = z_9^\beta / z_{11}^\gamma$, and that Θ meets the requirements stated in that theorem. The polyconvexity of the remaining items in last line of (4.3) are proved similarly, which completes the proof of the polyconvexity of the list (4.3). Then also each linear combination of the functions in (4.3) with positive coefficients (and of course with different values of the exponents) is polyconvex. This can be used to produce a big supply of isotropic polyconvex functions.

The following theorem analyzes the polyconvexity of the isotropic invariants from the list (1.7).

Theorem 4.4

(i) *The invariants* $I_1, I_2, I_3, K_1^{\mathrm{e}}, K_2^{\mathrm{e}}, K_1^{\mathrm{m}}, K_2^{\mathrm{m}}$, *are polyconvex on* $\mathbb{D}$ (*actually even their square roots are polyconvex*);

(ii) *the invariants* K_3^{e}, K_3^{m} *and* M^{em} *are not quasiconvex and hence not polyconvex on* $\mathbb{D}$; *their quasiconvex envelopes* $\mathbb{Q}K_3^{\mathrm{e}}$, $\mathbb{Q}K_3^{\mathrm{m}}$, *and* $\mathbb{Q}M^{\mathrm{em}}$ *on* $\mathbb{D}$ *are polyconvex and given by*

$$\mathbb{Q}K_3^{\mathrm{e}} = |Fd|^4, \quad \mathbb{Q}K_3^{\mathrm{m}} = |Fb|^4, \quad \mathbb{Q}M^{\mathrm{em}} = -\infty \tag{4.4}$$

for every $(F, d, b) \in \mathbb{D}_+$.

Since the above quasiconvexifications in (ii) are different from the originals, those originals are not quasiconvex and hence not polyconvex. Since any nondecreasing convex function of a family of convex functions is convex, see [33; Exercise 2.20(c)], one finds that if $\Psi : [0,\infty)^7 \to \mathbb{R}$ is a convex function nondecreasing in each argument, then

$$\psi(F,d,b) = \Psi(I_1, I_2, I_3, K_1^{\mathrm{e}}, K_2^{\mathrm{e}}, K_1^{\mathrm{m}}, K_2^{\mathrm{m}})$$

is an isotropic polyconvex function. An example of an polyconvex energy expressed through these invariants is a "Mooney–Rivlin magnetoelastic solid" [29]

$$\psi = \frac{1}{4}\mu\big((1+\gamma)(I_1-3) + (1-\gamma)(I_2-3)\big) + \alpha K_1^{\mathrm{m}} + \beta K_2^{\mathrm{m}}$$

where μ is the shear modulus, $\alpha \ge 0$, $\beta \ge 0$ magnetoelastic coupling parameters, and γ an additional parameter, with $|\gamma| \le 1$. Of course, many other proposals occur in the literature, some polyconvex and others not.

5 Proofs

Proof of Theorem 4.1 Only an outline will be given, since the details are notationally complicated (too many variables).

Clearly, the invariance of the variables v_1, v_2, v_3, $d^\flat$, $b^\flat$, $d^\sharp$, $b^\sharp$ under the passage $(F,d,b) \mapsto (QFR^{\mathrm{T}}, Rd, Rb)$ where Q, R are proper ortogonal tensors implies (1.6) and so ψ is an objective and isotropic function.

To prove the polyconvexity of ψ, we note that the result of Rosakis [34] implies that for each fixed $d^\flat, b^\flat, d^\sharp, b^\sharp \in [0,\infty)$ the function

$$F \mapsto \Theta(v_1, v_2, v_3, v_1v_2, v_1v_3, v_2v_3, d^\flat, b^\flat, d^\sharp, b^\sharp, v_1v_2v_3)$$

is polyconvex in Ball's original sense. Next we note that the functions

$$(F,d,b) \mapsto |d|,\ |b|,\ |Fd|,\ |Fb|$$

are all polyconvex since the norm $|\cdot|$ is convex. Furthermore, using the fact that a nondecreasing convex function of a family of convex functions is convex, we deduce that for each fixed $z_1, \dots, z_6, z_{11}$ the function

$$(F, d, b) \mapsto \Theta(z_1, z_2, z_3, z_4, z_5, z_6, |d|, |b|, |Fd|, |Fb|, z_{11})$$

is polyconvex. The full polyconvexity is then essentially a combination of the two particular polyconvexity results stated above. □

Proof of Theorem 4.4, Part (i) We apply Theorem 4.1. One finds that the invariants $I_1, I_2, I_3, K_1^{\mathrm{e}}, K_2^{\mathrm{e}}, K_1^{\mathrm{m}}, K_2^{\mathrm{m}}$, have the representations (4.1) with $\Theta(z_1, \dots, z_{11})$ given by

$$z_1^2 + z_2^2 + z_3^2, \quad z_4^2 + z_5^2 + z_6^2, \quad z_{11}^2, \quad z_7^2, \quad z_8^2, \quad z_9^2, \quad z_{10}^2,$$

respectively. It is easily seen that all these functions satisfy the requirements of Theorem 4.1 and hence its conclusion gives the assertions of Theorem 4.4, Part (i). □

To proceed to the proof of Theorem 4.4, Part (ii), we recall the following notions.

Definitions 5.1 (Eg., [1, 4, 22]) Let $\eta : \mathbb{M}^{3\times 3} \to \mathbb{R}$.

(i) The function η is said to be rank 1 convex if

$$\eta(F + tp \otimes q) \le (1 - t)\eta(F) + t\eta(F + p \otimes q) \tag{5.1}$$

for every $t \in (0,1)$, every $F \in \mathbb{M}^{3\times 3}$ and every $p, q \in \mathbb{R}^3$.

(ii) The function η is said to be rank 1 affine if it taken only finite values and (5.1) holds with the equality sign for every t, F, p, and q as in (i).

(iii) The rank 1 convex envelope $\mathbb{R}\eta : \mathbb{M}^{3\times 3} \to \mathbb{R}$ of η is defined by

$$\mathbb{R}\eta(F) = \sup\{\zeta(F) : \zeta \text{ is rank 1 convex and } \zeta \le \eta \text{ on } \mathbb{M}^{3\times 3}\},$$

$F \in \mathbb{M}^{3\times 3}$.

Theorem 5.2 ([31], [20; Theorem 2]) *Let $\eta : \mathbb{M}^{3\times 3} \to \mathbb{R}$ be of the form*

$$\eta(F) = \theta(F^{\mathrm{T}}F) \tag{5.2}$$

$F \in \mathbb{M}^{3\times 3}$, *where θ is a convex function on the set $\mathbb{S}_+^{3\times 3}$ of all positive semidefinite tensors. Then*

$$\mathbb{R}\eta(F) = \inf\{\theta(F^{\mathrm{T}}F + Y) : Y \in \mathbb{S}_+^{3\times 3}\}, \tag{5.3}$$

$F \in \mathbb{M}^{3\times 3}$, *and $\mathbb{R}\eta$ is convex on $\mathbb{M}^{3\times 3}$.*

Lemma 5.3 *Let $\eta : \mathbb{M}^{3\times 3} \to \mathbb{R}$ be given by $\eta(F) = |F^{\mathrm{T}}Fd|^2$, $F \in \mathbb{M}^{3\times 3}$, where $d \in \mathbb{R}^3$ is a fixed unit vector. Then*

$$\mathbb{R}\eta(F) = |Fd|^4 \tag{5.4}$$

for every $F \in \mathbb{M}^{3\times 3}$.

Proof We have (5.2) with $\theta(C) = |Cd|^2$, $C \in \mathbb{S}_+^{3\times 3}$. We observe that θ is convex since the function $p : C \mapsto |Cd|$, being a seminorm, is convex and $\theta = p^2$ is then convex as well. To prove (5.4), we note that $\eta(F) \ge |Fd|^4$ and hence there is nothing to prove if $\eta(F) = |Fd|^4$; accordingly, assume that $\eta(F) > |Fd|^4$. We shall employ (5.3) with

$$Y = \tau^{-1}(Cd - (Cd \cdot d)d - \tau d) \otimes (Cd - (Cd \cdot d)d - \tau d)$$

where $\tau > 0$. One finds that $\theta(C + Y) = (|Fd|^2 + \tau)^2$ and hence letting $\tau \to 0$, we obtain from (5.3) that $\mathbb{R}\eta(F) \le |Fd|^4$. However, the function $F \mapsto |Fd|^4$ is convex, hence rank 1 convex and thus we have (5.4). □

Proof of Theorem 4.4, Part (ii) Since for any function $\psi = \psi(F, d, b)$ on $\mathbb{D}$ the electro-magneto-elastic quasiconvexity in the sense of Definition 3.1 implies Morrey's quasiconvexity of $\psi(\cdot, d, b)$ for every $d, b \in \mathbb{R}^3$, and since Morrey's quasiconvexity implies rank 1 convexity in the sense of Definition 5.1(i), we conclude that

$$(\mathbb{Q}\psi)(\cdot, d, b) \le \mathbb{R}\psi(\cdot, d, b) \tag{5.5}$$

for every $d, b \in \mathbb{R}^3$. Fixing d and b and introducing $\eta_d, \eta_b : \mathbb{M}^{3\times3} \to \mathbb{R}$ by

$$g_d(F) := |F^{\mathrm{T}}Fd|^2, \quad g_b(F) := |F^{\mathrm{T}}Fb|^2,$$

$F \in \mathbb{M}^{3\times3}$ we obtain from Lemma 5.3 the formulas

$$\mathbb{R}\eta_d(F) = |Fd|^4, \quad \mathbb{R}\eta_b = |Fb|^4$$

and thus (5.5) yields

$$|Fd|^4 \ge \mathbb{Q}K_3^{\mathrm{e}}, \quad |Fb|^4 \ge \mathbb{Q}K_3^{\mathrm{m}}. \tag{5.6}$$

By a happy coincidence, the functions $(F, d, b) \mapsto |Fd|^4$, $(F, d, b) \mapsto |Fb|^4$ are polyconvex by Theorem 4.1; hence we have the equality signs throughout (5.6), which completes the proof of $(4.4)_{1,2}$.

Finally, $(4.4)_3$ is proved by an elementary observation that there is no quasiconvex function below M^{em}. Indeed, assume that $\psi : \mathbb{D} \to \mathbb{R}$ is a quasiconvex function below M^{em}. Then (3.1) gives

$$\begin{aligned}\psi(F, d, b) &\le \int_Q \psi(F + \varphi(x), d + \delta(x), b + \beta(x))\,dx \\ &\le \int_Q M^{\mathrm{em}}(F + \varphi(x), d + \delta(x), b + \beta(x))\,dx \\ &= \int_Q \big(d + \delta(x)\big) \cdot \big(b + \beta(x)\big)\,dx\end{aligned}$$

for each triple of constant values of $(F, d, b) \in \mathbb{D}$ and for each triplet (φ, δ, β) of smooth functions on $\mathbb{R}^3$, periodic with respect to Q, satisfying (3.2) and (3.3). Replacing δ and β by $t\delta$ and $t\beta$ where $t > 0$ and dividing by t^2, we obtain

$$\psi(F, d, b)/t^2 \le \int_Q \big(d/t + \delta(x)\big) \cdot \big(b/t + \beta(x)\big)\,dx$$

and letting $t \to \infty$ we obtain

$$0 \le \int_Q \delta(x) \cdot \beta(x)\,dx$$

for each pair (δ, β) of smooth functions on $\mathbb{R}^3$, periodic with respect to Q, satisfying $(3.2)_{2,3}$ and $(3.3)_{2,3}$. This is impossible since one can replace δ by $-\delta$ to obtain the opposite inequality and hence

$$\int_Q \delta(x) \cdot \beta(x)\,dx = 0$$

which is absurd. □

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