Zeitschrift für Angewandte Mathematik und Mechanik,
https://doi.org/10.1002/zamm.201800324

Consequences of the Coleman-Noll inequality for isotropic materials

Miroslav Šilhavý
Institute of Mathematics, CAS
Žitná 25
115 67 Prague 1
Czech Republic
e-mail: silhavy@math.cas.cz

Abstract The paper establishes necessary and sufficient conditions for the Coleman-Noll inequality for isotropic materials. The novel part in these conditions is based on a formula that was not available at the time of writings of Coleman and Noll on the subject.

AMS 2010 classification 74B20 · 74A20

Keywords Coleman-Noll inequality · isotropy · second derivative of the stored energy · Legendre-Hadamard condition · Baker-Ericksen inequalities

Contents

1 Introduction

Towards the end of the formatting period of the nonlinear elasticity, in 1956, in his paper *Das ungelöste Hauptproblem der endlichen Elastizitätstheorie,* [16], Truesdell points out that the nonlinear elasticity misses conditions on the strain-energy that

Acknowledgment This research was supported by RVO 67985840.

would guarantee reasonable behavior (existence of solutions, stability, reality of wave speeds, uniqueness etc.). These restrictions should be nonlinear counterparts of the well-known inequalities on Lamé constants in linear elasticity. After a debate that lasted more than two decades, the final solution of the *Hauptproblem* came in 1977 with the work J. M. Ball [1]. He showed that the triplet of weakened convexity conditions, viz., *polyconvexity, Morrey's quasiconvexity, and the rank 1 convexity* (the Legendre-Hadamard condition, the nonstrict version of the strong ellipticity) is exactly what is missing. The quasiconvexity and its consequence rank 1 convexity occur in basic aspects such as the *consequences* of the existence of minima, lower semicontinuity etc. On the other hand, the *sufficient condition* for the existence of minima under the condition on the density of the stored energy

$$\bar{\psi}(\boldsymbol{F}) \to \infty \quad \text{as} \quad \det \boldsymbol{F} \to 0$$

require the polyconvexity, a condition stronger than quasiconvexity.

The early paper of Coleman & Noll [8] (1959) is, on the contrary, probably the first response to Truesdell's paper. The authors postulated an inequality that now bears their name, and derived some of its consequences for various classes of materials. The Coleman-Noll inequality triggered a lively and fruitful discussion, which finally showed the untenability of the inequality. A number of objections have been raised, of which perhaps the most important is the one (apparently due to R. S. Rivlin) which points out that the Coleman-Noll inequality cannot hold for an almost incompressible isotropic material, due to the non-convexity of the surface $v_1 v_2 v_3 = 1$.⋆ Furthermore, after the recognition of the importance of the polyconvexity, quasiconvexity, and the rank 1 convexity, it turned out that the Coleman-Noll inequality is not entirely compatible with them (see below).⋆⋆

Nevertheless, in this note I discuss the consequences of the Coleman-Noll inequality for isotropic materials. This class of materials is treated in the original paper by Coleman & Noll [8; §12], where the authors show that the Coleman-Noll inequality *implies* the convexity of the free energy in the principal stretches , i.e., Condition (i) of Theorem 1, below. Here I add Condition (ii) to get conditions that are simultaneously necessary and sufficient. Some consequences are derived, and the relationship to the Legendre-Hadamard condition is briefly discussed.

2 Coleman-Noll inequality for isotropic materials

We consider an isothermal material governed by the constitutive equations for the specific free energy ψ and Piola-Kirchhoff stress $\boldsymbol{S}$ of the form

$$\psi = \bar{\psi}(\boldsymbol{F}), \quad \boldsymbol{S} = \bar{\boldsymbol{S}}(\boldsymbol{F}),$$

where $\boldsymbol{F}$ is the deformation gradient, the response functions $\bar{\psi} : \mathrm{Lin}^+ \to \mathbb{R}$ and $\bar{\boldsymbol{S}} : \mathrm{Lin}^+ \to \mathrm{Lin}$ are twice continuously differentiable and continuously differentiable,

⋆ See also [10].

⋆⋆ Detailed accounts of the works of B. D. Coleman and W. Noll in continuum mechanics and thermodynamics are found in [15] and [9], respectively.

respectively. Here Lin is the set of all of second-order tensors and Lin^+ the subset of tensors with positive determinant. The symmetric Cauchy stress $\boldsymbol{T}$ is given by

$$\boldsymbol{T} = \bar{\boldsymbol{T}}(\boldsymbol{F})$$

where the function $\bar{\boldsymbol{T}}$ is given by

$$\bar{\boldsymbol{T}}(\boldsymbol{F}) = \bar{\boldsymbol{S}}(\boldsymbol{F})\boldsymbol{F}^{\mathrm{T}}/\det\boldsymbol{F}. \tag{1}$$

Coleman-Noll inequality We say that the material satisfies the Coleman-Noll inequality (briefly, the CNI) if for every $\boldsymbol{F}, \boldsymbol{F}^* \in \mathrm{Lin}^+$ such that

$$\boldsymbol{F}^* = \boldsymbol{G}\boldsymbol{F}$$

for some symmetric positive definite tensor $\boldsymbol{G}$ we have

$$\bar{\psi}(\boldsymbol{F}^*) \geq \bar{\psi}(\boldsymbol{F}) + \bar{\boldsymbol{S}}(\boldsymbol{F}) \cdot (\boldsymbol{F}^* - \boldsymbol{F}). \tag{2}$$

Proposition 1 *The material satisfies the* CNI *if and only if the following two relations hold for all* $\boldsymbol{F} \in \mathrm{Lin}^+$:

$$\bar{\boldsymbol{S}}(\boldsymbol{F}) = \mathrm{D}\bar{\psi}(\boldsymbol{F}) \tag{3}$$

and

$$\mathrm{D}^2\bar{\psi}(\boldsymbol{F})[\boldsymbol{H}\boldsymbol{F}, \boldsymbol{H}\boldsymbol{F}] \geq 0 \tag{4}$$

for all symmetric second-order tensors $\boldsymbol{H}$.*

Here $\mathrm{D}\bar{\psi}(\boldsymbol{F})$ is the first derivative of $\bar{\psi}$ with respect to $\boldsymbol{F}$ at $\boldsymbol{F}$ and $\mathrm{D}^2\bar{\psi}(\boldsymbol{F})[\cdot,\cdot]$ is the second derivative of $\bar{\psi}$ at $\boldsymbol{F}$, interpreted as a quadratic form.

Proof If $\boldsymbol{H}$ is any symmetric second-order tensor, then $\boldsymbol{G}(t) := \boldsymbol{1} + t\boldsymbol{H}$ is a positive definite tensor for all $t \in \mathbb{R}$ sufficiently close to 0. Hence if (2) holds for the given $\boldsymbol{F}$ and $\boldsymbol{F}^*(t) := \boldsymbol{G}(t)\boldsymbol{F}$, the excess function

$$\phi(t) = \bar{\psi}\big(\boldsymbol{F}^*(t)\big) - \bar{\psi}(\boldsymbol{F}) - \bar{\boldsymbol{S}}(\boldsymbol{F}) \cdot \big(\boldsymbol{F}^*(t) - \boldsymbol{F}\big)$$

has a local minimum at $t = 0$. The conditions $\dot{\phi}(0) = 0$ gives the stress relation (3) while $\ddot{\phi}(0) \geq 0$ gives (4).

Conversely, assume that (3) and (4) hold for all $\boldsymbol{F} \in \mathrm{Lin}^+$. Let $\boldsymbol{G}$ be a positive definite and symmetric tensor, let $\boldsymbol{H} := \boldsymbol{G} - \boldsymbol{1}$, and define $\boldsymbol{F}(t) = \boldsymbol{F} + t\boldsymbol{H}\boldsymbol{F} \equiv \big((1-t)\boldsymbol{1} + t\boldsymbol{G}\big)\boldsymbol{F}$, where $t \in [0,1]$. Then $\det\boldsymbol{F}(t) = \det\big((1-t)\boldsymbol{1} + t\boldsymbol{G}\big)\det\boldsymbol{F} > 0$ as the tensor $(1-t)\boldsymbol{1} + t\boldsymbol{G}$ is positive definite symmetric and $\det\boldsymbol{F} > 0$ by definition. Thus $\boldsymbol{F}(t) \in \mathrm{Lin}^+$ for all $t \in [0,1]$ and we can define $\omega(t) := \bar{\psi}(\boldsymbol{F}(t))$ for $t \in [0,1]$. One finds that

$$\ddot{\omega}(t) = \mathrm{D}^2\bar{\psi}(\boldsymbol{F}(t))[\boldsymbol{H}\boldsymbol{F}(t), \boldsymbol{H}\boldsymbol{F}(t)] \geq 0$$

where the last inequality is Inequality (4) with $\boldsymbol{F}$ replaced by $\boldsymbol{F}(t)$. Thus ω is convex on $[0,1]$ and hence $\omega(1) \geq \omega(0) + \dot{\omega}(1)$. The last inequality gives (2). □

The rest of this note discusses the CNI for isotropic materials, i.e., for materials that satisfy

$$\bar{\psi}(\boldsymbol{Q}\boldsymbol{F}\boldsymbol{R}) = \bar{\psi}(\boldsymbol{F}) \tag{5}$$

* Equation (4) is close to [8; Theorem 5], which, though, is somewhat less explicit.

for every $\boldsymbol{F} \in \mathrm{Lin}^+$ and every two proper orthogonal tensors $\boldsymbol{Q}$ and $\boldsymbol{R}$. Equation (5) is equivalent to the existence a symmetric function $\Psi : (0,\infty)^3 \to \mathbb{R}$ such that

$$\bar{\psi}(\boldsymbol{F}) = \Psi(v_1, v_2, v_3) \tag{6}$$

for every $\boldsymbol{F} \in \mathrm{Lin}^+$, where v_1, v_2, v_3 are the principal stretches of $\boldsymbol{F}$, i.e., the eigenvalues of $\sqrt{\boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}}$. The symmetry of Ψ means that $\Psi(v_1, v_2, v_3)$ is invariant under any permutation of $(v_1, v_2, v_3) \in (0,\infty)^3$. Clearly, if the function Ψ exists, it is given by

$$\Psi(v_1, v_2, v_3) = \bar{\psi}(\mathrm{diag}(v_1, v_2, v_3))$$

for any $(v_1, v_2, v_3) \in (0,\infty)^3$.

To avoid repeated hypotheses, *until the end of this note it is assumed that $\bar{\psi}$ and Ψ are the functions associated with an isotropic material as just described.* We now apply the following assertion:

Proposition 2★ *We have the following assertions*:
(i) *the function Ψ is twice continuously differentiable as a consequence of the same property of $\bar{\psi}$*;
(ii) *if $\boldsymbol{F} = \mathrm{diag}(v_1, v_2, v_3) \in \mathrm{Lin}^+$ then*

$$\mathrm{D}\bar{\psi}(\boldsymbol{F}) = \mathrm{diag}(\Psi_1, \Psi_2, \Psi_3),$$

where the subscripts indicate the partial derivatives of Ψ at (v_1, v_2, v_3);
(iii) *if $\boldsymbol{F} = \mathrm{diag}(v_1, v_2, v_3) \in \mathrm{Lin}^+$ and the numbers v_1, v_2, v_3 are distinct, then*

$$\mathrm{D}^2\bar{\psi}(\boldsymbol{F})[\boldsymbol{B},\boldsymbol{B}] = \sum_{i,j=1}^{3} \Psi_{ij} B_{ii} B_{jj} + \sum_{1 \le i \ne j \le 3} \left(M_{ij}^{+} B_{ij}^2 + M_{ij}^{-} B_{ij} B_{ji} \right) \tag{7}$$

for any second-order tensor $\boldsymbol{B}$, where

$$M_{ij}^{+} = \frac{v_i \Psi_i - v_j \Psi_j}{v_i^2 - v_j^2}, \quad M_{ij}^{-} = \frac{v_j \Psi_i - v_i \Psi_j}{v_i^2 - v_j^2}.$$

The following is the main result of this note.

Theorem 1 *If $\boldsymbol{F} = \mathrm{diag}(v_1, v_2, v_3)$ has distinct diagonal elements, Inequality* (4) *is equivalent to the simultaneous validity of the following two assertions:*
(i)★★ *the matrix $[\Psi_{ij}]_{i,j=1}^3$ is positive-semidefinite at (v_1, v_2, v_3);*
(ii) *for each pair of indices i, j, $1 \le i \ne j \le 3$, we have*

$$\frac{(3v_j^2 + v_i^2) v_i \Psi_i - (3v_i^2 + v_j^2) v_j \Psi_j}{v_i^2 - v_j^2} \ge 0. \tag{8}$$

Proof Assume that (4) holds and let $\boldsymbol{H}$ be a symmetric tensor. Observing that for $\boldsymbol{F} = \mathrm{diag}(v_1, v_2, v_3)$ we have $(\boldsymbol{H}\boldsymbol{F})_{ij} = H_{ij} v_j$, we see from (7) that Condition (4) reads

$$\mathrm{D}^2\bar{\psi}(\boldsymbol{F})[\boldsymbol{H}\boldsymbol{F},\boldsymbol{H}\boldsymbol{F}] = \sum_{i,j=1}^{3} \Psi_{ij} v_i v_j H_{ii} H_{jj} + \sum_{1 \le i \ne j \le 3} C_{ij} H_{ij}^2 \ge 0$$

★ [3–6, 2, 13–14].
★★ [8; §12].

for every choice of H_{ij} with $H_{ij} = H_{ji}$, where

$$C_{ij} = M_{ij}^{+} v_j^2 + M_{ij}^{-} v_i v_j = \frac{2v_j^2 v_i \Psi_i - v_j^3 \Psi_j - v_i^2 v_j \Psi_j}{v_i^2 - v_j^2}.$$

The choice $\boldsymbol{H} = \operatorname{diag}(\lambda_1/v_1, \lambda_2/v_2, \lambda_3/v_3)$ where $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ gives

$$\sum_{i,j=1}^{3} \Psi_{ij} \lambda_i \lambda_j \geq 0,$$

i.e., (i). On the other hand, taking any pair of indices i, j, $1 \leq i \neq j \leq 3$, and H such that $H_{ij} = H_{ji} = 1$ while the remaining components of H vanish, we obtain $C_{ij} + C_{ji} \geq 0$, i.e., (ii).

Thus (4) ⇒ (i) and (ii). The converse is immediate. □

We summarize.

Theorem 2★ *An isotropic material satisfies the* CNI *if and only if the following three assertions hold simultaneously*:

(i) *we have*

$$\bar{\boldsymbol{S}}(\boldsymbol{F}) = \operatorname{diag}(\Psi_1, \Psi_2, \Psi_3) \tag{9}$$

for every $\boldsymbol{F} = \operatorname{diag}(v_1, v_2, v_3) \in \mathrm{Lin}^+$;

(ii) *the function* Ψ *is convex*;

(iii) *Condition* (ii) *of Theorem* 1 *holds for all* $(v_1, v_2, v_3) \in (0, \infty)$ *with distinct components.*

The convexity of Ψ *implies the ordered-forces inequality*

$$(v_i - v_j)(\Psi_i - \Psi_j) \geq 0 \tag{10}$$

for every (v_1, v_2, v_3) *and for every pair of indices* i, j, $1 \leq i \neq j \leq 3$.

Proof The direct implication is immediate.

Conversely, (9) gives (3). Furthermore, if Ψ is convex and Condition (ii) of Theorem 1 holds, one obtains (4) for all diagonal tensors $\boldsymbol{F} \in \mathrm{Lin}^+$ with distinct principal stretches. Hence for all diagonal $\boldsymbol{F} \in \mathrm{Lin}^+$ by continuity and density. Since the material is isotropic, the validity of (4) for all diagonal tensors implies the validity of (4) for all tensors.

The consequence (10) is well-known. □

If $\boldsymbol{F} = \operatorname{diag}(v_1, v_2, v_3) \in \mathrm{Lin}^+$ is diagonal, then $\bar{\boldsymbol{S}}(\boldsymbol{F})$ is diagonal as well; its diagonal elements s_i are called the principal forces. One has

$$s_i = \Psi_i.$$

On the other hand, $\bar{\boldsymbol{T}}(\boldsymbol{F})$ is symmetric for all $\boldsymbol{F} \in \mathrm{Lin}^+$, and its eigenvalues t_i are called the principal stresses. By (1) we have

$$t_i = v_i \Psi_i / v_1 v_2 v_3.$$

In terms of the principal stresses, Condition (8) reads

$$\frac{(3v_j^2 + v_i^2) t_i - (3v_i^2 + v_j^2) t_j}{v_i^2 - v_j^2} \geq 0. \tag{11}$$

★ For Items (i) and (ii) and for Inequality (10) see [8; §12].

3 Particular case: fluids

In this section we illustrate the consequences of the CNI on fluids, a subclass of isotropic materials, since they are particularly simple. We start with a consequence of CNI for all isotropic materials.

Consequence 1 (Nonnegativity of pressure) *If an isotropic material satisfies the* CNI *and* **if** *the stress at a given* $\boldsymbol{F} \in \mathrm{Lin}^+$ *reduces to the hydrostatic pressure, i.e.,*

$$\bar{\boldsymbol{T}}(\boldsymbol{F}) = -p\mathbf{1}$$

for some $p \in \mathbb{R}$, *then* $p \geq 0$.

Proof It suffices to consider $\boldsymbol{F} = \mathrm{diag}(v_1, v_2, v_3)$. If at least two principal stretches are distinct, say v_i, v_j, then the insertion $t_i = t_j = -p$ into (11) provides $p \geq 0$. If $v_1 = v_2 = v_3$ one perform a limit of (11) for $v_i \to v_j$ using the l'Hopital rule. □

By definition, fluids satisfy the requirement

$$\bar{\psi}(\boldsymbol{QFH}) = \bar{\psi}(\boldsymbol{F})$$

for every $\boldsymbol{F} \in \mathrm{Lin}^+$, every proper orthogonal tensors $\boldsymbol{Q}$, and every second-order tensor $\boldsymbol{H}$ with $\det \boldsymbol{H} = 1$. Hence, in particular, (5) holds, i.e., fluids are isotropic materials. A material is a fluid if and only if there exists a function $\Phi : (0, \infty) \to \mathbb{R}$ such that

$$\bar{\psi}(\boldsymbol{F}) = \Phi(v)$$

for every $\boldsymbol{F} \in \mathrm{Lin}^+$ where

$$v = \det \boldsymbol{F} = v_1 v_2 v_3$$

is the specific volume. Thus we have (6) with

$$\Psi(v_1, v_2, v_3) = \Phi(v_1 v_2 v_3).$$

Consequence 2★ *A fluid satisfies the* CNI *if and only if the following two conditions hold simultaneously*:

(i) *the stress reduces to the hydrostatic pressure, i.e., we have*

$$\bar{\boldsymbol{T}}(\boldsymbol{F}) = -p(v)\mathbf{1}$$

for every $\boldsymbol{F} \in \mathrm{Lin}^+$, *where* $p = -\Phi'$ *and* $p \geq 0$ *on* $(0, \infty)$;

(ii) *we have*

$$p(v) + \frac{3}{2} v p'(v) \leq 0 \tag{12}$$

on $(0, \infty)$; *equivalently,* $\Phi(v)$ *is a convex function of* $\sqrt[3]{v}$ *on* $(0, \infty)$.

4 Coleman-Noll inequality and the Legendre-Hadamard condition

In this section we present some remarks on the relationship between the CNI and the Legendre-Hadamard condition.

★ [8; §12], [7; §4].

Legendre-Hadamard condition A material with the stored energy $\bar{\psi}$ is said to satisfy the Legendre-Hadamard condition (briefly, the LHC) if

$$\mathrm{D}^2\bar{\psi}(\boldsymbol{F})[\boldsymbol{a}\otimes\boldsymbol{b},\boldsymbol{a}\otimes\boldsymbol{b}] \ge 0 \tag{13}$$

for every $\boldsymbol{F} \in \mathrm{Lin}^+$ and every $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$. If the inequality is strict when $\boldsymbol{a} \neq \boldsymbol{0} \neq \boldsymbol{b}$, we say that the material satisfies the strong ellipticity condition.

Remark 1 It is well-known that a fluid satisfies the LHC if and only if the function Φ is convex on $(0,\infty)$, i.e., if $\Phi'' \ge 0$ or equivalently $p' \le 0$.⋆ We thus see from the inequalities $p \ge 0$ and (12) that *for fluids the* CNI *implies the* LHC*, but not conversely.*

Actually, none of the CNI and LHC implies the other, for materials of any symmetry and for isotropic materials. The explicit form of the LHC for isotropic materials in 3D is complicated [12; Proposition 6.4]. However, this condition includes the simple Baker-Ericksen inequalities (14) (below), to which we mainly restrict below. We also illustrate the CNI and LHC on a model stored energy of a hypothetical Hadamard material.

Consequence 3 *Consider an isotropic material that satisfies the* CNI*, let* $\boldsymbol{F} \in \mathrm{Lin}^+$ *be a deformation gradient with the principal stretches* v_1, v_2, v_3*, the principal stresses* t_1, t_2, t_3*, and let* i, j *be a pair of indices,* $1 \le i \neq j \le 3$.

(i) *If*

$$t_i \ge 0, \quad t_j \ge 0,$$

then we have

$$(v_i - v_j)(t_i - t_j) \ge 0; \tag{14}$$

(ii) *if* (14) *is violated, i.e., if*

$$(v_i - v_j)(t_i - t_j) < 0 \tag{15}$$

then

$$t_i < 0, \quad t_j < 0. \tag{16}$$

Proof Without any loss of generality we can assume that $v_i > v_j$. To prove Assertion (i), note that Inequality (11) reduces to

$$(3v_j^2 + v_i^2)t_i - (3v_i^2 + v_j^2)t_j \ge 0. \tag{17}$$

If $t_j \ge 0$, then $(3v_i^2 + v_j^2)t_j \ge (3v_j^2 + v_i^2)t_j$ and hence (17) gives

$$t_i - t_j \ge 0.$$

Inequality (14) follows. To prove Assertion (ii), note that for $v_i > v_j$, Inequality (15) reduces to

$$t_i - t_j < 0. \tag{18}$$

Then $(3v_j^2 + v_i^2)t_i < (3v_j^2 + v_i^2)t_j$ and one finds that (11) gives $(2v_j^2 - 2v_i^2)t_j > 0$, i.e., $t_j < 0$, and $t_i < 0$ follows from (18). Thus we have (16). □

⋆ See, e.g., [11].

Remark 2 *Let M be a 3 by 3 matrix of the form*

$$M = \operatorname{diag}(a) + bc \otimes c \tag{19}$$

where $a = (a_1, a_2, a_3) \in \mathbb{R}^3$, $b \in \mathbb{R}$, and $c = (1, 1, 1)$, i.e., let

$$M = \begin{bmatrix} a_1 + b & b & b \\ b & a_2 + b & b \\ b & b & a_3 + b \end{bmatrix}.$$

Then M is positive semidefinite if and only if the following set of inequalities holds:

$$\left.\begin{array}{ll} a_i + b \geq 0, & i = 1, 2, 3, \\ a_i a_j + (a_i + a_j) b \geq 0, & 1 \leq i \neq j \leq 3, \\ a_1 a_2 a_3 + b(a_1 a_2 + a_2 a_3 + a_1 a_3) \geq 0. & \end{array}\right\} \tag{20}$$

This is Sylvester's criterion: the principal subdeterminants of orders 1, 2, and 3 must be positive.

Example *The stored energy of Hadamard's material is given by*

$$\bar{\psi}(\boldsymbol{F}) = \frac{1}{2} |\boldsymbol{F}|^2 + \Phi(\det \boldsymbol{F})$$

for every $\boldsymbol{F} \in \mathrm{Lin}^+$, where $\Phi : (0, \infty) \to \mathbb{R}$ is a twice continuously differentiable function. Then

(i) *the material satisfies the* LHC *if and only if*

$$\Phi'' \geq 0 \tag{21}$$

on $(0, \infty)$;

(ii) *the material satisfies the* CNI *if and only if*

$$\Phi' \leq 0, \quad v^2 \Phi' + 2v\Phi'' \geq 0, \tag{22}$$

and

$$v^2 + v^3 \Phi'^2 (2\Phi' + 3v\Phi'') - v^2 \Phi'(\Phi' + 2v\Phi'')\mathbf{t} + v^2 \Phi'' \mathbf{c} \geq 0 \tag{23}$$

for every $(v_1, v_2, v_3) \in (0, \infty)$, where $v^2 = v_1^2 v_2^2 v_3^2$ and

$$\mathbf{t} = v_1^2 + v_2^2 + v_3^2, \quad \mathbf{c} = v_1^2 v_2^2 + v_2^2 v_3^2 + v_1^2 v_3^2.$$

Inequality (23) is hard to verify. However, the nonnegativity of $\mathbf{t}$ and $\mathbf{c}$ shows that a sufficient condition for (23) to hold is to strengthen Inequalities (22) to

$$\Phi' \leq 0, \quad v^2 \Phi' + \frac{3}{2} v\Phi'' \geq 0.$$

Also, it is noted that Inequalities (22) imply (21). Thus, for this class of materials, the CNI implies the LHC, but not conversely.

Proof (i): The derivatives $\mathrm{D} \det \boldsymbol{F}$ and $\mathrm{D}^2 \det \boldsymbol{F}$ of the function $\boldsymbol{F} \mapsto \det \boldsymbol{F}$ with respect to $\boldsymbol{F}$ at $\boldsymbol{F}$ satisfy

$$\mathrm{D} \det \boldsymbol{F} = \operatorname{cof} \boldsymbol{F} = (\det \boldsymbol{F}) \boldsymbol{F}^{-\mathrm{T}}, \quad \mathrm{D}^2 \det \boldsymbol{F}[\boldsymbol{a} \otimes \boldsymbol{b}, \boldsymbol{a} \otimes \boldsymbol{b}] = 0 \tag{24}$$

for every $\boldsymbol{F} \in \mathrm{Lin}^+$ and every $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$. The formula $(24)_1$ is standard while the equality $(24)_2$ follows from the fact that $\det \boldsymbol{F}$ is a null lagrangian [1]. From (24) one deduces that

$$\mathrm{D}\bar{\psi}(\boldsymbol{F}) = \boldsymbol{F} + v\Phi'(v)\boldsymbol{F}^{-\mathrm{T}},$$
$$\mathrm{D}^2\bar{\psi}(\boldsymbol{F})[\boldsymbol{a}\otimes\boldsymbol{b}, \boldsymbol{a}\otimes\boldsymbol{b}] = |\boldsymbol{a}|^2|\boldsymbol{b}|^2 + v^2\Phi''(v)(\boldsymbol{F}^{-\mathrm{T}}\boldsymbol{b}\cdot\boldsymbol{a})^2$$

where $v = \det \boldsymbol{F} = v_1v_2v_3$. To prove that the LHC implies (21), we fix any $v > 0$ and take $\boldsymbol{a} = \boldsymbol{b} = (1,0,0)$ and $\boldsymbol{F} = \mathrm{diag}(\alpha, v\alpha^{-1}, 1)$ with $\alpha > 0$. For these values Inequality (13) reads $1 + v^2\Phi''(v)/\alpha^2 \ge 0$, i.e., $\alpha^2 + v^2\Phi''(v) \ge 0$. The limit $\alpha \to 0$ gives (21). Conversely, (21) implies (13) by (24).

(ii): One has $\Psi(v_1, v_2, v_3) = (1/2)(v_1^2 + v_2^2 + v_3^2) + \Phi(v_1v_2v_3)$ and hence

$$\Psi_i = v_i + v\Phi'(v)/v_i, \quad \Psi_{ij} = \delta_{ij} + \big((1-\delta_{ij})v\Phi' + v^2\Phi''\big)/v_iv_j. \tag{25}$$

Assume that the material satisfies the CNI. A calculation using these formulas gives that

$$\frac{(3v_j^2 + v_i^2)v_i\Psi_i - (3v_i^2 + v_j^2)v_j\Psi_j}{v_i^2 - v_j^2} = v_i^2 + v_j^2 - 2v\Phi'(v).$$

Thus (8) reads

$$v_i^2 + v_j^2 - 2v\Phi'(v) \ge 0.$$

Taking $i = 1, j = 2$, fixing $v > 0$, and letting $v_1 \to 0$, $v_2 \to 0$ and $v_3 \to \infty$ in such a way that $v_1v_2v_3 = v = \text{const}$, we obtain $(22)_1$. The second consequence of the CNI is the convexity of Ψ. Thus the matrix $A = [\Psi_{ij}]$ with Ψ_{ij} given by $(25)_2$ is positive semidefinite. This equivalent to the requirement that the matrix $M = [m_{ij}]$ with $m_{ij} = v_iv_j\delta_{ij} + (1-\delta_{ij})v\Phi' + v^2\Phi''$ is positive semidefinite. The matrix M takes the form (19) where

$$a_i = v_i^2 - v\Phi', \quad b = v(\Phi' + v\Phi'').$$

By Remark 2, the nonegative semidefiniteness of M is equivalent to the system of inequalities (20), which in the present case read:

$$\left.\begin{array}{ll} v_i^2 + v^2\Phi'' \ge 0, & i = 1,2,3, \\ v_i^2v_j^2 - v^2\Phi'^2 + (v_i^2 + v_j^2 - 2v\Phi')v^2\Phi'' \ge 0, & 1 \le i \ne j \le 3, \\ v^2 + v^3\Phi'^2(2\Phi' + 3v\Phi'') - v^2\Phi'(\Phi' + 2v\Phi'')\mathbf{t} + v^2\Phi''\mathbf{c}. & \end{array}\right\} \tag{26}$$

Let us now take $(26)_2$ with $i = 1, j = 2$ and fix $v > 0$. Letting $v_1 \to 0$, $v_2 \to 0$ and $v_3 \to \infty$ in such a way that $v_1v_2v_3 = v = \text{const}$, we obtain $(22)_2$. Thus the CNI implies the inequalities in (ii). The converse is immediate. □

References

1 Ball, J. M.: *Convexity conditions and existence theorems in nonlinear elasticity* Arch. Rational Mech. Anal. **63** (1977) 337–403

2 Ball, J. M.: *Differentiability properties of symmetric and isotropic functions* Duke Math. J. **51** (1984) 699–728

3 Bowen, R. M.; Wang, C.-C.: *Acceleration waves in inhomogeneous isotropic elastic bodies* Arch. Rational Mech. Anal. **38** (1970) 13–45

4 Bowen, R. M.; Wang, C.-C.: *Acceleration waves in inhomogeneous isotropic elastic bodies. Corrigendum* Arch. Rational Mech. Anal. **40** (1971) 403

5 Chadwick, P.; Ogden, R. W.: *On the definition of elastic moduli* Arch. Rational Mech. Anal. **44** (1971) 41–53

6 Chadwick, P.; Ogden, R. W.: *A theorem of tensor calculus and its application to isotropic elasticity* Arch. Rational Mech. Anal. **44** (1971) 54–68

7 Coleman, B. D.: *Mechanical and Thermodynamical Admissibility of Stress–Strain Functions* Arch. Rational Mech. Anal. **9** (1962) 172–186

8 Coleman, B. D.; Noll, W.: *On the thermostatics of continuous media* Arch. Rational Mech. Anal. **4** (1959) 97–128

9 Podio-Guidugli, P.; Virga, E. G.: *Scientific Life and Works of Walter Noll* J. Elasticity (2019) (Online: https://doi.org/10.1007/s10659-019-09728-w)

10 Sidoroff, F.: *Sur les restrictions a imposer a l'énergie de déformation d'un matériau hyperélastique* C. R. Acad. Sc. Paris **279** (1974) 379–382

11 Šilhavý, M.: *The mechanics and thermodynamics of continuous media* Springer, Berlin 1997

12 Šilhavý, M.: *On isotropic rank 1 convex functions* Proc. Royal Soc. Edinburgh **129A** (1999) 1081–1105

13 Šilhavý, M.: *Differentiability properties of isotropic functions* Duke Math. J. **104** (2000) 367–373

14 Šilhavý, M.: *Differentiability properties of objective isotropic functions* J. Elasticity **58** (2000) 225–232

15 Šilhavý, M.: *The Scientific Work of Bernard D. Coleman* Mathematics and Mechanics of Solids **24** (2019) 736–747 `https://doi.org/10.1177/1081286519825544`

16 Truesdell, C.: *Das ungelöste Hauptproblem der endlichen Elastizitätstheorie* Z. angew. Math. Mech. **36** (1956) 97–103