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Abstract

We consider the Navier–Stokes–Fourier system with the inhomogeneous boundary conditions for the velocity and the temperature. We show that solutions emanating from sufficiently regular data remain regular as long as the density ϱ, the absolute temperature ϑ, and the modulus of the fluid velocity $|\mathbf{u}|$ remain bounded.

Keywords: Navier–Stokes–Fourier system, conditional regularity, blow–up criterion, regular solution

1 Introduction

Standard systems of equations in fluid mechanics including the Navier–Stokes–Fourier system governing the motion of a compressible, viscous, and heat conducting fluid are well posed in the class of strong solutions on a possibly short time interval $[0, T_{\max})$. The recent results of Merle at al. [16], [17] strongly indicate that $T_{\max}$ may be finite, at least in the idealized case of "isentropic" viscous flow. Conditional regularity results guarantee that a blow up will not occur as soon as some lower order norms of solutions are controlled.

We consider the *Navier–Stokes–Fourier system* governing the time evolution of the mass density $\varrho = \varrho(t, x)$, the (absolute) temperature $\vartheta = \vartheta(t, x)$, and the velocity $\mathbf{u} = \mathbf{u}(t, x)$ of a compressible, viscous, and heat conducting fluid:

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$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{u}) + \varrho \mathbf{f},\ \mathbb{D}_x \mathbf{u} = \frac{1}{2}\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}\right), \tag{1.2}$$

$$\partial_t(\varrho e(\varrho, \vartheta)) + \mathrm{div}_x(\varrho e(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \mathbf{q}(\nabla_x \vartheta) = \mathbb{S}(\mathbb{D}_x \mathbf{u}) : \mathbb{D}_x \mathbf{u} - p(\varrho, \vartheta)\mathrm{div}_x \mathbf{u}. \tag{1.3}$$

The fluid is Newtonian, the viscous stress $\mathbb{S}$ is given by Newton's rheological law

$$\mathbb{S}(\mathbb{D}_x \mathbf{u}) = 2\mu \left(\mathbb{D}_x \mathbf{u} - \frac{1}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I},\ \mu > 0,\ \eta \geq 0. \tag{1.4}$$

The heat flux obeys Fourier's law

$$\mathbf{q}(\nabla_x \vartheta) = -\kappa \nabla_x \vartheta,\ \kappa > 0. \tag{1.5}$$

The equation of state for the pressure p and the internal energy e is given by the standard Boyle–Mariotte law of perfect gas,

$$p(\varrho, \vartheta) = \varrho \vartheta,\ e(\varrho, \vartheta) = c_v \vartheta,\ c_v > 0. \tag{1.6}$$

For the sake of simplicity, we suppose that the viscosity coefficients μ, η, the heat conductivity coefficient κ as well as the specific heat at constant volume c_v are constant.

There is a large number of recent results concerning conditional regularity for the Navier–Stokes–Fourier system in terms of various norms. Fan, Jiang, and Ou [4] consider a bounded fluid domain $\Omega \subset R^3$ with the conservative boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0,\ \nabla_x \vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{1.7}$$

The same problem is studied by Sun, Wang, and Zhang [19] and later by Huang, Li, Wang [14]. There are results for the Cauchy problem $\Omega = R^3$ by Huang and Li [13], and Jiu, Wang and Ye [15]. Possibly the best result so far has been established in [11], where the blow up criterion for both the Cauchy problem and the boundary value problem (1.7) is formulated in terms of the maximum of the density and a Serrin type regularity for the temperature:

$$\limsup_{t \to T_{\max}-} \left(\|\varrho(t, \cdot)\|_{L^\infty} + \|\vartheta - \vartheta_\infty\|_{L^s(0,t)(L^r)} \right) = \infty,\ \frac{3}{2} < r \leq \infty,\ 1 \leq s \leq \infty,\ \frac{2}{s} + \frac{3}{r} \leq 2,$$

where ϑ_∞ denotes the far field temperature in the Cauchy problem, cf. also the previous results by Wen and Zhu [23], [24].

Much less is known in the case of the Dirichlet boundary conditions

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_B,\ \vartheta|_{\partial\Omega} = \vartheta_B. \tag{1.8}$$

Fan, Zhi, and Zhang [5] showed that a strong solution of the Navier–Stokes–Fourier system remains regular up to a time $T > 0$ if (i) $\Omega \subset R^2$ is a bounded domain, (ii) $\mathbf{u}_B = 0$, $\vartheta_B = 0$, and (iii)

$$\limsup_{t \to T-} \left(\|\varrho\|_{L^\infty} + \|\vartheta\|_{L^\infty} \right) < \infty. \tag{1.9}$$

All results mentioned above describe fluids in a conservative regime, meaning solutions are close to equilibrium in the long run. However, many real world applications concern fluids out of equilibrium driven by possibly large driving forces $\mathbf{f}$ and/or inhomogeneous boundary conditions. The iconic examples are the Rayleigh–Bénard and Taylor–Couette flows where the fluid is driven to a turbulent regime by a large temperature gradient and large boundary velocity, respectively, see Davidson [3].

Motivated by these physically relevant examples, we consider a fluid confined to a bounded domain $\Omega \subset R^3$ with *impermeable boundary*, where the temperature and the (tangential) velocity are given on $\partial\Omega$,

$$\vartheta|_{\partial\Omega} = \vartheta_B,\ \vartheta_B = \vartheta_B(x),\ \vartheta_B > 0 \text{ on } \partial\Omega, \tag{1.10}$$

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_B,\ \mathbf{u}_B = \mathbf{u}_B(x),\ \mathbf{u}_B \cdot \mathbf{n} = 0 \text{ on } \partial\Omega. \tag{1.11}$$

The initial state of the fluid is prescribed:

$$\varrho(0, \cdot) = \varrho_0,\ \varrho_0 > 0 \text{ in } \overline{\Omega},\ \vartheta(0, \cdot) = \vartheta_0,\ \vartheta_0 > 0 \text{ in } \overline{\Omega},\ \mathbf{u}(0, \cdot) = \mathbf{u}_0. \tag{1.12}$$

The initial and boundary data are supposed to satisfy suitable *compatibility conditions* specified below.

The existence of local in time strong solutions for the problem (1.1)–(1.6), endowed with the inhomogeneous boundary conditions (1.10), (1.11) was established by Valli [20], [21] , see also Valli and Zajaczkowski [22]. The solution exists on a maximal time interval $[0, T_{\max})$, $T_{\max} > 0$. Our goal is to show that if $T_{\max} < \infty$, then necessarily

$$\limsup_{t \to T_{\max}-} \Big(\|\varrho(t, \cdot)\|_{L^\infty(\Omega)} + \|\vartheta(t, \cdot)\|_{L^\infty(\Omega)} + \|\mathbf{u}(t, \cdot)\|_{L^\infty(\Omega; R^3)} \Big) = \infty. \tag{1.13}$$

The proof is based on deriving suitable *a priori* bounds assuming boundedness of all norms involved in (1.13) as well as the norm of the initial/boundary data in a suitable function space. Although approach shares some similarity with Fang, Zi, and Zhang [5], essential modifications must be made to accommodate the inhomogeneous boundary data as well as the driving force $\mathbf{f}$. The importance of conditional regularity results in numerical analysis of flows with uncertain initial data was discussed recently in [7].

The paper is organized as follows. In Section 2, we introduce the class of strong solutions to the Navier–Stokes–Fourier system and state our main result concerning conditional regularity. The remaining part of the paper is devoted to the proof of the main result – deriving suitable *a priori* bounds. In Section 3 we recall the standard energy estimates that hold even in the class of weak solutions. Section 4 is the heart of the paper. We establish the necessary estimates on the velocity gradient by means of the celebrated Gagliardo–Nirenberg interpolation inequality. In Section 5, higher order estimates on the velocity gradient are derived, and, finally, the estimates are closed by proving bounds on the temperature time derivative in Section 6. This last part borrows the main ideas from [9].

2 Strong solutions, main result

We start the analysis by recalling the concept of strong solution introduced by Valli [21]. Similarly to the boundary data $\mathbf{u}_B$, ϑ_B we suppose that the driving force $\mathbf{f} = \mathbf{f}(x)$ is independent of time, meaning we deal with an autonomous problem. Following [21], we suppose that $\Omega \subset R^3$ is a bounded domain with $\partial\Omega$ of class C^4.

We assume the data belong to the following class:

$$
\begin{aligned}
\varrho_0 &\in W^{3,2}(\Omega),\ 0 < \underline{\varrho}_0 \leq \min_{x\in\Omega} \varrho_0(x),\\
\vartheta_0 &\in W^{3,2}(\Omega),\ 0 < \underline{\vartheta}_0 \leq \min_{x\in\Omega} \vartheta_0(x),\\
\mathbf{u}_0 &\in W^{3,2}(\Omega; R^3),\\
\vartheta_B &\in W^{\frac{7}{2}}(\partial\Omega),\ 0 < \underline{\vartheta}_B \leq \min_{x\in\partial\Omega} \vartheta_B(x),\\
\mathbf{u}_B &\in W^{\frac{7}{2}}(\partial\Omega; R^3),\ \mathbf{u}_B \cdot \mathbf{n} = 0,\\
\mathbf{f} &\in W^{2,2}(\Omega; R^3).
\end{aligned}
\tag{2.1}
$$

In addition, the data must satisfy the compatibility conditions

$$
\begin{aligned}
\vartheta_0 = \vartheta_B,\ \mathbf{u}_0 &= \mathbf{u}_B \text{ on } \partial\Omega,\\
\varrho_0\mathbf{u}_0 \cdot \nabla_x \mathbf{u}_0 + \nabla_x p(\varrho_0, \vartheta_0) &= \mathrm{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{u}_0) + \varrho_0 \mathbf{f} \text{ on } \partial\Omega,\\
\varrho_0\mathbf{u}_0 \cdot \nabla_x \vartheta_0 + \mathrm{div}_x \mathbf{q}(\vartheta_0) &= \mathbb{S}(\mathbb{D}_x \mathbf{u}_0) : \mathbb{D}_x \mathbf{u}_0 - p(\varrho_0, \vartheta_0) \mathrm{div}_x \mathbf{u}_0 \text{ on } \partial\Omega.
\end{aligned}
\tag{2.2}
$$

We set

$$
\mathcal{D}_0 = \max \left\{ \|(\varrho_0, \vartheta_0, \mathbf{u}_0)\|_{W^{3,2}(\Omega;R^5)}, \frac{1}{\underline{\varrho}_0},\ \frac{1}{\underline{\vartheta}_0}, \frac{1}{\underline{\vartheta}_B}, \|\vartheta_B\|_{W^{\frac{7}{2}}(\partial\Omega)}, \|\mathbf{u}_B\|_{W^{\frac{7}{2}}(\partial\Omega;R^3)}, \|\mathbf{f}\|_{W^{2,2}(\Omega;R^3)} \right\}.
\tag{2.3}
$$

2.1 Local existence

The following result was proved by Valli [21, Theorem A] (see also [20]).

Theorem 2.1. (Local existence of strong solutions) *Let* $\Omega \subset R^3$ *be a bounded domain of class* C^4*. Suppose that the data* $(\varrho_0, \vartheta_0, \mathbf{u}_0)$, $(\vartheta_B, \mathbf{u}_B)$ *and* $\mathbf{f}$ *belong to the class* (2.1) *and satisfy the compatibility conditions* (2.2)*.*

Then there exists a maximal time $T_{\max} > 0$ *such that the Navier–Stokes–Fourier system* (1.1)–(1.6)*, with the boundary conditions* (1.10), (1.11)*, and the initial conditions* (1.12) *admits a solution* $(\varrho, \vartheta, \mathbf{u})$ *in* $[0, T_{\max}) \times \Omega$ *unique in the class*

$$\begin{aligned} \varrho,\ \vartheta \in C([0,T]; W^{3,2}(\Omega)),\ \mathbf{u} \in C([0,T]; W^{3,2}(\Omega; R^3)), \\ \vartheta \in L^2(0,T; W^{4,2}(\Omega)),\ \mathbf{u} \in L^2(0,T; W^{4,2}(\Omega; R^3)) \end{aligned} \tag{2.4}$$

for any $0 < T < T_{\max}$*. The existence time* $T_{\max}$ *is bounded below by a quantity* $c(\mathcal{D}_0)$ *depending solely on the norms of the data specified in* (2.3)*. In particular,*

$$\lim_{\tau \to T_{\max}-} \|(\varrho, \vartheta, \mathbf{u})(\tau, \cdot)\|_{W^{3,2}(\Omega; R^5)} = \infty. \tag{2.5}$$

2.2 Blow up criterion, conditional regularity

Our goal is to show the following result.

Theorem 2.2. (Blow up criterion) *Under the hypotheses of Theorem 2.1, suppose that the maximal existence time* $T_{\max} < \infty$ *is finite.*

Then

$$\limsup_{\tau \to T_{\max}-} \|(\varrho, \vartheta, \mathbf{u})(\tau, \cdot)\|_{L^\infty(\Omega; R^5)} = \infty. \tag{2.6}$$

Theorem 2.2 is in the spirit of the blow up criteria for general parabolic systems – the solution remains regular as long as it is bounded. Of course, our problem in question is of mixed hyperbolic–parabolic type.

The proof of Theorem 2.2 follows from suitable *a priori* bounds applied on a compact time interval.

Proposition 2.3. (Conditional regularity)
Under the hypotheses of Theorem 2.1, let $(\varrho, \vartheta, \mathbf{u})$ *be the strong solution of the Navier–Stokes–Fourier system belonging to the class* (2.4) *and satisfying*

$$\sup_{(\tau,x)\in[0,T)\times\Omega} \varrho(\tau,x) \le \overline{\varrho}, \quad \sup_{(\tau,x)\in[0,T)\times\Omega} \vartheta(\tau,x) \le \overline{\vartheta}, \quad \sup_{(\tau,x)\in[0,T)\times\Omega} |\mathbf{u}(\tau,x)| \le \overline{u} \tag{2.7}$$

for some $T < T_{\max}$.

Then there is a quantity $c(T, \mathcal{D}_0, \overline{\varrho}, \overline{\vartheta}, \overline{u})$, bounded for bounded arguments, such that

$$\sup_{\tau\in[0,T)} \max \left\{ \|(\varrho, \vartheta, \mathbf{u})(\tau, \cdot)\|_{W^{3,2}(\Omega;R^5)}; \sup_{x\in\Omega} \frac{1}{\varrho(\tau, x)}; \sup_{x\in\Omega} \frac{1}{\vartheta(\tau, x)} \right\} \leq c(T, \mathcal{D}_0, \overline{\varrho}, \overline{\vartheta}, \overline{u}). \tag{2.8}$$

In view of Theorem 2.1, the conclusion of Theorem 2.2 follows from Proposition 2.3. The rest of the paper is therefore devoted to the proof of Proposition 2.3.

Remark 2.4. As observed in [8], the conditional regularity results established in Proposition 2.3 gives rise to *stability* with respect to the data. More specifically, the maximal existence time $T_{\max}$ is a lower semicontinuous function of the data with respect to the topologies in (2.1).

Remark 2.5. Conditional regularity results in combination with the weak–strong uniqueness principle in the class of measure–valued solutions is an efficient tool for proving convergence of numerical schemes, see [6, Chapter 11]. The concept of measure–valued solutions to the Navier–Stokes–Fourier system with inhomogeneous Dirichlet boundary conditions has been introduced recently by Chaudhuri [1].

3 Energy estimates

To begin, it is suitable to extend the boundary data into Ω. For definiteness, we consider the (unique) solutions of the Dirichlet problem

$$\begin{aligned} \Delta_x \tilde{\vartheta} = 0 \text{ in } \Omega,\ \tilde{\vartheta}|_{\partial\Omega} = \vartheta_B, \\ \mathrm{div}_x \mathbb{S}(\mathbb{D}_x \tilde{\mathbf{u}}) = 0 \text{ in } \Omega,\ \tilde{\mathbf{u}}|_{\partial\Omega} = \mathbf{u}_B. \end{aligned} \tag{3.1}$$

By abuse of notation, we use the same symbol ϑ_B, $\mathbf{u}_B$ for both the boundary values and their C^1 extensions $\tilde{\vartheta} = \tilde{\vartheta}(x)$, $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}(x)$ inside Ω.

We start with the ballistic energy equality, see [2, Section 2.4],

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{u}_B|^2 + \varrho e - \vartheta_B \varrho s \right) \ \mathrm{d}x + \int_\Omega \frac{\vartheta_B}{\vartheta} \left(\mathbb{S}(\mathbb{D}_x \mathbf{u}) : \mathbb{D}_x \mathbf{u} + \kappa \frac{|\nabla_x \vartheta|^2}{\vartheta} \right) \ \mathrm{d}x \\ &= - \int_\Omega \Big(\varrho \mathbf{u} \otimes \mathbf{u} + p \mathbb{I} - \mathbb{S}(\mathbb{D}_x \mathbf{u}) \Big) : \mathbb{D}_x \mathbf{u}_B \ \mathrm{d}x + \frac{1}{2} \int_\Omega \varrho \mathbf{u} \cdot \nabla_x |\mathbf{u}_B|^2 \ \mathrm{d}x \\ &+ \int_\Omega \varrho (\mathbf{u} - \mathbf{u}_B) \cdot \mathbf{f} \ \mathrm{d}x - \int_\Omega \varrho s \mathbf{u} \cdot \nabla_x \vartheta_B \ \mathrm{d}x + \kappa \int_\Omega \frac{\nabla_x \vartheta}{\vartheta} \cdot \nabla_x \vartheta_B \ \mathrm{d}x, \end{aligned} \tag{3.2}$$

where we have introduced the entropy

$$s = c_v \log(\vartheta) - \log(\varrho).$$

Thus the choice (3.1) yields the following bounds

$$\sup_{t\in[0,T)} \int_\Omega \varrho|\log(\vartheta)|(t,\cdot)\ \mathrm{d}x \leq c(T,\mathcal{D}_0,\overline{\varrho},\overline{\vartheta},\overline{u}), \tag{3.3}$$

$$\int_0^T\int_\Omega |\nabla_x\mathbf{u}|^2\ \mathrm{d}x\,\mathrm{d}t \leq C(\overline{\varrho},\overline{\vartheta},\overline{u};\mathrm{data}) \ \Rightarrow\ \int_0^T \|\mathbf{u}\|^2_{W^{1,2}(\Omega;R^3)}\,\mathrm{d}t \leq c(T,\mathcal{D}_0,\overline{\varrho},\overline{\vartheta},\overline{u}), \tag{3.4}$$

$$\int_0^T\int_\Omega \left(|\nabla_x\vartheta|^2 + |\nabla_x\log(\vartheta)|^2\right)\ \mathrm{d}x\,\mathrm{d}t \leq c(T,\mathcal{D}_0,\overline{\varrho},\overline{\vartheta},\overline{u}),$$

$$\Rightarrow \int_0^T \|\vartheta\|^2_{W^{1,2}(\Omega)}\,\mathrm{d}t + \int_0^T \|\log(\vartheta)\|^2_{W^{1,2}(\Omega)}\,\mathrm{d}t \leq c(T,\mathcal{D}_0,\overline{\varrho},\overline{\vartheta},\overline{u}). \tag{3.5}$$

4 Estimates of the velocity gradient

This section is the heart of the paper. In principle, we follow the arguments similar to Fang, Zi, and Zhang [5, Section 3] but here adapted to the inhomogeneous boundary conditions.

4.1 Estimates of the velocity material derivative

Let us introduce the material derivative of a function g,

$$D_t g = \partial_t g + \mathbf{u}\cdot\nabla_x g.$$

Accordingly, we may rewrite the momentum equation (1.2) as

$$\varrho D_t\mathbf{u} + \nabla_x p = \mathrm{div}_x\mathbb{S} + \varrho\mathbf{f}. \tag{4.1}$$

Now, consider the scalar product of the momentum equation (4.1) with $D_t(\mathbf{u}-\mathbf{u}_B)$,

$$\varrho|D_t\mathbf{u}|^2 + \nabla_x p\cdot D_t(\mathbf{u}-\mathbf{u}_B) = \mathrm{div}_x\mathbb{S}(\mathbb{D}_x\mathbf{u})\cdot D_t(\mathbf{u}-\mathbf{u}_B) + \varrho\mathbf{f}\cdot D_t(\mathbf{u}-\mathbf{u}_B) + \varrho D_t\mathbf{u}\cdot D_t\mathbf{u}_B. \tag{4.2}$$

The next step is integrating (4.2) over Ω. Here and hereafter we use the hypothesis $\mathbf{u}_B\cdot\mathbf{n}|_{\partial\Omega} = 0$ yielding

$$D_t(\mathbf{u}-\mathbf{u}_B)|_{\partial\Omega} = \left(\partial_t\mathbf{u} - \mathbf{u}\cdot\nabla_x(\mathbf{u}-\mathbf{u}_B)\right)|_{\partial\Omega} = -\mathbf{u}_B\cdot\nabla_x(\mathbf{u}-\mathbf{u}_B)|_{\partial\Omega} = 0. \tag{4.3}$$

Writing

$$\mathrm{div}_x\mathbb{S}(\mathbb{D}_x\mathbf{u}) = \mu\Delta_x\mathbf{u} + \left(\eta + \frac{\mu}{3}\right)\nabla_x\mathrm{div}_x\mathbf{u},$$

and making use of (4.3) we obtain

$$\int_\Omega \mathrm{div}_x\mathbb{S}(\mathbb{D}_x\mathbf{u})\cdot D_t(\mathbf{u}-\mathbf{u}_B)\ \mathrm{d}x$$

$$
\begin{aligned}
&= -\int_\Omega \mathbb{S}(\mathbb{D}_x\mathbf{u}) : \nabla_x \partial_t \mathbf{u} \ \mathrm{d}x \\
&\quad - \mu \int_\Omega \nabla_x \mathbf{u} : \nabla_x \big(\mathbf{u}\cdot\nabla_x(\mathbf{u}-\mathbf{u}_B)\big) \ \mathrm{d}x - \Big(\eta + \frac{\mu}{3}\Big)\int_\Omega \mathrm{div}_x\mathbf{u} \ \mathrm{div}_x\big(\mathbf{u}\cdot\nabla_x(\mathbf{u}-\mathbf{u}_B)\big) \ \mathrm{d}x \\
&= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathbb{S}(\mathbb{D}_x\mathbf{u}) : \mathbb{D}_x\mathbf{u} \ \mathrm{d}x \\
&\quad - \mu \int_\Omega \nabla_x \mathbf{u} : \nabla_x \big(\mathbf{u}\cdot\nabla_x(\mathbf{u}-\mathbf{u}_B)\big) \ \mathrm{d}x - \Big(\eta + \frac{\mu}{3}\Big)\int_\Omega \mathrm{div}_x\mathbf{u} \ \mathrm{div}_x\big(\mathbf{u}\cdot\nabla_x(\mathbf{u}-\mathbf{u}_B)\big) \ \mathrm{d}x, \qquad (4.4)
\end{aligned}
$$

where, furthermore,

$$
\begin{aligned}
\int_\Omega \nabla_x\mathbf{u} : \nabla_x(\mathbf{u}\cdot\nabla_x\mathbf{u}) \ \mathrm{d}x &= \int_\Omega \nabla_x\mathbf{u} : (\nabla_x\mathbf{u}\cdot\nabla_x\mathbf{u}) \ \mathrm{d}x + \frac{1}{2}\int_\Omega \mathbf{u}\cdot\nabla_x|\nabla_x\mathbf{u}|^2 \ \mathrm{d}x \\
&= \int_\Omega \nabla_x\mathbf{u} : (\nabla_x\mathbf{u}\cdot\nabla_x\mathbf{u}) \ \mathrm{d}x - \frac{1}{2}\int_\Omega \mathrm{div}_x\mathbf{u}|\nabla_x\mathbf{u}|^2 \ \mathrm{d}x \qquad (4.5)
\end{aligned}
$$

Note carefully we have used $\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0$ in the last integration. Similarly,

$$
\int_\Omega \mathrm{div}_x\mathbf{u} \ \mathrm{div}_x(\mathbf{u}\cdot\nabla_x\mathbf{u}) \ \mathrm{d}x = \int_\Omega \mathrm{div}_x\mathbf{u} \ \nabla_x\mathbf{u} : \nabla_x^t\mathbf{u} \ \mathrm{d}x - \frac{1}{2}\int_\Omega (\mathrm{div}_x\mathbf{u})^3 \ \mathrm{d}x. \qquad (4.6)
$$

Thus summing up the previous observations, we get

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathbb{S}(\mathbb{D}_x\mathbf{u}) : \mathbb{D}_x\mathbf{u} \ \mathrm{d}x + \frac{1}{2}\int_\Omega \varrho|D_t\mathbf{u}|^2 \ \mathrm{d}x + \int_\Omega \nabla_x p \cdot D_t(\mathbf{u}-\mathbf{u}_B) \ \mathrm{d}x& \\
\le c(T,\mathcal{D}_0,\overline{\varrho},\overline{\vartheta},\overline{u})\left(1 + \int_\Omega |\nabla_x\mathbf{u}|^3 \ \mathrm{d}x\right).& \qquad (4.7)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\int_\Omega \nabla_x p\cdot D_t(\mathbf{u}-\mathbf{u}_B) \ \mathrm{d}x &= -\int_\Omega p \ \mathrm{div}_x(D_t(\mathbf{u}-\mathbf{u}_B)) \ \mathrm{d}x \\
&= -\int_\Omega p \ \mathrm{div}_x D_t\mathbf{u} \ \mathrm{d}x + \int_\Omega p \ \mathrm{div}_x(\mathbf{u}\cdot\nabla_x\mathbf{u}_B) \ \mathrm{d}x, \qquad (4.8)
\end{aligned}
$$

where

$$
\begin{aligned}
p \ \mathrm{div}_x D_t\mathbf{u} &= \partial_t(p \ \mathrm{div}_x\mathbf{u}) - \big(\partial_t p + \mathrm{div}_x(p\mathbf{u})\big)\mathrm{div}_x\mathbf{u} + \mathrm{div}_x(p\mathbf{u})\mathrm{div}_x\mathbf{u} + p \ \mathrm{div}_x(\mathbf{u}\cdot\nabla_x\mathbf{u}) \\
&= \partial_t(p \ \mathrm{div}_x\mathbf{u}) - \big(\partial_t p + \mathrm{div}_x(p\mathbf{u})\big)\mathrm{div}_x\mathbf{u} + p\nabla_x\mathbf{u} : \nabla_x^t\mathbf{u} + \mathrm{div}_x\big(p\mathbf{u} \ \mathrm{div}_x\mathbf{u}\big).
\end{aligned}
$$

As $\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0$, we have

$$
\int_\Omega \mathrm{div}_x\big(p\mathbf{u} \ \mathrm{div}_x\mathbf{u}\big) \ \mathrm{d}x = 0,
$$

and the above estimates together with (4.7) give rise to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathbb{S}(\mathbb{D}_x\mathbf{u}):\mathbb{D}_x\mathbf{u}\ \mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega p\mathrm{div}_x\mathbf{u}\ \mathrm{d}x + \frac{1}{2}\int_\Omega \varrho|D_t\mathbf{u}|^2\ \mathrm{d}x$$
$$\leq c(T,\mathcal{D}_0,\overline{\varrho},\overline{\vartheta},\overline{u})\left(1+\int_\Omega |\nabla_x\mathbf{u}|^3\ \mathrm{d}x\right) - \int_\Omega \Big(\partial_t p + \mathrm{div}_x(p\mathbf{u})\Big)\mathrm{div}_x\mathbf{u}\ \mathrm{d}x.$$

Finally, we realize

$$\partial_t p + \mathrm{div}_x(p\mathbf{u}) = \varrho D_t\vartheta$$

to conclude

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathbb{S}(\mathbb{D}_x\mathbf{u}):\mathbb{D}_x\mathbf{u}\ \mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega p\mathrm{div}_x\mathbf{u}\ \mathrm{d}x + \frac{1}{2}\int_\Omega \varrho|D_t\mathbf{u}|^2\ \mathrm{d}x$$
$$\leq c(T,\mathcal{D}_0,\overline{\varrho},\overline{\vartheta},\overline{u})\left(1+\int_\Omega \varrho|D_t\vartheta||\nabla_x\mathbf{u}|\ \mathrm{d}x + \int_\Omega |\nabla_x\mathbf{u}|^3\ \mathrm{d}x\right). \tag{4.9}$$

4.2 Higher order velocity material derivative estimates

Following [5, Section 3, Lemma 3.3], see also Hoff [12], we deduce

$$\varrho D_t^2\mathbf{u} + \nabla_x\partial_t p + \mathrm{div}_x(\nabla_x p\otimes\mathbf{u})$$
$$= \mu\Big(\Delta_x\partial_t\mathbf{u} + \mathrm{div}_x(\Delta_x\mathbf{u}\otimes\mathbf{u})\Big) + \left(\eta+\frac{\mu}{3}\right)\Big(\nabla_x\mathrm{div}_x\partial_t\mathbf{u} + \mathrm{div}_x\left((\nabla_x\mathrm{div}_x\mathbf{u})\otimes\mathbf{u}\right)\Big) + \varrho\mathbf{u}\cdot\nabla_x\mathbf{f}. \tag{4.10}$$

Next, we compute

$$D_t\mathbf{u}_B = \mathbf{u}\cdot\nabla_x\mathbf{u}_B,\quad D_t^2\mathbf{u}_B = \partial_t\mathbf{u}\cdot\nabla_x\mathbf{u}_B + \mathbf{u}\cdot\nabla_x(\mathbf{u}\cdot\nabla_x\mathbf{u}_B)$$
$$= D_t\mathbf{u}\cdot\nabla_x\mathbf{u}_B - (\mathbf{u}\cdot\nabla_x\mathbf{u})\cdot\nabla_x\mathbf{u}_B + \mathbf{u}\cdot\nabla_x(\mathbf{u}\cdot\nabla_x\mathbf{u}_B)$$
$$= D_t\mathbf{u}\cdot\nabla_x\mathbf{u}_B + (\mathbf{u}\otimes\mathbf{u}):\nabla_x^2\mathbf{u}_B. \tag{4.11}$$

Consequently, we may rewrite (4.10) in the form

$$\varrho D_t^2(\mathbf{u}-\mathbf{u}_B) + \nabla_x\partial_t p + \mathrm{div}_x(\nabla_x p\otimes\mathbf{u})$$
$$= \mu\Big(\Delta_x\partial_t\mathbf{u} + \mathrm{div}_x(\Delta_x\mathbf{u}\otimes\mathbf{u})\Big) + \left(\eta+\frac{\mu}{3}\right)\Big(\nabla_x\mathrm{div}_x\partial_t\mathbf{u} + \mathrm{div}_x\left((\nabla_x\mathrm{div}_x\mathbf{u})\otimes\mathbf{u}\right)\Big) + \varrho\mathbf{u}\cdot\nabla_x\mathbf{f}$$
$$- \varrho D_t\mathbf{u}\cdot\nabla_x\mathbf{u}_B - \varrho(\mathbf{u}\otimes\mathbf{u}):\nabla_x^2\mathbf{u}_B. \tag{4.12}$$

The next step is considering the scalar product of (4.12) with $D_t(\mathbf{u}-\mathbf{u}_B)$ and integrating over Ω. The resulting integrals can be handled as follows:

$$\varrho D_t^2(\mathbf{u}-\mathbf{u}_B)\cdot D_t(\mathbf{u}-\mathbf{u}_B) = \varrho\frac{1}{2}D_t|D_t(\mathbf{u}-\mathbf{u}_B)|^2$$

$$= \frac{1}{2}\varrho \left(\partial_t |D_t(\mathbf{u}-\mathbf{u}_B)|^2 + \mathbf{u}\cdot\nabla_x |D_t(\mathbf{u}-\mathbf{u}_B)|^2\right)$$
$$= \frac{1}{2}\partial_t \left(\varrho |D_t(\mathbf{u}-\mathbf{u}_B)|^2\right) + \frac{1}{2}{\rm div}_x \left(\varrho\mathbf{u} |D_t(\mathbf{u}-\mathbf{u}_B)|^2\right),$$

where we have used the equation of continuity (1.1). Seeing that $\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0$ we get

$$\int_\Omega \varrho D_t^2(\mathbf{u}-\mathbf{u}_B)\cdot D_t(\mathbf{u}-\mathbf{u}_B)\ {\rm d}x = \frac{{\rm d}}{{\rm d}t}\frac{1}{2}\int_\Omega \varrho |D_t(\mathbf{u}-\mathbf{u}_B)|^2\ {\rm d}x. \tag{4.13}$$

Similarly,

$$\int_\Omega \Big(\nabla_x\partial_t p + {\rm div}_x(\nabla_x p\otimes\mathbf{u})\Big)\cdot D_t(\mathbf{u}-\mathbf{u}_B)\ {\rm d}x$$
$$= -\int_\Omega \Big(\partial_t p + {\rm div}_x(p\mathbf{u})\Big){\rm div}_x D_t(\mathbf{u}-\mathbf{u}_B)\ {\rm d}x$$
$$+ \int_\Omega \Big({\rm div}_x(p\mathbf{u}){\rm div}_x D_t(\mathbf{u}-\mathbf{u}_B) - \nabla_x p\otimes\mathbf{u} : \nabla_x D_t(\mathbf{u}-\mathbf{u}_B)\Big)\ {\rm d}x, \tag{4.14}$$

where

$$\int_\Omega \nabla_x p\otimes\mathbf{u} : \nabla_x D_t(\mathbf{u}-\mathbf{u}_B)\ {\rm d}x$$
$$= -\int_\Omega p\nabla_x\mathbf{u} : \nabla_x D_t(\mathbf{u}-\mathbf{u}_B)\ {\rm d}x + \int_\Omega \nabla_x(p\mathbf{u}) : \nabla_x D_t(\mathbf{u}-\mathbf{u}_B)\ {\rm d}x.$$

In addition, as $D_t(\mathbf{u}-\mathbf{u}_B)$ vanishes on $\partial\Omega$, we can perform by parts integration in the last integral obtaining

$$\int_\Omega \nabla_x(p\mathbf{u}) : \nabla_x D_t(\mathbf{u}-\mathbf{u}_B)\ {\rm d}x = \int_\Omega {\rm div}_x(p\mathbf{u}){\rm div}_x D_t(\mathbf{u}-\mathbf{u}_B)\ {\rm d}x.$$

Thus, similarly to the preceding section, we conclude

$$\int_\Omega \Big(\nabla_x\partial_t p + {\rm div}_x(\nabla_x p\otimes\mathbf{u})\Big)\cdot D_t(\mathbf{u}-\mathbf{u}_B)\ {\rm d}x$$
$$= -\int_\Omega \varrho D_t\vartheta {\rm div}_x D_t(\mathbf{u}-\mathbf{u}_B)\ {\rm d}x\ + \int_\Omega p\nabla_x\mathbf{u} : \nabla_x D_t(\mathbf{u}-\mathbf{u}_B)\ {\rm d}x. \tag{4.15}$$

Analogously,

$$\int_\Omega \Big(\Delta_x\partial_t\mathbf{u} + {\rm div}_x(\Delta_x\mathbf{u}\otimes\mathbf{u})\Big)\cdot D_t(\mathbf{u}-\mathbf{u}_B)\ {\rm d}x$$
$$= -\int_\Omega \nabla_x\partial_t\mathbf{u} : \nabla_x D_t(\mathbf{u}-\mathbf{u}_B)\ {\rm d}x - \int_\Omega (\Delta_x\mathbf{u}\otimes\mathbf{u}) : \nabla_x D_t(\mathbf{u}-\mathbf{u}_B)\ {\rm d}x$$
$$= -\int_\Omega \nabla_x D_t\mathbf{u} : \nabla_x D_t(\mathbf{u}-\mathbf{u}_B)\ {\rm d}x - \int_\Omega \Big(\Delta_x\mathbf{u}\otimes\mathbf{u} - \nabla_x(\mathbf{u}\cdot\nabla_x\mathbf{u})\Big) : \nabla_x D_t(\mathbf{u}-\mathbf{u}_B)\ {\rm d}x, \tag{4.16}$$

where, using summation convention,

$$
\begin{aligned}
&\int_\Omega \big(\Delta_x \mathbf{u} \otimes \mathbf{u}\big) : \nabla_x D_t(\mathbf{u} - \mathbf{u}_B) \ \mathrm{d}x \\
&= \int_\Omega \partial_{x_k} \Big(u_j \partial_{x_k} u_i \Big) \partial_{x_j} D_t(\mathbf{u} - \mathbf{u}_B)_i \ \mathrm{d}x - \int_\Omega \partial_{x_k} u_i \partial_{x_k} u_j \partial_{x_j} D_t(\mathbf{u} - \mathbf{u}_B)_i \ \mathrm{d}x \\
&= \int_\Omega \partial_{x_j} \Big(u_j \partial_{x_k} u_i \Big) \partial_{x_k} D_t(\mathbf{u} - \mathbf{u}_B)_i \ \mathrm{d}x - \int_\Omega \partial_{x_k} u_i \partial_{x_k} u_j \partial_{x_j} D_t(\mathbf{u} - \mathbf{u}_B)_i \ \mathrm{d}x \\
&= \int_\Omega \mathrm{div}_x \mathbf{u} \ \nabla_x \mathbf{u} : \nabla_x D_t(\mathbf{u} - \mathbf{u}_B) \ \mathrm{d}x \\
&+ \int_\Omega \Big(u_j \partial_{x_k} \partial_{x_j} u_i \Big) \partial_{x_k} D_t(\mathbf{u} - \mathbf{u}_B)_i \ \mathrm{d}x - \int_\Omega \partial_{x_k} u_i \partial_{x_k} u_j \partial_{x_j} D_t(\mathbf{u} - \mathbf{u}_B)_i \ \mathrm{d}x \\
&= \int_\Omega \nabla_x (\mathbf{u} \cdot \nabla_x \mathbf{u}) : \nabla_x D_t(\mathbf{u} - \mathbf{u}_B) \ \mathrm{d}x + \int_\Omega \mathrm{div}_x \mathbf{u} \ \nabla_x \mathbf{u} : \nabla_x D_t(\mathbf{u} - \mathbf{u}_B) \ \mathrm{d}x \\
&- \int_\Omega \partial_{x_j} u_i \partial_{x_k} u_j \partial_{x_k} D_t(\mathbf{u} - \mathbf{u}_B)_i \ \mathrm{d}x - \int_\Omega \partial_{x_k} u_i \partial_{x_k} u_j \partial_{x_j} D_t(\mathbf{u} - \mathbf{u}_B)_i \ \mathrm{d}x.
\end{aligned} \tag{4.17}
$$

Summing up (4.16), (4.17) we conclude

$$
\begin{aligned}
&\int_\Omega \Big(\Delta_x \partial_t \mathbf{u} + \mathrm{div}_x (\Delta_x \mathbf{u} \otimes \mathbf{u}) \Big) \cdot D_t(\mathbf{u} - \mathbf{u}_B) \ \mathrm{d}x \\
&= - \int_\Omega \nabla_x D_t \mathbf{u} : \nabla_x D_t(\mathbf{u} - \mathbf{u}_B) \ \mathrm{d}x - \int_\Omega \mathrm{div}_x \mathbf{u} \ \nabla_x \mathbf{u} : \nabla_x D_t(\mathbf{u} - \mathbf{u}_B) \ \mathrm{d}x \\
&+ \int_\Omega \partial_{x_j} u_i \partial_{x_k} u_j \partial_{x_k} D_t(\mathbf{u} - \mathbf{u}_B)_i \ \mathrm{d}x + \int_\Omega \partial_{x_k} u_i \partial_{x_k} u_j \partial_{x_j} D_t(\mathbf{u} - \mathbf{u}_B)_i \ \mathrm{d}x.
\end{aligned} \tag{4.18}
$$

Estimating the remaining integrals in (4.12) in a similar manner we may infer

$$
\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \varrho |D_t(\mathbf{u} - \mathbf{u}_B)|^2 \ \mathrm{d}x + \mu \int_\Omega |\nabla_x D_t(\mathbf{u} - \mathbf{u}_B)|^2 \ \mathrm{d}x + \Big(\eta + \frac{\mu}{3} \Big) \int_\Omega |\mathrm{div}_x D_t(\mathbf{u} - \mathbf{u}_B)|^2 \ \mathrm{d}x \\
&\quad \leq c(T, \mathcal{D}_0, \overline{\varrho}, \overline{\vartheta}, \overline{u}) \left(1 + \int_\Omega \varrho |D_t \vartheta|^2 \ \mathrm{d}x + \int_\Omega |\nabla_x \mathbf{u}|^4 \ \mathrm{d}x + \int_\Omega \varrho |D_t \mathbf{u}|^2 \ \mathrm{d}x \right).
\end{aligned} \tag{4.19}
$$

cf. [5, Section 3, Lemma 3.3].

4.3 Velocity decomposition

Following the original idea of Sun, Wang, and Zhang [18], we decompose the velocity field in the form:

$$
\mathbf{u} = \mathbf{v} + \mathbf{w}, \tag{4.20}
$$

$$
\mathrm{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{v}) = \nabla_x p \text{ in } (0, T) \times \Omega, \ \ \mathbf{v}|_{\partial\Omega} = 0, \tag{4.21}
$$

$$\mathrm{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{w}) = \varrho D_t \mathbf{u} - \varrho \mathbf{f} \ \text{in} \ (0,T)\times\Omega, \ \mathbf{w}|_{\partial\Omega} = \mathbf{u}_B. \tag{4.22}$$

Since

$$\mathrm{div}_x \mathbb{S}(\mathbb{D}_x \partial_t \mathbf{v}) = \nabla_x \partial_t p \ \text{in} \ (0,T)\times\Omega, \ \mathbf{v}|_{\partial\Omega} = 0,$$

we get

$$\int_\Omega \partial_t p \ \mathrm{div}_x \mathbf{v} \ \mathrm{d}x = -\int_\Omega \nabla_x \partial_t p \cdot \mathbf{v} \ \mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathbb{S}(\mathbb{D}_x \mathbf{v}) : \mathbb{D}_x \mathbf{v} \ \mathrm{d}x. \tag{4.23}$$

Moreover, the standard elliptic estimates for the Lamé operator yield:

$$\|\mathbf{v}\|_{W^{1,q}(\Omega;R^3)} \le c(q,\overline{\varrho},\overline{\vartheta}) \ \text{for all} \ 1 \le q < \infty, \tag{4.24}$$

$$\|\mathbf{v}\|_{W^{2,q}(\Omega;R^3)} \le c(q,\overline{\varrho},\overline{\vartheta}) \left(\|\nabla_x \varrho\|_{L^q(\Omega;R^3)} + \|\nabla_x \vartheta\|_{L^q(\Omega;R^3)} \right), \ 1 < q < \infty. \tag{4.25}$$

Similarly,

$$\|\mathbf{w}\|_{W^{2,2}(\Omega;R^3)} \le c(T,\mathcal{D}_0,\overline{\varrho},\overline{\vartheta},\overline{u}) \left(1 + \|\sqrt{\varrho}\partial_t \mathbf{u}\|_{L^2(\Omega;R^3)} + \|\nabla_x \mathbf{u}\|_{L^2(\Omega;R^{3\times3})} \right). \tag{4.26}$$

The estimates (4.24)–(4.26) are uniform in the time interval $[0,T)$.

4.4 Temperature estimates

Similarly to Fang, Zi, Zhang [5, Section 3, Lemma 3.4] we multiply the internal energy equation (1.3) on $\partial_t \vartheta$ and integrate over Ω obtaining

$$\begin{aligned}
c_v \int_\Omega \varrho |D_t \vartheta|^2 \ \mathrm{d}x &+ \frac{\kappa}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla_x \vartheta|^2 \ \mathrm{d}x \\
&= c_v \int_\Omega \varrho D_t \vartheta \ \mathbf{u}\cdot\nabla_x \vartheta \ \mathrm{d}x - \int_\Omega \varrho\vartheta \ \mathrm{div}_x \mathbf{u} \ D_t \vartheta \ \mathrm{d}x + \int_\Omega \varrho\vartheta \ \mathrm{div}_x \mathbf{u} \ \mathbf{u}\cdot\nabla_x \vartheta \ \mathrm{d}x \\
&+ \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \vartheta \ \mathbb{S}(\mathbb{D}_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x \\
&- \mu \int_\Omega \vartheta \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I} \right) : \left(\nabla_x \partial_t \mathbf{u} + \nabla_x^t \partial_t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \partial_t \mathbf{u}\mathbb{I} \right) \ \mathrm{d}x \\
&- 2\eta \int_\Omega \vartheta \ \mathrm{div}_x \mathbf{u} \ \mathrm{div}_x \partial_t \mathbf{u} \ \mathrm{d}x.
\end{aligned} \tag{4.27}$$

Indeed the term involving the boundary integral is handled as

$$-\kappa \int_\Omega \Delta_x \vartheta \ \partial_t \vartheta \ \mathrm{d}x = -\kappa \int_{\partial\Omega} \partial_t \vartheta_B \nabla_x \vartheta \cdot \mathbf{n} \ \mathrm{d}S_x + \frac{\kappa}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla_x \vartheta|^2 \ \mathrm{d}x,$$

where

$$\int_{\partial\Omega} \partial_t \vartheta_B \nabla_x \vartheta \cdot \mathbf{n} \ \mathrm{d}S_x = 0$$

as the boundary temperature is independent of t.

Similarly to Fang, Zi, Zhang [5, Section 3, Lemma 3.4], we have to show that the intergrals

$$\int_\Omega \vartheta\ \nabla_x \mathbf{u} : \nabla_x \partial_t \mathbf{u}\ \mathrm{d}x,\ \int_\Omega \vartheta\ \nabla_x \mathbf{u} : \nabla_x^t \partial_t \mathbf{u}\ \mathrm{d}x,\ \text{and}\ \int_\Omega \vartheta\ \mathrm{div}_x \mathbf{u}\ \mathrm{div}_x \partial_t \mathbf{u}\ \mathrm{d}x$$

can be rewritten in the form compatible with (4.19), meaning with the time derivatives replaced by material derivatives. Fortunately, this step can be carried out in the present setting using only the boundary condition $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$. Indeed we get

$$\int_\Omega \vartheta\ \nabla_x \mathbf{u} : \nabla_x \partial_t \mathbf{u}\ \mathrm{d}x = \int_\Omega \vartheta\ \nabla_x \mathbf{u} : \nabla_x (D_t \mathbf{u})\ \mathrm{d}x - \int_\Omega \vartheta\ \nabla_x \mathbf{u} : \nabla_x (\mathbf{u} \cdot \nabla_x \mathbf{u})\ \mathrm{d}x,$$

where

$$\begin{aligned}
&\int_\Omega \vartheta\ \nabla_x \mathbf{u} : \nabla_x (\mathbf{u} \cdot \nabla_x \mathbf{u})\ \mathrm{d}x \\
&= \int_\Omega \vartheta\ \nabla_x \mathbf{u} : (\nabla_x \mathbf{u} \cdot \nabla_x \mathbf{u})\ \mathrm{d}x + \frac{1}{2} \int_\Omega \vartheta\ \mathbf{u} \cdot \nabla_x |\nabla_x \mathbf{u}|^2\ \mathrm{d}x \\
&= \int_\Omega \vartheta\ \nabla_x \mathbf{u} : (\nabla_x \mathbf{u} \cdot \nabla_x \mathbf{u})\ \mathrm{d}x - \frac{1}{2} \int_\Omega |\nabla_x \mathbf{u}|^2\ \nabla_x \vartheta \cdot \mathbf{u}\ \mathrm{d}x - \frac{1}{2} \int_\Omega |\nabla_x \mathbf{u}|^2\ \vartheta \mathrm{div}_x \mathbf{u}\ \mathrm{d}x.
\end{aligned}$$

Similarly,

$$\int_\Omega \vartheta\ \nabla_x \mathbf{u} : \nabla_x^t \partial_t \mathbf{u}\ \mathrm{d}x = \int_\Omega \vartheta\ \nabla_x \mathbf{u} : \nabla_x^t (D_t \mathbf{u})\ \mathrm{d}x - \int_\Omega \vartheta\ \nabla_x \mathbf{u} : \nabla_x^t (\mathbf{u} \cdot \nabla_x \mathbf{u})\ \mathrm{d}x,$$

where

$$\begin{aligned}
&\int_\Omega \vartheta\ \nabla_x \mathbf{u} : \nabla_x^t (\mathbf{u} \cdot \nabla_x \mathbf{u})\ \mathrm{d}x \\
&= \int_\Omega \vartheta\ \nabla_x \mathbf{u} : (\nabla_x^t \mathbf{u} \cdot \nabla_x^t \mathbf{u})\ \mathrm{d}x + \frac{1}{2} \int_\Omega \vartheta\ \mathbf{u} \cdot \nabla_x (\nabla_x \mathbf{u} : \nabla_x^t \mathbf{u})\ \mathrm{d}x \\
&= \int_\Omega \vartheta\ \nabla_x \mathbf{u} : (\nabla_x^t \mathbf{u} \cdot \nabla_x^t \mathbf{u})\ \mathrm{d}x - \frac{1}{2} \int_\Omega (\nabla_x \mathbf{u} : \nabla_x^t \mathbf{u})\ \nabla_x \vartheta \cdot \mathbf{u}\ \mathrm{d}x - \frac{1}{2} \int_\Omega (\nabla_x \mathbf{u} : \nabla_x^t \mathbf{u})\ \vartheta \mathrm{div}_x \mathbf{u}\ \mathrm{d}x.
\end{aligned}$$

Finally,

$$\int_\Omega \vartheta\ \mathrm{div}_x \mathbf{u}\ \mathrm{div}_x \partial_t \mathbf{u}\ \mathrm{d}x = \int_\Omega \vartheta\ \mathrm{div}_x \mathbf{u}\ \mathrm{div}_x D_t \mathbf{u}\ \mathrm{d}x - \int_\Omega \vartheta\ \mathrm{div}_x \mathbf{u}\ \mathrm{div}_x (\mathbf{u} \cdot \nabla_x \mathbf{u})\ \mathrm{d}x,$$

where

$$\int_\Omega \vartheta\ \mathrm{div}_x \mathbf{u}\ \mathrm{div}_x (\mathbf{u} \cdot \nabla_x \mathbf{u})\ \mathrm{d}x$$

$$
= \int_\Omega \vartheta \ \mathrm{div}_x \mathbf{u} \ (\nabla_x \mathbf{u} : \nabla_x^t \mathbf{u}) \ \mathrm{d}x + \frac{1}{2} \int_\Omega \vartheta \mathbf{u} \cdot \nabla_x |\mathrm{div}_x \mathbf{u}|^2 \ \mathrm{d}x
$$

$$
= \int_\Omega \vartheta \ \mathrm{div}_x \mathbf{u} \ (\nabla_x \mathbf{u} : \nabla_x^t \mathbf{u}) \ \mathrm{d}x - \frac{1}{2} \int_\Omega |\mathrm{div}_x \mathbf{u}|^2 \ \nabla_x \vartheta \cdot \mathbf{u} \ \mathrm{d}x - \frac{1}{2} \int_\Omega |\mathrm{div}_x \mathbf{u}|^2 \ \vartheta \mathrm{div}_x \mathbf{u} \ \mathrm{d}x.
$$

We conclude, using (4.7), (4.19), and (4.27),

$$
\int_\Omega |\nabla_x \vartheta|^2 (\tau, \cdot) \ \mathrm{d}x + \int_0^\tau \int_\Omega \varrho |D_t \vartheta|^2 \ \mathrm{d}x \, \mathrm{d}t \\
\leq c(T, \mathcal{D}_0, \overline{\varrho}, \overline{\vartheta}, \overline{u}) \left(1 + \int_0^\tau \int_\Omega |\nabla_x \mathbf{u}|^4 \ \mathrm{d}x \, \mathrm{d}t \right). \tag{4.28}
$$

Next, by virtue of the decomposition $\mathbf{u} = \mathbf{v} + \mathbf{w}$ and the bound (4.24),

$$
\int_\Omega |\nabla_x \mathbf{u}|^4 \ \mathrm{d}x \lesssim \int_\Omega |\nabla_x \mathbf{v}|^4 \ \mathrm{d}x + \int_\Omega |\nabla_x \mathbf{w}|^4 \ \mathrm{d}x \leq c(T, \mathcal{D}_0, \overline{\varrho}, \overline{\vartheta}, \overline{u}) \left(1 + \int_\Omega |\nabla_x \mathbf{w}|^4 \ \mathrm{d}x \right), \tag{4.29}
$$

and, similarly,

$$
\|\mathbf{w}\|_{L^\infty(\Omega; R^3)} \leq \|\mathbf{u}\|_{L^\infty(\Omega; R^3)} + \|\mathbf{v}\|_{L^\infty(\Omega; R^3)} \leq c(T, \mathcal{D}_0, \overline{\varrho}, \overline{\vartheta}, \overline{u}). \tag{4.30}
$$

Recalling the Gagliardo–Nirenberg interpolation inequality in the form

$$
\|\nabla_x U\|^2_{L^4(\Omega; R^3)} \leq \|U\|_{L^\infty(\Omega)} \|\Delta_x U\|_{L^2(\Omega)} \text{ whenever } U|_{\partial\Omega} = 0, \tag{4.31}
$$

we may use (4.29), (4.30) to rewrite (4.28) in the form

$$
\int_\Omega |\nabla_x \vartheta|^2 (\tau, \cdot) \ \mathrm{d}x + \int_0^\tau \int_\Omega \varrho |D_t \vartheta|^2 \ \mathrm{d}x \, \mathrm{d}t \\
\leq c(T, \mathcal{D}_0, \overline{\varrho}, \overline{\vartheta}, \overline{u}) \left(1 + \int_0^\tau \int_\Omega |\nabla_x \vartheta|^2 \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \|\mathbf{w}\|^2_{W^{2,2}(\Omega; R^3)} \, \mathrm{d}t \right). \tag{4.32}
$$

Finally, we use the elliptic estimates (4.26) to conclude

$$
\int_\Omega |\nabla_x \vartheta|^2 (\tau, \cdot) \ \mathrm{d}x + \int_0^\tau \int_\Omega \varrho |D_t \vartheta|^2 \ \mathrm{d}x \, \mathrm{d}t \\
\leq c(T, \mathcal{D}_0, \overline{\varrho}, \overline{\vartheta}, \overline{u}) \left(1 + \int_0^\tau \int_\Omega \left(|\nabla_x \vartheta|^2 + |\nabla_x \mathbf{u}|^2 \right) \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \|\sqrt{\varrho} \partial_t \mathbf{u}\|^2_{L^2(\Omega; R^3)} \, \mathrm{d}t \right). \tag{4.33}
$$

Summing up (4.7), (4.19), and (4.33) we may apply Gronwall's lemma to obtain the following bounds:

$$
\sup_{t \in [0,T)} \|\mathbf{u}(t, \cdot)\|_{W^{1,2}(\Omega; R^3)} \leq c(T, \mathcal{D}_0, \overline{\varrho}, \overline{\vartheta}, \overline{u}), \tag{4.34}
$$

$$
\sup_{t \in [0,T)} \|\sqrt{\varrho} D_t \mathbf{u}(t, \cdot)\|_{L^2(\Omega; R^3)} \leq c(T, \mathcal{D}_0, \overline{\varrho}, \overline{\vartheta}, \overline{u}), \tag{4.35}
$$

$$\sup_{t\in[0,T)} \|\vartheta(t,\cdot)\|_{W^{1,2}(\Omega)} \leq c(T,\mathcal{D}_0,\overline{\varrho},\overline{\vartheta},\overline{u}), \tag{4.36}$$

$$\int_0^T \int_\Omega |\nabla_x D_t \mathbf{u}|^2 \ \mathrm{d}x\,\mathrm{d}t \leq c(T,\mathcal{D}_0,\overline{\varrho},\overline{\vartheta},\overline{u}), \tag{4.37}$$

$$\int_0^T \int_\Omega \varrho |D_t \vartheta|^2 \ \mathrm{d}x\,\mathrm{d}t \leq c(T,\mathcal{D}_0,\overline{\varrho},\overline{\vartheta},\overline{u}). \tag{4.38}$$

Moreover, it follows from (4.24), (4.31), (4.35)

$$\sup_{t\in[0,T)} \|\nabla_x \mathbf{u}(t,\cdot)\|_{L^4(\Omega;R^{3\times 3})} \leq c(T,\mathcal{D}_0,\overline{\varrho},\overline{\vartheta},\overline{u}). \tag{4.39}$$

In addition, (4.38), (4.39) and the standard parabolic estimates applied to the internal energy balance (1.3) yield

$$\int_0^T \|\vartheta\|^2_{W^{2,2}(\Omega)}\,\mathrm{d}t \leq c(T,\mathcal{D}_0,\overline{\varrho},\overline{\vartheta},\overline{u}). \tag{4.40}$$

5 Second energy bound

It follows from (4.26), (4.35) that

$$\sup_{t\in[0,T)} \|\mathbf{w}(t,\cdot)\|_{W^{2,2}(\Omega;R^3)} \leq c(T,\mathcal{D}_0,\overline{\varrho},\overline{\vartheta},\overline{u}); \tag{5.1}$$

whence, by virtue of (4.24) and Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$,

$$\sup_{t\in[0,T)} \|\nabla_x \mathbf{u}(t,\cdot)\|^2_{L^6(\Omega;R^{3\times 3})} \leq c(T,\mathcal{D}_0,\overline{\varrho},\overline{\vartheta},\overline{u}). \tag{5.2}$$

Moreover, as a consequence of (4.37), $D_t\mathbf{u}$ is bounded in $L^2(L^6)$, which, combined with (5.2), gives rise to

$$\int_0^T \|\partial_t \mathbf{u}\|^2_{L^6(\Omega;R^3)}\ \mathrm{d}t \leq c(T,\mathcal{D}_0,\overline{\varrho},\overline{\vartheta},\overline{u}). \tag{5.3}$$

Finally, going back to (4.22) we conclude

$$\int_0^T \|\mathbf{w}\|^2_{W^{2,6}(\Omega;R^3)}\,\mathrm{d}t \leq c(T,\mathcal{D}_0,\overline{\varrho},\overline{\vartheta},\overline{u}), \tag{5.4}$$

and

$$\int_0^T \|\mathbf{u}\|^2_{W^{1,q}(\Omega;R^3)}\,\mathrm{d}t \leq c(T,\mathcal{D}_0,\overline{\varrho},\overline{\vartheta},\overline{u},q) \text{ for any } 1 \leq q < \infty. \tag{5.5}$$

6 Estimates of the derivatives of the density

Using (5.4), (5.5), we may proceed as in [19, Section 5] to deduce the bounds

$$\sup_{t\in[0,T)} \left(\|\partial_t \varrho(t,\cdot)\|_{L^6(\Omega)} + \|\varrho(t,\cdot)\|_{W^{1,6}(\Omega)} \right) \leq c(T, \mathcal{D}_0, \overline{\varrho}, \overline{\vartheta}, \overline{u}). \tag{6.1}$$

Revisiting the momentum equation (1.2) we use (6.1) together with the other bounds established above to obtain

$$\int_0^T \|\mathbf{u}\|^2_{W^{2,6}(\Omega;R^3)} \, \mathrm{d}t \leq c(T, \mathcal{D}_0, \overline{\varrho}, \overline{\vartheta}, \overline{u}). \tag{6.2}$$

6.1 Positivity of the density and temperature

It follows from (6.2) that $\mathrm{div}_x \mathbf{u}$ is bounded in $L^1(0,T; L^\infty(\Omega))$. Thus the equation of continuity (1.1) yields a positive lower bound on the density

$$\inf_{(t,x)\in[0,T)\times\overline{\Omega}} \varrho(t,x) \geq \underline{\varrho} > 0, \tag{6.3}$$

where the lower bound depends on the data as well as on the length T of the time interval.

Similarly, rewriting the internal energy balance equation (1.3) in the form

$$c_v \left(\partial_t \vartheta + \mathbf{u} \cdot \nabla_x \vartheta \right) - \frac{\kappa}{\varrho} \Delta_x \vartheta = \frac{1}{\varrho} \mathbb{S} : \mathbb{D}_x \mathbf{u} - \vartheta \mathrm{div}_x \mathbf{u} \tag{6.4}$$

we may apply the standard parabolic maximum/minimum principle to deduce

$$\inf_{(t,x)\in[0,T)\times\overline{\Omega}} \vartheta(t,x) \geq \underline{\vartheta} > 0. \tag{6.5}$$

7 Parabolic regularity for the heat equation

We rewrite the parabolic equation (6.4) in terms of $\Theta = \vartheta - \vartheta_B$. Recalling $\Delta_x \vartheta_B = 0$ we get

$$c_v \left(\partial_t \Theta + \mathbf{u} \cdot \nabla_x \vartheta \right) - \frac{\kappa}{\varrho} \Delta_x \Theta = \frac{1}{\varrho} \mathbb{S} : \mathbb{D}_x \mathbf{u} - \vartheta \mathrm{div}_x \mathbf{u} \tag{7.1}$$

with the *homogeneous* Dirichlet boundary conditions

$$\Theta|_{\partial\Omega} = 0. \tag{7.2}$$

Now, we can apply all arguments of [10, Sections 4.6, 4.7] to Θ obtaining the bounds

$$\|\vartheta\|_{C^\alpha([0,T]\times\overline{\Omega})} \leq c(T, \mathcal{D}_0, \overline{\varrho}, \overline{\vartheta}, \overline{u}) \text{ for some } \alpha > 0, \tag{7.3}$$

$$\|\vartheta\|_{L^p(0,T;W^{2,3}(\Omega))} + \|\partial_t \vartheta\|_{L^p(0,T;L^3(\Omega))} \leq c(T, \mathcal{D}_0, \overline{\varrho}, \overline{\vartheta}, \overline{u}) \text{ for all } 1 \leq p < \infty, \tag{7.4}$$

together with

$$\|\mathbf{u}\|_{L^p(0,T;W^{2,6}(\Omega;R^3))} + \|\partial_t \mathbf{u}\|_{L^p(0,T;L^6(\Omega;R^3))} \leq c(T, \mathcal{D}_0, \overline{\varrho}, \overline{\vartheta}, \overline{u}) \text{ for any } 1 \leq p < \infty. \tag{7.5}$$

8 Final estimates

The bounds (7.5) imply, in particular,

$$\sup_{(t,x)\in[0,T)\times\overline{\Omega}} |\nabla_x \mathbf{u}(t,x)| \leq c(T, \mathcal{D}_0, \underline{\varrho}, \overline{\vartheta}, \overline{u}). \tag{8.1}$$

Thus the desired higher order estimates can be obtained exactly as in [9, Section 4.6]. Indeed the arguments of [9, Section 4.6] are based on differentiating the equation (7.1) with respect to time which gives rise to a parabolic problem for $\partial_t \vartheta$ with the *homogeneous* Dirichlet boundary conditions $\partial_t \vartheta|_{\partial\Omega} = 0$. Indeed we get

$$\begin{aligned} c_v \partial^2_{tt}\vartheta + c_v \mathbf{u}\cdot\nabla_x \partial_t \vartheta - \frac{\kappa}{\varrho}\Delta_x \partial_t \vartheta =& -c_v \partial_t \mathbf{u}\cdot\nabla_x \vartheta - \frac{1}{\varrho^2}\partial_t \varrho \left(\kappa \Delta_x \vartheta + \mathbb{S}(\mathbb{D}_x \mathbf{u}) : \mathbb{D}_x \mathbf{u}\right) \\ &+ \frac{2}{\varrho}\, \mathbb{S}(\mathbb{D}_x \mathbf{u}) : \mathbb{D}_x \partial_t \mathbf{u} - \partial_t \vartheta \operatorname{div}_x \mathbf{u} - \vartheta \operatorname{div}_x \partial_t \mathbf{u}. \end{aligned}$$

The estimates obtained in the previous sections imply that the right–hand side of the above equation is bounded in $L^2(0,T;L^2(\Omega))$. Thus multiplying the equation on $\Delta_x \partial_t \vartheta$ and performing the standard by parts integration, we get the desired estimates as in [9, Section 4.6].

The remaining estimates are obtained exactly as in [9, Section 4.6] :

$$\sup_{t\in[0,T)} \|\vartheta(t,\cdot)\|_{W^{3,2}(\Omega)} + \sup_{t\in[0,T)} \|\partial_t \vartheta(t,\cdot)\|_{W^{1,2}(\Omega)} \leq c(T, \mathcal{D}_0, \underline{\varrho}, \overline{\vartheta}, \overline{u}), \tag{8.2}$$

$$\int_0^T \left(\|\partial_t \vartheta\|^2_{W^{2,2}(\Omega)} + \|\vartheta\|^2_{W^{4,2}(\Omega)} \right) \mathrm{d}t \leq c(T, \mathcal{D}_0, \underline{\varrho}, \overline{\vartheta}, \overline{u}), \tag{8.3}$$

$$\sup_{t\in[0,T)} \|\mathbf{u}(t,\cdot)\|_{W^{3,2}(\Omega;R^3)} + \sup_{t\in[0,T)} \|\partial_t \mathbf{u}(t,\cdot)\|_{W^{1,2}(\Omega;R^3)} \leq c(T, \mathcal{D}_0, \underline{\varrho}, \overline{\vartheta}, \overline{u}), \tag{8.4}$$

$$\int_0^T \left(\|\partial_t \mathbf{u}\|^2_{W^{2,2}(\Omega;R^3)} + \|\mathbf{u}\|^2_{W^{4,2}(\Omega;R^3)} \right) \mathrm{d}t \leq c(T, \mathcal{D}_0, \underline{\varrho}, \overline{\vartheta}, \overline{u}), \tag{8.5}$$

and

$$\sup_{t\in[0,T)} \|\varrho(t,\cdot)\|_{W^{3,2}(\Omega)} \leq c(T, \mathcal{D}_0, \underline{\varrho}, \overline{\vartheta}, \overline{u}). \tag{8.6}$$

We have completed the proof of Proposition 2.3.

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