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Abstract

We consider the motion of N rigid bodies – compact sets $(\mathcal{S}_\varepsilon^1, \dots, \mathcal{S}_\varepsilon^N)_{\varepsilon>0}$ – immersed in a viscous incompressible fluid contained in a domain in the Euclidean space \mathbb{R}^d , $d = 2, 3$. We show the fluid flow is not influenced by the presence of the bodies in the asymptotic limit as $\varepsilon \rightarrow 0$ as soon as

$$\text{diam}[\mathcal{S}_\varepsilon^i] \rightarrow 0 \text{ as } \varepsilon \rightarrow 0, \quad i = 1, \dots, N.$$

The result depends solely on the geometry of the bodies and is independent of their mass densities. Collisions are allowed and the initial data are arbitrary with finite energy.

Keywords: Navier-Stokes system, body–fluid interaction problem, small light rigid body

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1 Introduction

There is a number of studies concerning the impact of a small rigid body immersed in a viscous fluid on the fluid motion. A general approach used so far is based on the idea that if the body is small but “heavy”, meaning its mass density $\rho_{\mathcal{S}}^\varepsilon$ is large, its velocity can be controlled and the resulting situation is therefore close to the rigid obstacle problem. He and Iftimie [10], [11] exploited this idea to handle the case when the body mass density satisfies $\rho_{\mathcal{S}}^\varepsilon \rightarrow \infty$, while the rigid body diameter is proportional to a small number ε . More recently, Bravin and Nečasová [2] showed that if the density is “very large”, the rigid object keeps moving with its initial velocity not being influenced by the fluid. Note that these results are slightly at odds with a physically relevant hypothesis that the body density should be at least bounded and also with a commonly accepted scenario that a *light* particle should not have any major impact on the fluid motion.

To the best of our knowledge, the only available result concerning a body with a constant density was obtained by Lacave and Takahashi [18] in the case of the planar motion. Their technique, similarly to a more recent paper by Tucsnak et al. [5], is based on the $L^p - L^q$ theory for the associated solution semigroup and requires smallness of the initial fluid velocity. The authors also note that the result can be extended to the case of several “massive” bodies up to the first contact.

Our goal is to extend the main result of [18] to the case of several bodies $(\mathcal{S}_\varepsilon^1, \dots, \mathcal{S}_\varepsilon^N)_{\varepsilon>0}$ that may collide in the evolution process. More specifically, our result holds under the following hypotheses:

- The fluid is confined to an arbitrary domain $\Omega \subset \mathbb{R}^d$, $d=2,3$.
- The mass density of the bodies is irrelevant. The only restriction concerns their shape. Specifically, we suppose

$$D_\varepsilon \equiv \max_{i=1, \dots, N} \{\text{diam}[\mathcal{S}_\varepsilon^i]\} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,$$

$$0 < \lambda D_\varepsilon^\beta \leq |\mathcal{S}_\varepsilon| \text{ as } \varepsilon \rightarrow 0, \quad d \leq \beta < \begin{cases} 15 \text{ if } d = 3, \\ \text{arbitrary finite if } d = 2, \end{cases} \quad (1.1)$$

for some $\lambda > 0$ independent of ε .

- The result is global in time and holds in the class of weak solutions and for any finite-energy initial data.

Very roughly indeed, we may conclude that the effect of a finite number of rigid bodies is negligible as soon as their diameters are small whereas their mass densities are irrelevant. In addition, hypothesis (1.1) allows different bodies to shrink to zero in different order of scaling.

Let us mention that there are some results in the context of rigid obstacles in viscous Newtonian fluids. The flow around a small rigid obstacle was studied by Iftimie et al. [12]. Lacave [17] studies the limit of a viscous fluid flow in the exterior of a thin obstacle shrinking to a curve.

We use the framework of *weak solutions* in the spirit of Judakov [16], Gunzburger, Lee and Seregin [15] or Galdi [13]. The relevant existence theory for the fluid structure interaction problem was developed by San Martin, Starovoitov, and Tucsnak [19] for $d = 2$ and in [6] for $d = 3$. In both cases, the solutions are global-in-time and allow for possible collisions of the bodies and also collisions with the domain boundary.

Similarly to the companion paper [8] concerning compressible fluids, our approach is based on a new *restriction operator* that assigns a given function its “projection” on the space of rigid motions attached to the bodies. We point out that accommodation of several bodies needs a nontrivial modification of the construction presented in [8]. In addition, we show new “negative norm” estimates of the restriction operator that are of independent interest and can be used in problems involving compressible flows.

The new restriction operator improves considerably the error estimates necessary to perform the asymptotic limit. Another new ingredient is that we use the dissipation energy rather than the energy itself to obtain suitable bounds on the rigid body translation velocity. This is why the result is independent of the mass densities of the bodies.

The paper is organized as follows. In Section 2, we formulate the problem and state our main result. Next, in Section 3, we derive uniform bounds on the sequence of solutions to the fluid-structure interaction problem independent of the scaling parameter. In Section 4, we introduce a restriction operator suitable for modifying the test function in the weak formulation of the problem. The convergence analysis and the proof of the main result are done in Section 5.

2 Problem formulation, main result

We consider a domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, containing a viscous, incompressible Newtonian fluid. Accordingly the fluid velocity \mathbf{u} satisfies the Navier–Stokes system of equations

$$\operatorname{div}_x \mathbf{u} = 0, \tag{2.1}$$

$$\partial_t \mathbf{u} + \operatorname{div}_x(\mathbf{u} \otimes \mathbf{u}) + \nabla_x \Pi = \operatorname{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{u}) + \mathbf{g}, \tag{2.2}$$

$$\mathbb{S}(\mathbb{D}_x \mathbf{u}) = \mu \mathbb{D}_x \mathbf{u}, \quad \mathbb{D}_x \mathbf{u} = \frac{\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}}{2}, \quad \mu > 0, \tag{2.3}$$

where Π is the pressure and the function \mathbf{g} denotes an external volume force.

The rigid bodies are represented by compact connected sets $\mathcal{S}^i \subset \mathbb{R}^d$, $i = 1, \dots, N$. We suppose the bodies are immersed in the fluid and their position at a time $t > 0$ is determined by a family of affine isometries $(\sigma^i(t))_{t \geq 0}$,

$$\mathcal{S}^i(t) = \sigma^i[\mathcal{S}^i], \quad \sigma^i(t)x = \mathbb{O}_i(t)x + \mathbf{h}_i(t), \quad \mathbb{O}_i \in SO(d), \quad t \geq 0, \quad i = 1, \dots, N.$$

In addition, we introduce the associated rigid velocity fields,

$$\mathbf{u}_{S^i}(t, x) = \mathbf{Y}_i + \mathbb{Q}_i(x - \mathbf{h}_i), \quad \mathbf{Y}_i(t) = \frac{d}{dt}\mathbf{h}_i(t), \quad \mathbb{Q}_i(t) = \frac{d}{dt}\mathbb{O}_i(t) \circ \mathbb{O}_i^{-1}(t). \quad (2.4)$$

Finally, we identify the fluid region

$$\Omega_f(t) = \Omega \setminus \cup_{i=1}^N \mathcal{S}^i(t), \quad Q_f = \left\{ (t, x) \mid t \in (0, T), x \in \Omega_f(t) \right\}$$

2.1 Weak formulation

We suppose that the rigid bodies are immersed in the fluid. As the fluid is viscous, a natural working hypothesis asserts that both the velocity and the momentum coincide on the body boundary, see e.g. Galdi [13]. Accordingly, a suitable weak formulation of the fluid–structure interaction problem (see [6]) is based on the quantities $[\rho, \mathbf{u}, (\mathbf{h}^1 \cdots \mathbf{h}^N), (\mathbb{O}^1 \cdots \mathbb{O}^N)]$. In this context, it is convenient to consider both the mass density $\rho = \rho(t, x)$ and the velocity $\mathbf{u} = \mathbf{u}(t, x)$ as functions defined for all $x \in \mathbb{R}^d$.

2.1.1 Regularity

- The mass density ρ is non–negative,

$$\rho \in L^\infty((0, T) \times \mathbb{R}^d) \cap C^1([0, T]; L^1(\Omega)). \quad (2.5)$$

- The velocity \mathbf{u} belongs to the Ladyzhenskaya class

$$\mathbf{u} \in L^\infty(0, T; L^2(\mathbb{R}^d; \mathbb{R}^d)) \cap L^2(0, T; W^{1,2}(\mathbb{R}^d; \mathbb{R}^d)), \quad \operatorname{div}_x \mathbf{u} = 0. \quad (2.6)$$

- The affine isometries are Lipschitz continuous in time,

$$\mathbf{h}_i \in W^{1,\infty}(0, T; \mathbb{R}^d), \quad \mathbb{O}_i \in W^{1,\infty}(0, T; SO(d)), \quad i = 1, \dots, N. \quad (2.7)$$

2.1.2 Compatibility

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$$\begin{aligned} \rho(t, x) &= 1 \text{ for a.a. } x \in \Omega_f(t), \\ \rho(t, x) &= \rho_{S^i} \text{ for a.a. } x \in \mathcal{S}^i(t), \quad i = 1, \dots, N \end{aligned} \quad (2.8)$$

for any $t \in [0, T]$.

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$$\mathbf{u}(t, \cdot) \in W_0^{1,2}(\Omega; \mathbb{R}^d) \quad (2.9)$$

$$(\mathbf{u} - \mathbf{u}_{S^i})(t, \cdot) \in W_0^{1,2}(\mathbb{R}^d \setminus \mathcal{S}^i(t); \mathbb{R}^d) \quad (2.10)$$

$$\text{for a.a. } t \in (0, T), \quad i = 1, \dots, N. \quad (2.11)$$

2.1.3 Mass conservation

The equation of continuity

$$\int_0^T \int_{\mathbb{R}^d} [\rho \partial_t \varphi + \rho \mathbf{u} \cdot \nabla_x \varphi] dt = - \int_{\mathbb{R}^d} \rho_0 \varphi(0, \cdot) \quad (2.12)$$

holds for any $\varphi \in C_c^1([0, T) \times \mathbb{R}^d)$.

2.1.4 Momentum balance

The momentum equation

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d} [\rho \mathbf{u} \cdot \partial_t \varphi + \rho \mathbf{u} \otimes \mathbf{u} : \mathbb{D}_x \varphi] dt \\ &= \int_0^T \int_{\mathbb{R}^d} [\mathbb{S}(\mathbb{D}_x \mathbf{u}) : \mathbb{D}_x \varphi - \rho \mathbf{g} \cdot \varphi] dt - \int_{\mathbb{R}^d} \rho_0 \mathbf{u}_0 \cdot \varphi \end{aligned} \quad (2.13)$$

holds for any function $\varphi \in C_c^1([0, T) \times \Omega; \mathbb{R}^d)$, $\operatorname{div}_x \varphi = 0$ satisfying

$$\mathbb{D}_x \varphi(t, \cdot) = 0 \text{ on an open neighbourhood of } \mathcal{S}^i(t) \text{ for any } t \in [0, T), i = 1, \dots, N. \quad (2.14)$$

2.1.5 Total energy dissipation

The energy inequality

$$\int_{\mathbb{R}^d} \rho |\mathbf{u}|^2(\tau, \cdot) + \int_0^\tau \int_{\mathbb{R}^d} \mathbb{S}(\mathbb{D}_x \mathbf{u}) : \mathbb{D}_x \mathbf{u} \leq \int_{\mathbb{R}^d} \rho_0 |\mathbf{u}_0|^2 + \int_0^\tau \int_{\mathbb{R}^d} \rho \mathbf{g} \cdot \mathbf{u} dt \quad (2.15)$$

holds for a.a. $\tau \in (0, T)$.

2.2 Main result

We are ready to state our main result.

Theorem 2.1 (Asymptotic limit). *Let $\mathcal{S}_\varepsilon^i \subset \mathbb{R}^d$, $i = 1, \dots, N$, $d = 2, 3$ be a family of compact connected sets satisfying*

$$\begin{aligned} D_\varepsilon &\equiv \max_{i=1, \dots, N} \operatorname{diam}[\mathcal{S}_\varepsilon^i] \rightarrow 0 \text{ as } \varepsilon \rightarrow 0, \\ 0 &< \lambda D_\varepsilon^\beta \leq |\mathcal{S}_\varepsilon^i| \text{ as } \varepsilon \rightarrow 0, \quad d \leq \beta < \begin{cases} 15 & \text{if } d = 3, \\ \text{arbitrary finite} & \text{if } d = 2, \end{cases} \end{aligned} \quad (2.16)$$

for some $\lambda > 0$ independent of ε , $i = 1, \dots, N$. Suppose the rigid body densities $\rho_{\mathcal{S}^i}^\varepsilon$, $i = 1, \dots, N$ are constant,

$$0 < \rho_{\mathcal{S}^i}^\varepsilon \leq \bar{\rho} \text{ uniformly for } \varepsilon \rightarrow 0, \quad i = 1, \dots, N \quad (2.17)$$

for some $\bar{\rho} > 0$ independent of ε .

Let $[\rho^\varepsilon, \mathbf{u}^\varepsilon, (\mathbf{h}_i^\varepsilon, \dots, \mathbf{h}_N^\varepsilon), (\mathbb{O}_1^\varepsilon, \dots, \mathbb{O}_N^\varepsilon)]_{\varepsilon > 0}$ be the associated sequence of weak solutions to the fluid–structure interaction problem specified in Section 2.1, with the initial data $(\rho_0^\varepsilon, \mathbf{u}_0^\varepsilon)_{\varepsilon > 0}$ satisfying

$$\rho_0^\varepsilon(x) = \begin{cases} \rho_{\mathcal{S}^i}^\varepsilon & \text{if } x \in \mathcal{S}_\varepsilon^i(0), \quad i = 1, \dots, N, \\ 1 & \text{otherwise,} \end{cases} \quad (2.18)$$

$$\mathbf{u}_0^\varepsilon \in L^2(\mathbb{R}^d; \mathbb{R}^d), \quad \rho_0^\varepsilon \mathbf{u}_0^\varepsilon \rightarrow \mathbf{u}_0 \text{ in } L^2(\mathbb{R}^d; \mathbb{R}^d), \quad (2.19)$$

where

$$\mathbf{u}_0 \in L^2(\Omega; \mathbb{R}^d), \quad \operatorname{div}_x \mathbf{u}_0 = 0. \quad (2.20)$$

Finally, suppose

$$\mathbf{g} = \mathbf{g}(x), \quad \mathbf{g} \in L^2 \cap L^\infty(\mathbb{R}^d; \mathbb{R}^d). \quad (2.21)$$

Then, up to a suitable subsequence,

$$\begin{aligned} \mathbf{u}^\varepsilon &\rightarrow \mathbf{u} \text{ weakly in } L^2(0, T; W_0^{1,2}(\Omega; \mathbb{R}^d)), \\ &\text{and in } L^2((0, T) \times \Omega; \mathbb{R}^d), \end{aligned} \quad (2.22)$$

where \mathbf{u} is a weak solution of the Navier–Stokes system (2.1)–(2.3), with the initial data \mathbf{u}_0 satisfying the energy inequality

$$\int_\Omega |\mathbf{u}|^2(\tau, \cdot) \, dx + \int_0^\tau \int_\Omega \mathbb{S}(\mathbb{D}_x \mathbf{u}) : \mathbb{D}_x \mathbf{u} \, dx \leq \int_\Omega |\mathbf{u}_0|^2 \, dx + \int_0^\tau \int_\Omega \mathbf{g} \cdot \mathbf{u} \, dx \, dt \quad (2.23)$$

for a.a. $\tau \in (0, T)$.

Remark 2.2. In fact, hypothesis (2.17) requiring uniform bounds on the body density is not restrictive. Indeed, if $\rho_{\mathcal{S}^i}^\varepsilon$ contains an unbounded sequence, then the relevant results have been already obtained by He and Iftimie [10], [11]. The main novelty of the present result is allowing the body density to be asymptotically small.

Remark 2.3. Without loss of generality, we set $D_\varepsilon = \varepsilon$. In accordance with hypothesis (2.16), there are balls $B_{r\varepsilon}[\mathbf{y}_i^\varepsilon]$ centred at $\mathbf{y}_i^\varepsilon \in \mathbb{R}^d$ and of radius $r\varepsilon$ such that

$$\mathcal{S}_\varepsilon^i \subset B_{r\varepsilon}[\mathbf{y}_i^\varepsilon].$$

In addition, choosing $r > 0$ large enough, we may suppose

$$\mathbf{y}_i^\varepsilon = \frac{1}{|\mathcal{S}_\varepsilon^i|} \int_{\mathcal{S}_\varepsilon^i} x \, dx$$

coincided with the barycenter of the body $\mathcal{S}_\varepsilon^i$. Accordingly, we may infer

$$\mathcal{S}_\varepsilon^i(t) \subset B_{r\varepsilon}[\mathbf{h}_i^\varepsilon(t)], \quad t \in [0, T], \quad i = 1, \dots, N, \quad (2.24)$$

where $(\mathbf{h}_i^\varepsilon)_{\varepsilon>0}$ are the rigid translation of the barycenters of the body. Finally, again without loss of generality, we suppose $r = 1$ therefore $\mathcal{S}_\varepsilon^i(t) \subset B_\varepsilon[\mathbf{h}_i^\varepsilon(t)], \quad t \in [0, T], \quad i = 1, \dots, N$.

It follows from (2.24) (with $r = 1$) and the weak formulation of the momentum equation (2.13) that the integral identity

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d} \left[\rho^\varepsilon \mathbf{u}^\varepsilon \cdot \partial_t \boldsymbol{\varphi} + \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon : \mathbb{D}_x \boldsymbol{\varphi} \right] \, dx \, dt \\ &= \int_0^T \int_{\mathbb{R}^d} \left[\mathbb{S}(\mathbb{D}_x \mathbf{u}^\varepsilon) : \mathbb{D}_x \boldsymbol{\varphi} - \rho^\varepsilon \mathbf{g} \cdot \boldsymbol{\varphi} \right] \, dx \, dt - \int_{\mathbb{R}^d} \rho_0^\varepsilon \mathbf{u}_0^\varepsilon \cdot \boldsymbol{\varphi} \, dx \end{aligned} \quad (2.25)$$

holds for any function $\boldsymbol{\varphi} \in C_c^1([0, T] \times \Omega; \mathbb{R}^d)$, $\operatorname{div}_x \boldsymbol{\varphi} = 0$ satisfying

$$\nabla_x \boldsymbol{\varphi}(t, \cdot) = 0 \text{ on } B_\varepsilon[\mathbf{h}_i^\varepsilon(t)] \text{ for any } t \in [0, T], \quad i = 1, \dots, N. \quad (2.26)$$

Obviously, the test functions satisfying (2.26) are constant on the shifted balls containing the rigid bodies. It is worth noting that the class of test functions (2.26) is much larger than its counterpart for the obstacle problem, where the test functions are supposed to vanish on the obstacle.

3 Uniform bounds

In this section, we derive suitable uniform bounds necessary for passing to the limit in the weak formulation of the momentum equation (2.13). All bounds used in the limit passage follow from the energy inequality (2.15).

Let $[\rho^\varepsilon, \mathbf{u}^\varepsilon, (\mathbf{h}_1^\varepsilon, \dots, \mathbf{h}_N^\varepsilon), (\mathbb{O}_1^\varepsilon, \dots, \mathbb{O}_N^\varepsilon)]_{\varepsilon>0}$ be the associated sequence of weak solutions to the fluid–structure interaction problem satisfying (2.5)–(2.15). As \mathbf{g} satisfies the hypothesis (2.21), we have

$$\int_{\mathbb{R}^d} \rho^\varepsilon \mathbf{g} \cdot \mathbf{u}^\varepsilon \, dx = \sum_{i=1}^N \int_{\mathcal{S}_\varepsilon^i(t)} \rho_{\mathcal{S}^i}^\varepsilon \mathbf{g} \cdot \mathbf{u}^\varepsilon \, dx + \int_{\mathbb{R}^d \setminus \cup_{i=1}^N \mathcal{S}_\varepsilon^i(t)} \mathbf{g} \cdot \mathbf{u}^\varepsilon$$

where, in accordance with the hypothesis (2.17), (2.21):

$$\int_{\mathcal{S}_\varepsilon^i(t)} \rho_{\mathcal{S}^i}^\varepsilon \mathbf{g} \cdot \mathbf{u}^\varepsilon \, dx \lesssim \int_{\mathcal{S}_\varepsilon^i(t)} \rho_{\mathcal{S}^i}^\varepsilon \, dx + \int_{\mathbb{R}^d} \rho^\varepsilon |\mathbf{u}^\varepsilon|^2 \, dx \lesssim 1 + \int_{\mathbb{R}^d} \rho^\varepsilon |\mathbf{u}^\varepsilon|^2 \, dx, \quad i = 1, \dots, N.$$

Here and hereafter, the symbol $a \lesssim b$ means $a \leq cb$, where c is a generic constant independent of the scaling parameter ε . Similarly, by virtue of hypothesis (2.21) and Cauchy–Schwartz inequality,

$$\int_{\mathbb{R}^d \setminus \cup_{i=1}^N \mathcal{S}_\varepsilon^i(t)} \mathbf{g} \cdot \mathbf{u}^\varepsilon \leq c(\delta) + \delta \int_{\mathbb{R}^d} |\mathbf{u}^\varepsilon|^2 \, dx$$

for any $\delta > 0$. In addition, using Korn–Poincaré inequality, we get

$$\int_{\mathbb{R}^d} |\mathbf{u}^\varepsilon|^2 \, dx \lesssim \int_{\mathbb{R}^d} \mathbb{S}(\mathbb{D}_x \mathbf{u}^\varepsilon) : \mathbb{D}_x \mathbf{u}^\varepsilon \, dx + \int_{\mathbb{R}^d} \rho^\varepsilon |\mathbf{u}^\varepsilon|^2 \, dx. \quad (3.1)$$

Finally, we apply Gronwall’s argument to the energy inequality (2.15) and deduce the following bounds

$$\operatorname{ess\,sup}_{t \in (0, T)} \int_{\mathbb{R}^d} \rho^\varepsilon |\mathbf{u}^\varepsilon|^2(t, \cdot) \, dx \lesssim 1, \quad \int_0^T \int_{\mathbb{R}^d} \mathbb{S}(\mathbb{D}_x \mathbf{u}^\varepsilon) : \mathbb{D}_x \mathbf{u}^\varepsilon \, dx \lesssim 1, \quad (3.2)$$

which, together with (3.1), yields

$$\int_0^T \int_{\mathbb{R}^d} |\mathbf{u}^\varepsilon|^2 + |\nabla_x \mathbf{u}^\varepsilon|^2 \, dx \, dt \lesssim 1. \quad (3.3)$$

3.1 Bounds on the rigid velocity $d = 2$

In view of (2.10), $\mathbf{u}^\varepsilon(t, \cdot) = \mathbf{u}_{\mathcal{S}_\varepsilon^i}^\varepsilon(t, \cdot)$ a.a. on $\mathcal{S}_\varepsilon^i(t)$ for a.a. $t \in (0, T)$, $i = 1, \dots, N$. As the rigid body densities are constant, the translational and rotational velocities are orthogonal on $\mathcal{S}_\varepsilon^i(t)$, specifically

$$\int_{\mathcal{S}_\varepsilon^i(t)} \mathbf{Y}_i^\varepsilon(t) \cdot \mathbb{Q}_i^\varepsilon(\cdot - \mathbf{h}_i^\varepsilon(t)) \, dx = 0.$$

Consequently, we have

$$\int_{\mathcal{S}_\varepsilon^i(t)} |\mathbf{u}^\varepsilon|^2 \, dx = \int_{\mathcal{S}_\varepsilon^i(t)} |\mathbf{u}_{\mathcal{S}_\varepsilon^i}^\varepsilon|^2 \, dx = \int_{\mathcal{S}_\varepsilon^i(t)} |\mathbf{Y}_i^\varepsilon(t) + \mathbb{Q}_i^\varepsilon(\cdot - \mathbf{h}_i^\varepsilon(t))|^2 \geq \int_{\mathcal{S}_\varepsilon^i(t)} |\mathbf{Y}_i^\varepsilon(t)|^2 \, dx \quad (3.4)$$

If $d = 2$, the standard Sobolev embedding relation yields

$$\|\mathbf{u}^\varepsilon\|_{L^q(\mathcal{S}_\varepsilon^i(t); \mathbb{R}^2)} \leq c(q) \|\mathbf{u}^\varepsilon\|_{W^{1,2}(\mathbb{R}^2; \mathbb{R}^2)} \text{ for any finite } 1 \leq q < \infty,$$

where the constant $c(q)$ is *independent* of ε , $i = 1, \dots, N$.

Next, by (3.4) and interpolation,

$$|\mathcal{S}_\varepsilon^i|^{\frac{1}{2}} |\mathbf{Y}_i^\varepsilon| \leq \|\mathbf{u}^\varepsilon\|_{L^2(\mathcal{S}_\varepsilon^i(t); \mathbb{R}^2)} \leq \|\mathbf{u}^\varepsilon\|_{L^q(\mathcal{S}_\varepsilon^i(t); \mathbb{R}^2)} |\mathcal{S}_\varepsilon^i|^{\frac{1}{2} - \frac{1}{q}} \leq c(q) \|\mathbf{u}^\varepsilon\|_{W^{1,2}(\mathbb{R}^2; \mathbb{R}^2)} |\mathcal{S}_\varepsilon^i|^{\frac{1}{2} - \frac{1}{q}}.$$

Consequently, in view of the bound (3.3), we may infer

$$\|\mathbf{Y}_i^\varepsilon\|_{L^2(0, T)} \lesssim |\mathcal{S}_\varepsilon^i|^{-\frac{1}{q}} \text{ for any finite } 1 \leq q < \infty, \quad i = 1, \dots, N, \quad d = 2. \quad (3.5)$$

3.2 Bounds on the rigid velocity $d = 3$

Now, we repeat the arguments of the previous section with $q = 6$ obtaining

$$\|\mathbf{Y}_i^\varepsilon\|_{L^2(0,T)} \lesssim |\mathcal{S}_\varepsilon^i|^{-\frac{1}{6}}, \quad i = 1, \dots, N, \quad d = 3. \quad (3.6)$$

4 Restriction operator

A suitable choice of the restriction operator is crucial in our analysis. In contrast to the overwhelming amount of the available literature, where the test functions are modified to vanish on the body, we take advantage of the freedom allowed by (2.25), (2.26) and replace the function on a ball of radius ε by its integral average over that ball. The same idea has already been exploited in [8], where detailed proofs of the statements collected below are available.

Consider a function

$$\begin{aligned} H &\in C^\infty(\mathbb{R}), \quad 0 \leq H(Z) \leq 1, \quad H'(Z) = H'(1-Z) \text{ for all } Z \in \mathbb{R}, \\ H(Z) &= 0 \text{ for } -\infty < Z \leq \frac{1}{4}, \quad H(Z) = 1 \text{ for } \frac{3}{4} \leq Z < \infty \end{aligned}$$

For $\varphi \in L^1_{\text{loc}}(\mathbb{R}^d)$, $r > 0$ we define E_r ,

$$E_r[\varphi](x) = \frac{1}{|B_r|} \int_{B_r} \varphi \, dz \, H\left(2 - \frac{|x|}{r}\right) + \varphi(x) H\left(\frac{|x|}{r} - 1\right). \quad (4.1)$$

where B_r denotes the ball centred at zero with the radius $r > 0$. Obviously, the operator E_r maps the space $C_c^\infty(\mathbb{R}^d)$ into itself, and, moreover

$$\|E_r[\varphi]\|_{L^p(\mathbb{R}^d)} \lesssim \|\varphi\|_{L^p(\mathbb{R}^d)}, \quad (4.2)$$

$$\|\nabla_x E_r[\varphi]\|_{L^p(\mathbb{R}^d; \mathbb{R}^d)} \lesssim \|\nabla_x \varphi\|_{L^p(\mathbb{R}^d; \mathbb{R}^d)} \text{ for any } 1 \leq p \leq \infty, \quad (4.3)$$

uniformly for $0 < r \leq 1$, see [8, Section 4.1].

4.1 Restriction operator in the class of solenoidal functions

The operator E_r does not preserve solenoidality if applied componentwise to a solenoidal function. To fix this problem, we introduce the operator

$$\mathbf{R}_r[\varphi] = E_r[\varphi] - \mathcal{B}_{2r,r} \left[\text{div}_x E_r[\varphi] \Big|_{B_{2r} \setminus B_r} \right], \quad (4.4)$$

where $\mathcal{B}_{2r,r}$ is a suitable branch of the inverse of the divergence operator defined on the annulus $B_{2r} \setminus B_r$. A possible construction of \mathcal{B} was proposed by Bogovskii [1] and later elaborated by Galdi [14] followed by Diening et al. [4], Geißert [9] et al. among others. In our setting, the operator $\mathcal{B}_{2r,r}$ can be constructed by a simple scaling argument:

- We start with the annulus

$$O = B_2 \setminus B_1 = \left\{ y \in \mathbb{R}^d \mid 1 < |y| < 2 \right\}.$$

Following Galdi [14, Chapter III, Section III.3], Diening et al. [4], we construct a linear operator \mathcal{B}_O defined *a priori* on smooth functions $g \in C_c^\infty(O)$, $\int_O g \, dy = 0$, enjoying the following properties:

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$$\mathcal{B}_O[g] \in C_c^\infty(O; \mathbb{R}^d), \quad \operatorname{div}_y \mathcal{B}_O[g] = g;$$

-

$$\|\nabla_y \mathcal{B}_O[g]\|_{L^p(O; \mathbb{R}^{d \times d})} \leq c_I(p, O) \|g\|_{L^p(O)} \text{ for any } 1 < p < \infty. \quad (4.5)$$

Thanks to this property, $\mathcal{B}_O[g]$ can be extended as a bounded linear operator on the space

$$L_0^p(O) = \left\{ g \in L^p(O) \mid \int_O g \, dy = 0 \right\}$$

ranging in $W_0^{1,p}(O, \mathbb{R}^d)$.

- If, in addition, the function g can be written as $g = \operatorname{div}_y \mathbf{f}$, where $\mathbf{f} \in L^q(O; \mathbb{R}^d)$ satisfies

$$\mathbf{f} \cdot \mathbf{n}|_{\partial O} = 0, \quad (4.6)$$

then

$$\|\mathcal{B}_O[\operatorname{div}_y \mathbf{f}]\|_{L^q(O; \mathbb{R}^d)} \leq c_{II}(q, O) \|\mathbf{f}\|_{L^q(O; \mathbb{R}^d)}, \quad 1 < q < \infty. \quad (4.7)$$

- If $g \in W_0^{l,p}$, then

$$\|\mathcal{B}_O[g]\|_{W_0^{l+1,p}(O; \mathbb{R}^d)} \leq c(l, p, O) \|g\|_{W_0^{l,p}(O; \mathbb{R}^d)}, \quad l = 0, 1, \dots, \quad 1 < p < \infty,$$

see Galdi [14, Theorem III.3.3].

- An analogue of \mathcal{B}_O on the domain $B_{2r} \setminus B_r$ can be now defined via scaling. To a given function g defined on $B_{2r} \setminus B_r$ and satisfying $\int_{B_{2r} \setminus B_r} g \, dx = 0$, we associate a function \tilde{g} on O ,

$$\tilde{g}(y) = g(ry), \quad y \in O.$$

We set

$$\mathcal{B}_{2r,r}[g](x) = r \mathcal{B}_O[\tilde{g}]\left(\frac{x}{r}\right), \quad x \in B_{2r} \setminus B_r.$$

Seeing that

$$\operatorname{div}_x \mathcal{B}_{2r,r}[g](x) = \operatorname{div}_y \mathcal{B}_O[\tilde{g}]\left(\frac{x}{r}\right), \quad \nabla_x \mathcal{B}_{2r,r}[g](x) = \nabla_y \mathcal{B}_O[\tilde{g}]\left(\frac{x}{r}\right),$$

and

$$\mathcal{B}_{2r,r}[\operatorname{div}_x \mathbf{f}](x) = \mathcal{B}_O[\operatorname{div}_y \tilde{\mathbf{f}}]\left(\frac{x}{r}\right)$$

we easily observe that $\mathcal{B}_{2r,r}$ shares all properties of \mathcal{B}_O on the domain $B_{2r} \setminus B_r$. Moreover, the bounds (4.5), (4.7) are satisfied with the same with the same constants $c_I(p, O)$, $c_{II}(p, O)$:

$$\begin{aligned} \|\nabla_x \mathcal{B}_{2r,r}[g]\|_{L^p(B_{2r} \setminus B_r; \mathbb{R}^{d \times d})} &\leq c_I(p) \|g\|_{L^p(B_{2r} \setminus B_r)} \text{ whenever } \int_{B_{2r} \setminus B_r} g \, dx = 0 \\ &\text{for any } 1 < p < \infty, \end{aligned} \quad (4.8)$$

and

$$\begin{aligned} \|\mathcal{B}_{2r,r}[\operatorname{div}_x \mathbf{f}]\|_{L^q(B_{2r} \setminus B_r; \mathbb{R}^d)} &\leq c_{II}(q) \|\mathbf{f}\|_{L^q(B_{2r} \setminus B_r; \mathbb{R}^d)}, \\ \text{whenever } \mathbf{f} \cdot \mathbf{n}|_{\partial(B_{2r} \setminus B_r)} &= 0 \text{ for any } 1 < q < \infty. \end{aligned} \quad (4.9)$$

with the constants c_I , c_{II} independent of $r > 0$.

Recalling the definition of the operator E_r as in (4.1), we have

$$E_r[\varphi] = \text{constant vector in } B_{\frac{5}{4}r} \setminus B_r, \quad E_r[\varphi] = \varphi \text{ in } B_{2r} \setminus B_{\frac{7}{4}r}. \quad (4.10)$$

In particular,

$$\int_{B_{2r} \setminus B_r} \operatorname{div}_x E_r[\varphi] \, dx = 0 \text{ whenever } \operatorname{div}_x \varphi = 0.$$

Therefore, going back to formula (4.4), \mathbf{R}_r is well-defined for solenoidal functions. Moreover, obviously,

$$\operatorname{div}_x \mathbf{R}_r[\varphi] = \operatorname{div}_x \varphi.$$

Using the uniform bound (4.8) together with (4.3) we deduce

$$\|\nabla_x \mathbf{R}_r[\varphi]\|_{L^p(\mathbb{R}^d, \mathbb{R}^{d \times d})} \lesssim \|\nabla_x \varphi\|_{L^p(\mathbb{R}^d, \mathbb{R}^{d \times d})}, \quad 1 < p < \infty \quad (4.11)$$

uniformly for $0 < r \leq 1$.

Finally, we claim the L^q -bound

$$\|\mathbf{R}_r[\varphi]\|_{L^q(\mathbb{R}^d, \mathbb{R}^d)} \lesssim \|\varphi\|_{L^q(\mathbb{R}^d, \mathbb{R}^d)}, \quad 1 < q < \infty \quad (4.12)$$

uniformly for $0 < r \leq 1$. Note that this *is not* a direct consequence of the “negative” estimates stated in (4.9) as $E_r[\boldsymbol{\varphi}] \cdot \mathbf{n}$ may not vanish on $\partial(B_{2r} \setminus B_r)$. To see (4.12), we first construct a family of cut-off functions

$$\psi_r \in C_c^\infty(B_{2r} \setminus B_r), \quad 0 \leq \psi_r \leq 1, \quad \psi_r(x) = 1 \text{ for all } \frac{5}{4}r \leq |x| \leq \frac{7}{4}r, \quad |\nabla_x \psi_r| \lesssim \frac{1}{r}.$$

In accordance with (4.10), we have

$$\operatorname{div}_x E_r[\boldsymbol{\varphi}] = \psi_r \operatorname{div}_x E_r[\boldsymbol{\varphi}] \text{ provided } \operatorname{div}_x \boldsymbol{\varphi} = 0.$$

Accordingly,

$$\mathcal{B}_{2r,r}[\operatorname{div}_x E_r[\boldsymbol{\varphi}]] = \mathcal{B}_{2r,r}[\psi_r \operatorname{div}_x E_r[\boldsymbol{\varphi}]] = \mathcal{B}_{2r,r}[\operatorname{div}_x(\psi_r E_r[\boldsymbol{\varphi}])] - \mathcal{B}_{2r,r}[\nabla_x \psi_r \cdot E_r[\boldsymbol{\varphi}]].$$

On the one hand, as $\psi_r \in C_c^\infty(B_{2r} \setminus B_r)$, we are allowed to apply the negative bound (4.9) to obtain

$$\|\mathcal{B}_{2r,r}[\operatorname{div}_x(\psi_r E_r[\boldsymbol{\varphi}])]\|_{L^q(\mathbb{R}^d; \mathbb{R}^d)} \lesssim \|\psi_r E_r[\boldsymbol{\varphi}]\|_{L^q(\mathbb{R}^d; \mathbb{R}^d)} \lesssim \|\boldsymbol{\varphi}\|_{L^q(\mathbb{R}^d; \mathbb{R}^d)}. \quad (4.13)$$

On the other hand, the gradient bounds (4.8) yield

$$\|\nabla_x \mathcal{B}_{2r,r}[\nabla_x \psi_r \cdot E_r[\boldsymbol{\varphi}]]\|_{L^q(\mathbb{R}^d; \mathbb{R}^{d \times d})} \lesssim \|\nabla_x \psi_r \cdot E_r[\boldsymbol{\varphi}]\|_{L^q(\mathbb{R}^d)} \lesssim \frac{1}{r} \|\boldsymbol{\varphi}\|_{L^q(\mathbb{R}^d; \mathbb{R}^d)}. \quad (4.14)$$

Next, by virtue of Poincaré inequality,

$$\|\mathcal{B}_{2r,r}[\nabla_x \psi_r \cdot E_r[\boldsymbol{\varphi}]]\|_{L^q(\mathbb{R}^d; \mathbb{R}^d)} \lesssim r \|\nabla_x \mathcal{B}_{2r,r}[\nabla_x \psi_r \cdot E_r[\boldsymbol{\varphi}]]\|_{L^q(\mathbb{R}^d; \mathbb{R}^{d \times d})}. \quad (4.15)$$

Combining the estimates (4.13)–(4.15) we obtain

$$\|\mathcal{B}_{2r,r}[\operatorname{div}_x E_r[\boldsymbol{\varphi}]]\|_{L^q(\mathbb{R}^d; \mathbb{R}^d)} \lesssim \|\boldsymbol{\varphi}\|_{L^q(\mathbb{R}^d; \mathbb{R}^d)},$$

which, together with (4.2) yields the desired conclusion (4.12).

4.2 Space shift

For $\mathbf{h} \in \mathbb{R}^d$, we set

$$\mathbf{R}_r(\mathbf{h})[\boldsymbol{\varphi}] = S_{-\mathbf{h}} \mathbf{R}_r[S_{\mathbf{h}}[\boldsymbol{\varphi}]], \quad (4.16)$$

where $S_{\mathbf{h}}$ is the shift operator given by

$$S_{\mathbf{h}}[f](x) = f(\mathbf{h} + x).$$

The basic properties of the operator $\mathbf{R}_r(\mathbf{h})$ are summarized below, cf. also [8, Proposition 5.1].

Proposition 4.1. *The operator $\mathbf{R}_r(\mathbf{h})$ is well defined for any function φ in the class*

$$\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^d), \quad \operatorname{div}_x \varphi = 0$$

and can be uniquely extended to functions

$$\varphi \in L_{\text{loc}}^p(\mathbb{R}^d; \mathbb{R}^d), \quad \operatorname{div}_x \varphi = 0 \text{ a.a.}$$

Moreover, the following holds:

•

$$\varphi \in C^\infty(\mathbb{R}^d; \mathbb{R}^d) \Rightarrow \mathbf{R}_r(\mathbf{h})[\varphi] \in C^\infty(\mathbb{R}^d; \mathbb{R}^d); \quad (4.17)$$

•

$$\operatorname{div}_x \mathbf{R}_r(\mathbf{h})[\varphi] = \operatorname{div}_x \varphi = 0; \quad (4.18)$$

•

$$\mathbf{R}_r(\mathbf{h})[\varphi] = \begin{cases} \frac{1}{|B_r(\mathbf{h})|} \int_{B_r(\mathbf{h})} \varphi \, dx & \text{if } |x - \mathbf{h}| < r, \\ \varphi(x) & \text{if } |x - \mathbf{h}| > 2r; \end{cases} \quad (4.19)$$

•

$$\mathbf{R}_r(\mathbf{h})[\varphi] = \varphi \text{ whenever } \varphi \text{ is a constant vector on } B_{2r}(\mathbf{h}); \quad (4.20)$$

•

$$\|\mathbf{R}_r(\mathbf{h})[\varphi]\|_{L^p(\mathbb{R}^d; \mathbb{R}^d)} \lesssim \|\varphi\|_{L^p(\mathbb{R}^d; \mathbb{R}^d)} \quad (4.21)$$

$$\|\nabla_x \mathbf{R}_r(\mathbf{h})[\varphi]\|_{L^p(\mathbb{R}^d; \mathbb{R}^{d \times d})} \lesssim \|\nabla_x \varphi\|_{L^p(\mathbb{R}^d; \mathbb{R}^{d \times d})} \quad (4.22)$$

for any $1 < p < \infty$ independently of $0 < r \leq 1$;

• *If φ is compactly supported, then so is $\mathbf{R}_r(\mathbf{h})[\varphi]$. Specifically,*

$$\operatorname{supp}[\mathbf{R}_r(\mathbf{h})[\varphi]] \subset \overline{\mathcal{U}_{2r}[\operatorname{supp}[\varphi]]} \quad (4.23)$$

where $\mathcal{U}_{2r}(O)$ denotes the $2r$ -neighbourhood of a set O .

Finally, we evaluate the differential of $\nabla_{\mathbf{h}} \mathbf{R}_r(\mathbf{h})[\varphi]$ for a given function φ . It follows from (4.17) that $\mathbf{R}_r(\mathbf{h})[\varphi]$ is a smooth function of x as long as φ is smooth and the differentiation can be performed in a direct manner. Consequently, we obtain

$$\nabla_{\mathbf{h}} \mathbf{R}_r(\mathbf{h})[\varphi] = \nabla_x (\mathbf{R}_r(\mathbf{h})[\varphi]) - \mathbf{R}_r(\mathbf{h})[\nabla_x \varphi]. \quad (4.24)$$

Note carefully that if φ is solenoidal, meaning $\operatorname{div}_x \varphi = 0$, then so is $\nabla_x \varphi$ (component-wise) and the right-hand side of (4.24) is well defined. Using the bounds (4.21), (4.22), the formula (4.24) can be extended to solenoidal functions $\varphi \in W_{\text{loc}}^{1,p}(\mathbb{R}^d; \mathbb{R}^d)$ by density argument.

4.3 Composition

As we are facing the N -body problem, it is convenient to consider the composition of N restriction operators

$$\mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_N)[\varphi] = \mathbf{R}_\varepsilon(\mathbf{h}_1) \circ \mathbf{R}_{5\varepsilon}(\mathbf{h}_2) \circ \dots \circ \mathbf{R}_{5^{N-1}\varepsilon}(\mathbf{h}_N)[\varphi] \quad (4.25)$$

for arbitrary $(\mathbf{h}_1, \dots, \mathbf{h}_N)$. In view of the property (4.18), the operator $\mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_N)[\varphi]$ is well defined for any solenoidal φ and

$$\operatorname{div}_x \mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_N)[\varphi] = 0.$$

The following result is crucial.

Lemma 4.2. *Suppose $\mathbf{h}_1, \dots, \mathbf{h}_N$ are N points in \mathbb{R}^d , $N \geq 1$. Let r_1, \dots, r_N be positive numbers,*

$$r_{n+1} \geq 5r_n, \quad n = 1, 2, \dots, N-1. \quad (4.26)$$

Then

$$\mathbf{R}_{r_1}(\mathbf{h}_1) \circ \dots \circ \mathbf{R}_{r_N}(\mathbf{h}_N)[\varphi] = \Lambda_i \text{ on } B_{r_1}(\mathbf{h}_i), \quad i = 1, \dots, N, \quad (4.27)$$

where Λ_i are constant vectors.

Proof. The proof can be done by induction with respect to N . In view of (4.19), the result obviously holds for $N = 1$.

Suppose we have already shown the conclusion for N points and consider

$$\mathbf{w} = \mathbf{R}_{r_1}(\mathbf{h}_1) \circ \dots \circ \mathbf{R}_{r_N}(\mathbf{h}_N) \circ \mathbf{R}_{r_{N+1}}(\mathbf{h}_{N+1})[\varphi].$$

In view of the induction hypothesis, it is enough to show

$$\mathbf{w} = \Lambda_{N+1} \text{ on } B_{r_1}(\mathbf{h}_{N+1}). \quad (4.28)$$

Denote

$$\mathbf{v} = \mathbf{R}_{r_2}(\mathbf{h}_2) \circ \dots \circ \mathbf{R}_{r_N}(\mathbf{h}_N) \circ \mathbf{R}_{r_{N+1}}(\mathbf{h}_{N+1})[\varphi]$$

Using again the induction hypothesis we get

$$\mathbf{v} = \Lambda_{N+1} \text{ on } B_{r_2}(\mathbf{h}_{N+1}). \quad (4.29)$$

Now, consider two complementary cases

$$\mathbf{h}_{N+1} \in \mathbb{R}^d \setminus B_{3r_1}(\mathbf{h}_1) \text{ or } \mathbf{h}_{N+1} \in B_{3r_1}(\mathbf{h}_1).$$

If $\mathbf{h}_{N+1} \in \mathbb{R}^d \setminus B_{3r_1}(\mathbf{h}_1)$, then it follows from (4.19)

$$\mathbf{w} = \mathbf{R}_{r_1}(\mathbf{h}_1)[\mathbf{v}] = \mathbf{v} \text{ on } B_{r_1}(\mathbf{h}_{N+1})$$

and the desired conclusion follows from (4.29) as $r_2 > r_1$.

If $\mathbf{h}_{N+1} \in B_{3r_1}(\mathbf{h}_1)$, then it follows from (4.29) and the hypothesis $r_2 \geq 5r_1$ that \mathbf{v} equals Λ_{N+1} on $B_{2r_1}(\mathbf{h}_1)$. Consequently, by virtue of (4.20),

$$\mathbf{w} = \mathbf{R}_{r_1}(\mathbf{h}_1)[\mathbf{v}] = \mathbf{v}$$

and the desired result follows again from (4.29). □

Let us summarize the properties of the operator $\mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_N)$ that can be easily deduced from Proposition 4.1 and Lemma 4.2.

Proposition 4.3. *The operator $\mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_N)$ is well defined for any function φ in the class*

$$\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^d), \quad \operatorname{div}_x \varphi = 0$$

and can be uniquely extended to functions

$$\varphi \in L_{\text{loc}}^p(\mathbb{R}^d; \mathbb{R}^d), \quad \operatorname{div}_x \varphi = 0 \text{ a.a.}$$

Moreover, the following holds:

- $$\varphi \in C^\infty(\mathbb{R}^d; \mathbb{R}^d) \Rightarrow \mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_N) \in C^\infty(\mathbb{R}^d; \mathbb{R}^d); \quad (4.30)$$

- $$\operatorname{div}_x \mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_N)[\varphi] = \operatorname{div}_x \varphi = 0; \quad (4.31)$$

- $$\mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_N)[\varphi] = \Lambda_i - \text{a constant vector on } B_\varepsilon(\mathbf{h}_i), \quad i = 1, \dots, N; \quad (4.32)$$

- $$\mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_N)[\varphi](x) = \varphi(x) \text{ whenever } x \in \mathbb{R}^d \setminus \cup_{i=1}^N B_{2.5^{i-1}\varepsilon}(\mathbf{h}_i); \quad (4.33)$$

- $$\|\mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_N)[\varphi]\|_{L^p(\mathbb{R}^d; \mathbb{R}^d)} \leq c(p, N) \|\varphi\|_{L^p(\mathbb{R}^d; \mathbb{R}^d)} \quad (4.34)$$

$$\|\nabla_x \mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_N)[\varphi]\|_{L^p(\mathbb{R}^d; \mathbb{R}^{d \times d})} \leq c(p, N) \|\nabla_x \varphi\|_{L^p(\mathbb{R}^d; \mathbb{R}^{d \times d})} \quad (4.35)$$

for any $1 < p < \infty$ uniformly for $0 < \varepsilon \leq 1$;

- If φ is compactly supported, then so is $\mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_N)[\varphi]$. Specifically,

$$\operatorname{supp}[\mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_N)[\varphi]] \subset \overline{\mathcal{U}_{2.5^N \varepsilon}[\operatorname{supp}[\varphi]]}. \quad (4.36)$$

Similarly to the preceding part, we may compute the gradients with respect to the parameters \mathbf{h}_i , $i = 1, \dots, N$. A straightforward application of formula (4.24) yields:

$$\begin{aligned}
& \nabla_{\mathbf{h}_i} \mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_N)[\varphi] \\
&= \mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_{i-1}) \left[\nabla_x \mathbf{R}_{5^{i-1}\varepsilon}(\mathbf{h}_i, \dots, \mathbf{h}_N)[\varphi] - \mathbf{R}_{5^{i-1}\varepsilon}(\mathbf{h}_i) \left[\nabla_x \mathbf{R}_{5^i\varepsilon}(\mathbf{h}_{i+1}, \dots, \mathbf{h}_N)[\varphi] \right] \right] \\
&= \mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_{i-1}) \left[\nabla_x \mathbf{R}_{5^{i-1}\varepsilon}(\mathbf{h}_i, \dots, \mathbf{h}_N)[\varphi] \right] \\
&\quad - \mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_i) \left[\nabla_x \mathbf{R}_{5^i\varepsilon}(\mathbf{h}_{i+1}, \dots, \mathbf{h}_N)[\varphi] \right], \quad i = 1, \dots, N,
\end{aligned} \tag{4.37}$$

with the convention

$$\mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_0) = \mathbf{R}_{5^N\varepsilon}(\mathbf{h}_{N+1}, \dots, \mathbf{h}_N) = \text{Id}.$$

5 Convergence: Proof of Theorem 2.1

Let $\varphi \in C_c^\infty([0, T] \times \Omega; \mathbb{R}^d)$, $\text{div}_x \varphi = 0$ be a smooth solenoidal function. Our ultimate goal is to plug $\mathbf{R}_\varepsilon(\mathbf{h}_1^\varepsilon, \dots, \mathbf{h}_N^\varepsilon)[\varphi]$ as a test function in the “relaxed” momentum balance (2.25) and perform the limit $\varepsilon \rightarrow 0$. It follows from Proposition 4.3, notably (4.31), (4.32) and (4.36), that $\mathbf{R}_\varepsilon(\mathbf{h}_1^\varepsilon, \dots, \mathbf{h}_N^\varepsilon)[\varphi]$ belongs to the class (2.26) and therefore represent an eligible test function for (2.25) as long as we check the regularity of its time derivative. This will be done in the next section.

5.1 Error estimates for test functions

Given a test function $\varphi \in C_c^\infty([0, T] \times \Omega; \mathbb{R}^d)$, $\text{div}_x \varphi = 0$, the time derivative of its approximation $\mathbf{R}_\varepsilon(\mathbf{h}_1^\varepsilon, \dots, \mathbf{h}_N^\varepsilon)[\varphi]$ can be computed directly from formula (4.37):

$$\begin{aligned}
\partial_t \mathbf{R}_\varepsilon(\mathbf{h}_1^\varepsilon, \dots, \mathbf{h}_N^\varepsilon)[\varphi] &= \mathbf{R}_\varepsilon(\mathbf{h}_1^\varepsilon, \dots, \mathbf{h}_N^\varepsilon)[\partial_t \varphi] + \sum_{i=1}^N \nabla_{\mathbf{h}_i} \mathbf{R}_\varepsilon(\mathbf{h}_1^\varepsilon, \dots, \mathbf{h}_N^\varepsilon)[\varphi] \cdot \mathbf{Y}_i^\varepsilon \\
&= \mathbf{R}_\varepsilon(\mathbf{h}_1^\varepsilon, \dots, \mathbf{h}_N^\varepsilon)[\partial_t \varphi] \\
&\quad + \sum_{i=1}^N \mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_{i-1}) \left[\nabla_x \mathbf{R}_{5^{i-1}\varepsilon}(\mathbf{h}_i, \dots, \mathbf{h}_N)[\varphi] \cdot \mathbf{Y}_i^\varepsilon \right. \\
&\quad \left. - \mathbf{R}_{5^{i-1}\varepsilon}(\mathbf{h}_i) \left[\nabla_x \mathbf{R}_{5^i\varepsilon}(\mathbf{h}_{i+1}, \dots, \mathbf{h}_N)[\varphi] \cdot \mathbf{Y}_i^\varepsilon \right] \right]
\end{aligned} \tag{5.1}$$

As a matter of fact, the functions \mathbf{h}^ε are merely Lipschitz; whence the identity for time derivatives

$$\frac{d}{dt} \mathbf{h}_i^\varepsilon(t) = \mathbf{Y}_i^\varepsilon \text{ holds for only for a.a. } t \in (0, T).$$

Still formula (5.1) as well as eligibility of $\mathbf{R}_\varepsilon(\mathbf{h}_1^\varepsilon, \dots, \mathbf{h}_N^\varepsilon)[\varphi]$ as a test function in (2.25) can be verified by a density argument.

To derive the error estimates on the difference

$$\varphi - \mathbf{R}_\varepsilon(\mathbf{h}_1^\varepsilon, \dots, \mathbf{h}_N^\varepsilon)[\varphi], \quad \varphi \in C_c^\infty([0, T] \times \Omega; \mathbb{R}^d), \quad \operatorname{div}_x \varphi = 0$$

we use essentially two facts:

- For any fixed $t \in [0, T]$:

$$\varphi(t, \cdot) - \mathbf{R}_\varepsilon(\mathbf{h}_1^\varepsilon, \dots, \mathbf{h}_N^\varepsilon)[\varphi](t, \cdot) = 0 \text{ outside the union of balls } \cup_{i=1}^N B_{10^{-N}\varepsilon}[\mathbf{h}_i^\varepsilon(t)], \quad (5.2)$$

which far from being optimally stated consequence of (4.33);

- φ is smooth, in particular Lipschitz in $[0, T] \times \Omega$.

In view of the bounds (4.34), (4.35), we get

$$\|\mathbf{R}_\varepsilon(\mathbf{h}_1^\varepsilon, \dots, \mathbf{h}_N^\varepsilon)[\varphi]\|_{W^{1,p}(\mathbb{R}^d; \mathbb{R}^d)} \lesssim \|\varphi\|_{C_c^1(\Omega; \mathbb{R}^d)};$$

whence, by virtue of (5.2),

$$\|(\varphi - \mathbf{R}_\varepsilon(\mathbf{h}_1^\varepsilon, \dots, \mathbf{h}_N^\varepsilon)[\varphi])(t, \cdot)\|_{W^{1,p}(\mathbb{R}^d; \mathbb{R}^d)} \rightarrow 0 \text{ uniformly for } t \in [0, T] \text{ for any } 1 \leq p < \infty \quad (5.3)$$

for any $\varphi \in C_c^1([0, T] \times \Omega; \mathbb{R}^d)$, $\operatorname{div}_x \varphi = 0$.

As for the time derivative, we use formula (5.1) obtaining

$$\begin{aligned} \partial_t \varphi - \partial_t \mathbf{R}_\varepsilon(\mathbf{h}_1^\varepsilon, \dots, \mathbf{h}_N^\varepsilon)[\varphi] &= \partial_t \varphi - \mathbf{R}_\varepsilon(\mathbf{h}_1^\varepsilon, \dots, \mathbf{h}_N^\varepsilon)[\partial_t \varphi] \\ &+ \sum_{i=1}^N \mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_{i-1}) \left[\nabla_x \mathbf{R}_{5^{i-1}\varepsilon}(\mathbf{h}_i, \dots, \mathbf{h}_N)[\varphi] \cdot \mathbf{Y}_i^\varepsilon \right. \\ &\left. - \mathbf{R}_{5^{i-1}\varepsilon}(\mathbf{h}_i) \left[\nabla_x \mathbf{R}_{5^i\varepsilon}(\mathbf{h}_{i+1}, \dots, \mathbf{h}_N)[\varphi] \cdot \mathbf{Y}_i^\varepsilon \right] \right], \end{aligned}$$

where, similarly to the above,

$$\|(\partial_t \varphi - \mathbf{R}_\varepsilon(\mathbf{h}_1^\varepsilon, \dots, \mathbf{h}_N^\varepsilon)[\partial_t \varphi])(t, \cdot)\|_{W^{1,p}(\mathbb{R}^d; \mathbb{R}^d)} \rightarrow 0 \text{ uniformly in } t \in [0, T] \text{ for any } 1 \leq p < \infty. \quad (5.4)$$

Finally, the second error term can be estimated with the help of (4.33),

$$\left| \sum_{i=1}^N \mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_{i-1}) \left[\nabla_x \mathbf{R}_{5^{i-1}\varepsilon}(\mathbf{h}_i, \dots, \mathbf{h}_N)[\varphi] \cdot \mathbf{Y}_i^\varepsilon \right. \right.$$

$$\begin{aligned}
& - \mathbf{R}_{5^{i-1}\varepsilon}(\mathbf{h}_i) \left[\nabla_x \mathbf{R}_{5^i\varepsilon}(\mathbf{h}_{i+1}, \dots, \mathbf{h}_N)[\varphi] \cdot \mathbf{Y}_i^\varepsilon \right] \Big| \\
& \leq \mathbf{1}_{\cup_{i=1}^N B_{10N\varepsilon}(\mathbf{h}_i^\varepsilon)} \sum_{i=1}^N |\mathbf{Y}_i^\varepsilon| \left(\left| \mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_{i-1}) \left[\nabla_x \mathbf{R}_{5^{i-1}\varepsilon}(\mathbf{h}_i, \dots, \mathbf{h}_N)[\varphi] \right] \right| \right. \\
& \quad \left. + \mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_i) \left[\nabla_x \mathbf{R}_{5^i\varepsilon}(\mathbf{h}_{i+1}, \dots, \mathbf{h}_N)[\varphi] \right] \right)
\end{aligned} \tag{5.5}$$

where, by virtue of (4.34), (4.35),

$$\begin{aligned}
& \left\| \mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_{i-1}) \left[\nabla_x \mathbf{R}_{5^{i-1}\varepsilon}(\mathbf{h}_i, \dots, \mathbf{h}_N)[\varphi] \right] \right\|_{L^p(\mathbb{R}^d; \mathbb{R}^{d \times d})} \\
& + \left\| \mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_i) \left[\nabla_x \mathbf{R}_{5^i\varepsilon}(\mathbf{h}_{i+1}, \dots, \mathbf{h}_N)[\varphi] \right] \right\|_{L^p(\mathbb{R}^d; \mathbb{R}^{d \times d})} \\
& \lesssim \|\varphi\|_{W^{1,p}(\mathbb{R}^d; \mathbb{R}^d)} \text{ uniformly for } t \in (0, T)
\end{aligned} \tag{5.6}$$

and any $1 < p < \infty$.

5.2 Convergence

We know from estimates (3.2)–(3.3) that

$$\sqrt{\rho^\varepsilon} \mathbf{u}^\varepsilon \text{ is bounded in } L^\infty(0, T; L^2(\Omega; \mathbb{R}^d)), \tag{5.7}$$

$$\mathbf{u}^\varepsilon \text{ is bounded in } L^2(0, T; W_0^{1,2}(\Omega; \mathbb{R}^d)).$$

Thus there exists $\mathbf{u} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W_0^{1,2}(\Omega))$ such that, up to a subsequence,

$$\sqrt{\rho^\varepsilon} \mathbf{u}^\varepsilon \rightarrow \mathbf{u} \text{ weak-* in } L^\infty(0, T; L^2(\Omega; \mathbb{R}^d)), \tag{5.8}$$

$$\mathbf{u}^\varepsilon \rightarrow \mathbf{u} \text{ weakly in } L^2(0, T; W_0^{1,2}(\Omega; \mathbb{R}^d)). \tag{5.9}$$

5.2.1 Limit in the momentum equation

Our ultimate goal is to perform the limit in the momentum equation (2.25), with the test function $\mathbf{R}_\varepsilon(\mathbf{h}_1^\varepsilon, \dots, \mathbf{h}_N^\varepsilon)[\varphi]$.

Viscous term.

In view of (5.9) and the error estimate (5.3), it is easy to see

$$\int_0^T \int_\Omega \mathbb{S}(\mathbb{D}_x \mathbf{u}^\varepsilon) : \mathbb{D}_x(\mathbf{R}_\varepsilon(\mathbf{h}_1^\varepsilon, \dots, \mathbf{h}_N^\varepsilon)[\varphi]) \, dx \, dt \rightarrow \int_0^T \int_\Omega \mathbb{S}(\mathbb{D}_x \mathbf{u}) : \mathbb{D}_x \varphi \, dx \, dt \tag{5.10}$$

for any $\varphi \in C_c^1([0, T] \times \Omega; \mathbb{R}^d)$, $\operatorname{div}_x \varphi = 0$.

Convective term.

As ρ^ε is bounded and the uniform bounds (3.2), (3.3) hold, it is easy to check that

$$\rho^\varepsilon \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon \rightarrow \overline{\mathbf{u} \otimes \mathbf{u}} \text{ weakly in } L^p((0, T) \times \Omega; \mathbb{R}^{d \times d})$$

for some $p > 1$. Consequently, in view of (5.3),

$$\int_0^T \int_\Omega (\rho^\varepsilon \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon) : \nabla_x (\mathbf{R}_\varepsilon(\mathbf{h}_1^\varepsilon, \dots, \mathbf{h}_N^\varepsilon))[\varphi] \, dx \, dt \rightarrow \int_0^T \int_\Omega \overline{(\mathbf{u} \otimes \mathbf{u})} : \nabla_x \varphi \, dx \, dt \text{ as } \varepsilon \rightarrow 0 \quad (5.11)$$

for any $\varphi \in C_c^1([0, T] \times \Omega; \mathbb{R}^d)$, $\operatorname{div}_x \varphi = 0$.

Time derivative.

Our next goal is to establish the limit

$$\int_0^T \int_\Omega \rho^\varepsilon \mathbf{u}^\varepsilon \cdot \partial_t \mathbf{R}_\varepsilon(\mathbf{h}_1^\varepsilon, \dots, \mathbf{h}_N^\varepsilon)[\varphi] \, dx \, dt \rightarrow \int_0^T \int_\Omega \mathbf{u} \cdot \partial_t \varphi \, dx \, dt. \quad (5.12)$$

In view of the estimates (5.4), (5.5) this amounts to show

$$\int_0^T \int_\Omega \mathbf{1}_{\cup_{i=1}^N B_{10N\varepsilon}(\mathbf{h}_i^\varepsilon)} \rho^\varepsilon \mathbf{u}^\varepsilon \cdot \sum_{i=1}^N |\mathbf{Y}_i^\varepsilon| \mathbf{G}_i^\varepsilon \, dx \, dt \rightarrow 0, \quad (5.13)$$

where,

$$\mathbf{G}_i^\varepsilon = \left| \mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_{i-1}) \left[\nabla_x \mathbf{R}_{5^{i-1}\varepsilon}(\mathbf{h}_i, \dots, \mathbf{h}_N)[\varphi] \right] + \mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_i) \left[\nabla_x \mathbf{R}_{5^i\varepsilon}(\mathbf{h}_{i+1}, \dots, \mathbf{h}_N)[\varphi] \right] \right|.$$

By virtue of (5.6),

$$\|\mathbf{G}_i^\varepsilon\|_{L^\infty(0, T; L^p(\Omega; \mathbb{R}^d))} \lesssim \|\varphi\|_{L^\infty(0, T; W^{1, p}(\mathbb{R}^d; \mathbb{R}^d))} \text{ for any } 1 \leq p < \infty, \, i = 1, \dots, N. \quad (5.14)$$

Let us start with the case $d = 3$. In view of (5.5) and uniform boundedness of the density, we have

$$\begin{aligned} & \left| \int_\Omega \mathbf{1}_{\cup_{i=1}^N B_{10N\varepsilon}(\mathbf{h}_i^\varepsilon)} \rho^\varepsilon \mathbf{u}^\varepsilon \cdot \sum_{i=1}^N |\mathbf{Y}_i^\varepsilon| \mathbf{G}_i^\varepsilon \, dx \right| \\ & \lesssim c(\delta) \sum_{i=1}^N |\mathbf{Y}_i^\varepsilon(t)| \|\mathbf{u}^\varepsilon\|_{L^6(\Omega; \mathbb{R}^d)} |\cup_{j=1}^N B_{10N\varepsilon}(\mathbf{h}_j^\varepsilon)|^{\frac{5}{6}-\delta} \text{ for any } \delta > 0. \end{aligned} \quad (5.15)$$

Writing

$$\begin{aligned} & \sum_{i=1}^N |\mathbf{Y}_i^\varepsilon(t)| \|\mathbf{u}^\varepsilon\|_{L^6(\Omega; \mathbb{R}^d)} \left| \bigcup_{j=1}^N B_{10N\varepsilon}(\mathbf{h}_j^\varepsilon) \right|^{\frac{5}{6}-\delta} \\ &= \sum_{i=1}^N |\mathcal{S}_\varepsilon^i|^{\frac{1}{6}} |\mathbf{Y}_i^\varepsilon(t)| \|\mathbf{u}^\varepsilon\|_{L^6(\Omega; \mathbb{R}^d)} \left(\left| \bigcup_{j=1}^N B_{10N\varepsilon}(\mathbf{h}_j^\varepsilon) \right|^{\frac{5}{6}-\delta} |\mathcal{S}_\varepsilon^i|^{-\frac{1}{6}} \right), \end{aligned}$$

we deduce from the uniform bounds (3.3), (3.6)

$$\left\| |\mathbf{Y}_i^\varepsilon(t)| \|\mathbf{u}^\varepsilon\|_{L^6(\Omega; \mathbb{R}^d)} \right\|_{L^1(0, T)} \lesssim 1,$$

while, as a consequence of hypothesis (2.16),

$$\left| \bigcup_{j=1}^N B_{10N\varepsilon}(\mathbf{h}_j^\varepsilon) \right|^{\frac{5}{6}-\delta} |\mathcal{S}_\varepsilon^i|^{-\frac{1}{6}} \lesssim \varepsilon^{3(\frac{5}{6}-\delta)} |\mathcal{S}_\varepsilon^i|^{-\frac{1}{6}} \rightarrow 0 \text{ as long as } \delta > 0 \text{ is small enough.}$$

The same result can be obtained in the case $d = 2$ by means of the Sobolev embedding $W^{1,2} \subset L^p$ for any finite p .

5.2.2 Limit in the convective term

The only thing remaining is to establish the identity:

$$\overline{\mathbf{u} \otimes \mathbf{u}} = \mathbf{u} \otimes \mathbf{u}. \quad (5.16)$$

We consider the quantity

$$\varphi = \psi(t) \mathbf{R}_\varepsilon(\mathbf{h}_1^\varepsilon, \dots, \mathbf{h}_N^\varepsilon)(t)[\phi], \quad \psi = \psi(t) \in C_c^1[0, T], \quad \phi \in C_c^1(\Omega; \mathbb{R}^d), \quad \operatorname{div}_x \phi = 0,$$

as a test function in the momentum equation (2.25). It follows that the time distributional derivative of

$$t \in [0, T] \mapsto \int_{\mathbb{R}^d} \rho^\varepsilon \mathbf{u}^\varepsilon \cdot \mathbf{R}_\varepsilon(\mathbf{h}_1^\varepsilon, \dots, \mathbf{h}_N^\varepsilon)[\phi] \, dx$$

belongs to $L^q(0, T)$ for any $1 \leq q < 2$ and Arzelà–Ascoli theorem yields

$$\begin{aligned} t \in [0, T] \mapsto \int_{\mathbb{R}^d} \rho^\varepsilon \mathbf{u}^\varepsilon \cdot \mathbf{R}_\varepsilon(\mathbf{h}_1^\varepsilon, \dots, \mathbf{h}_N^\varepsilon)[\phi] \, dx & \text{ is precompact in } C[0, T] \\ & \text{ for any } \phi \in C_c^1(\Omega; \mathbb{R}^d), \operatorname{div}_x \phi = 0. \end{aligned} \quad (5.17)$$

Next, as $\rho^\varepsilon \mathbf{u}^\varepsilon$ is bounded in $L^\infty(0, T; L^2(\mathbb{R}^d; \mathbb{R}^d))$, we use the error estimates for the operator $\mathbf{R}_\varepsilon(\mathbf{h}_1^\varepsilon, \dots, \mathbf{h}_N^\varepsilon)$ established in (5.3) to show

$$\int_{\mathbb{R}^d} (\rho^\varepsilon \mathbf{u}^\varepsilon)(t, \cdot) \cdot \left(\mathbf{R}_\varepsilon(\mathbf{h}_1^\varepsilon, \dots, \mathbf{h}_N^\varepsilon)(t)[\phi] - \phi \right) \, dx \rightarrow 0 \text{ in } L^\infty(0, T),$$

which, combined with (5.17), implies

$$\int_{\mathbb{R}^d} (\rho^\varepsilon \mathbf{u}^\varepsilon)(t, \cdot) \cdot \phi \, dx \rightarrow \int_{\mathbb{R}^d} \mathbf{u}(t, \cdot) \cdot \phi \, dx \text{ in } L^\infty(0, T) \text{ for any } \phi \in C_c^1(\Omega; \mathbb{R}^d), \operatorname{div}_x \phi = 0. \quad (5.18)$$

Using a density argument, we deduce from (5.18) that

$$\int_{\Omega} (\rho^\varepsilon \mathbf{u}^\varepsilon)(t, \cdot) \cdot \phi \, dx \rightarrow \int_{\Omega} \mathbf{u}(t, \cdot) \cdot \phi \, dx \text{ in } L^\infty(0, T), \quad (5.19)$$

for any $\phi \in L^2(\Omega; \mathbb{R}^d)$, $\operatorname{div}_x \phi = 0$ in Ω , $\phi \cdot \mathbf{n} = 0$ on $\partial\Omega$.

For a general $\phi \in C_c^1(\Omega; \mathbb{R}^d)$, consider its Helmholtz decomposition in Ω ,

$$\phi = \mathbf{H}[\phi] + \nabla_x \Psi \text{ in } \Omega \text{ with } \nabla_x \Psi \cdot \mathbf{n} = 0 \text{ on } \partial\Omega.$$

Accordingly, we get

$$\int_{\Omega} (\rho^\varepsilon \mathbf{u}^\varepsilon)(t, \cdot) \cdot \phi \, dx = \int_{\Omega} (\rho^\varepsilon \mathbf{u}^\varepsilon)(t, \cdot) \cdot \mathbf{H}[\phi] \, dx + \int_{\Omega} (\rho^\varepsilon \mathbf{u}^\varepsilon)(t, \cdot) \cdot \nabla_x \Psi \, dx,$$

where, in accordance with (5.18),

$$\int_{\Omega} (\rho^\varepsilon \mathbf{u}^\varepsilon)(t, \cdot) \cdot \mathbf{H}[\phi] \, dx \rightarrow \int_{\Omega} \mathbf{u}(t, \cdot) \cdot \mathbf{H}[\phi] \, dx = \int_{\Omega} \mathbf{u}(t, \cdot) \cdot \phi \, dx \text{ in } L^\infty(0, T).$$

Moreover, as \mathbf{u}^ε is solenoidal

$$\begin{aligned} \int_{\Omega} (\rho^\varepsilon \mathbf{u}^\varepsilon)(t, \cdot) \nabla_x \Psi \, dx &= \int_{\Omega} (\rho^\varepsilon \mathbf{u}^\varepsilon - \mathbf{u}^\varepsilon)(t, \cdot) \cdot \nabla_x \Psi \, dx \\ &= \sum_{i=1}^N (\rho_{S^i}^\varepsilon - 1) \int_{S^i_\varepsilon} \mathbf{u}^\varepsilon \cdot \nabla_x \Psi \, dx \end{aligned}$$

for a.a. $t \in (0, T)$. Thus it follows from the uniform bounds established in (3.3) that

$$\int_{\Omega} (\rho^\varepsilon \mathbf{u}^\varepsilon)(t, \cdot) \cdot \nabla_x \Psi \, dx \rightarrow 0 \text{ in } L^2(0, T).$$

and we may infer that

$$\int_{\Omega} (\rho^\varepsilon \mathbf{u}^\varepsilon)(t, \cdot) \phi \, dx \rightarrow \int_{\Omega} \mathbf{u}(t, \cdot) \phi \, dx \text{ in } L^2(0, T) \text{ for any } \phi \in C_c^1(\Omega; \mathbb{R}^d).$$

By density, we extend the conclusion to square integrable function,

$$\int_{\Omega} (\rho^\varepsilon \mathbf{u}^\varepsilon)(t, \cdot) \cdot \phi \, dx \rightarrow \int_{\Omega} \mathbf{u}(t, \cdot) \cdot \phi \, dx \text{ in } L^2(0, T) \text{ for any } \phi \in L^2(\Omega; \mathbb{R}^d). \quad (5.20)$$

Equivalently, we can extend the function \mathbf{u}^ε by zero in $\mathbb{R}^d \setminus \Omega$ and obtain:

$$\rho^\varepsilon \mathbf{u}^\varepsilon \rightarrow \mathbf{u} \text{ in } L^2(0, T; L^2_{\text{weak}}(\mathbb{R}^d; \mathbb{R}^d)). \quad (5.21)$$

Since $L^2_{\text{weak}}(K; \mathbb{R}^d)$ is compactly embedded in the dual $W^{-1,2}(K; \mathbb{R}^d)$ for any compact $K \subset \mathbb{R}^d$, the desired conclusion

$$\int_0^T \int_{\mathbb{R}^d} \rho^\varepsilon \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon : \nabla_x \varphi \, dx \, dt \rightarrow \int_0^T \int_{\mathbb{R}^d} \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi \, dx \, dt \text{ for any } \varphi \in C_c^1([0, T) \times \Omega; \mathbb{R}^d) \quad (5.22)$$

follows.

We infer from the above discussion on the passing to the limit as $\varepsilon \rightarrow 0$ that the limit velocity of \mathbf{u}^ε is given by \mathbf{u} where \mathbf{u} is a weak solution of the Navier–Stokes system (2.1)–(2.3) satisfying the energy inequality (2.23).

6 Concluding Remarks

The approach proposed in the present paper can be applied to the case of a general *compressible* fluid. In view of the estimates obtained [8, Theorem 2.3], we anticipate the present result to hold in the range of the adiabatic exponent $\gamma > \frac{d}{2}$.

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