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STRONG TREE PROPERTIES, KUREPA TREES, AND GUESSING MODELS

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ABSTRACT. We investigate the generalized tree properties and guessing model properties introduced by Weiß and Viale, as well as natural weakenings thereof, studying the relationships among these properties and between these properties and other prominent combinatorial principles. We introduce a weakening of Viale and Weiß's Guessing Model Property, which we call the Almost Guessing Property, and prove that it provides an alternate formulation of the slender tree property in the same way that the Guessing Model Property provides and alternate formulation of the ineffable slender tree property. We show that instances of the Almost Guessing Property have sufficient strength to imply, for example, failures of square or the nonexistence of weak Kurepa trees. We show that these instances of the Almost Guessing Property hold in the Mitchell model starting from a strongly compact cardinal and prove a number of other consistency results showing that certain implications between the principles under consideration are in general not reversible. In the process, we provide a new answer to a question of Viale by constructing a model in which, for all regular $\theta \geq \omega_2$, there are stationarily many ω_2-guessing models $M \in \mathscr{P}_{\omega_2} H(\theta)$ that are not ω_1-guessing models.

CONTENTS

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1. Introduction

A central line of research in modern combinatorial set theory concerns the study of the extent to which various properties of large cardinals can consistently hold at small cardinals, and the exploration of the web of implications and non-implications existing among these principles. These large cardinal properties can typically be interpreted as assertions of various forms of *compactness*, i.e., statements asserting that the global behavior of certain combinatorial structures necessarily reflects their local behavior.

Among the more prominent of these principles in recent years have been various two-cardinal tree properties. Their study dates back at least to the 1970s, when they were used to provide combinatorial characterizations of strongly compact and supercompact cardinals. In modern terminology, Jech [11] proved that an uncountable cardinal κ is strongly compact if and only if, for all $\lambda \geq \kappa$, every (κ, λ)-list has a cofinal branch. Shortly thereafter, Magidor [20] proved that κ is supercompact if and only if, for all $\lambda \geq \kappa$, every (κ, λ)-list has an ineffable branch (see Section 4 for precise definitions of these terms).

The modern study of these two-cardinal tree properties at small cardinals was initiated in the 2000s by Weiß [26], who realized that, although it is impossible to have a small cardinal κ for which every (κ, λ)-list has a cofinal branch, if one restricts one's attention to certain interesting subclasses of (κ, λ)-lists, one obtains consistent yet nontrivial principles. In particular, Weiß introduced the notions of *thin* and *slender* (κ, λ)-lists (with slenderness being a weakening of thinness). These in turn gave rise to four compactness principles asserting that, for a given uncountable regular cardinal κ, for every $\lambda \geq \kappa$, every thin (or slender) (κ, λ)-list has a cofinal (or ineffable) branch. The strongest of these principles, denoted ISP_κ, is the assertion that every slender (κ, λ)-list has an ineffable branch. Weiß proved that these principles can consistently hold at small cardinals; for example, ISP_{ω_2} holds in the supercompact Mitchell model.

In [25], Viale and Weiß proved that ISP_{ω_2} also follows from the Proper Forcing Axiom (PFA), in the process giving a useful reformulation of ISP_κ in terms of the existence of powerful elementary substructures known as *guessing models* (again, see Section 4 for precise definitions). The guessing model formulation of ISP_κ has proven quite useful in efficiently establishing that a wide variety of other compactness principles are in fact consequences of ISP_κ. For example, the following is a partial but representative list of some of the consequences of ISP_{ω_2}:

- (Weiß [26]) the failure of $\square(\lambda, \omega_1)$ for every regular $\lambda \geq \omega_2$ (this only requires the weaker ITP_{ω_2});
- (Viale [24], Krueger [13]) the Singular Cardinals Hypothesis;
- (Lambie-Hanson [16] and Lücke [19], independently) the narrow system property $\mathsf{NSP}(\omega_1, \lambda)$ for every regular $\lambda \geq \omega_2$;
- (Cox–Krueger [2]) the negation of the weak Kurepa Hypothesis ($\neg\mathsf{wKH}$).

This paper is the first in a projected series of works further exploring the world of these two-cardinal tree properties and their relatives. A future installment [18] will deal primarily with questions connected to cardinal arithmetic and cardinal characteristics; the present entry can be seen as an investigation into natural weakenings of ISP_κ and the extent to which these weakenings still imply various combinatorial consequences of ISP_κ.

This investigation inevitably entails a more fine-grained analysis of the notions of *slenderness* and *guessing model*. Still leaving the formal definitions for Section 4, let us give a broad overview here. The standard notion of "slender" can naturally be seen as "ω_1-slender", and the standard notion of "guessing model" can naturally be seen as "ω_1-guessing model". One can obtain natural variations of these notions by replacing ω_1 by any other uncountable cardinal μ. Following [8], given cardinals $\mu \leq \kappa \leq \lambda$ with κ regular, let $\mathsf{SP}(\mu, \kappa, \lambda)$ (resp. $\mathsf{ISP}(\mu, \kappa, \lambda)$) be the assertion that every μ-slender (κ, λ)-list has a cofinal (resp. ineffable) branch. It will follow immediately from the definitions that these principles become stronger as μ decreases. Arguments of Viale and Weiss [25] show that, for uncountable cardinals $\mu \leq \kappa$, the following two statements are equivalent:

- $\mathsf{ISP}(\mu, \kappa, \lambda)$ holds for every $\lambda \geq \kappa$;
- the set of μ-guessing models is stationary in $\mathscr{P}_\kappa H(\theta)$ for every regular $\theta \geq \kappa$.

The statement in the second bullet point above is abbreviated here as $\mathsf{GMP}(\mu, \kappa, \theta)$ (GMP stands for "guessing model property"). In this paper, we isolate a weakening of GMP, which we denote by AGP (for "almost guessing property"), and we also introduce further weakenings of SP and AGP, wSP and wAGP respectively, by replacing all instances of "club" and "stationary" in the definitions by "strong club" and "weakly stationary", respectively. We then prove the following theorem.

Theorem A. *For all uncountable cardinals $\mu \leq \kappa$, with κ regular, the following are equivalent:*

(1) $\mathsf{SP}(\mu, \kappa, \lambda)$ (resp. $\mathsf{wSP}(\mu, \kappa, \lambda)$) holds for all $\lambda \geq \kappa$;
(2) $\mathsf{AGP}(\mu, \kappa, \theta)$ (resp. $\mathsf{wAGP}(\mu, \kappa, \theta)$) holds for all regular $\theta \geq \kappa$.

One reason for introducing these weakenings is the fact that, unlike ISP, they hold at cardinals that are strongly compact but not supercompact, or in the model obtained by Mitchell forcing starting with a cardinal that is strongly compact but not supercompact. For example, we obtain the following theorem, where, for a set $\mathcal{Y} \subseteq \mathscr{P}_\kappa \lambda$, the principle $\mathsf{wSP}_{\mathcal{Y}}(\mu, \kappa, \lambda)$ denotes the strengthening of $\mathsf{wSP}(\mu, \kappa, \lambda)$ in which expands the set of (κ, λ)-lists under consideration to include those indexed only by elements of $\mathcal{Y}$ rather than the full $\mathscr{P}_\kappa \lambda$.

Theorem B. *In the strongly compact Mitchell model, for every cardinal $\lambda \geq \omega_2$, $\mathsf{wSP}_{\mathcal{Y}}(\omega_1, \omega_2, \lambda)$ holds, where $\mathcal{Y} := \{x \in \mathscr{P}_\kappa \lambda \mid \mathrm{cf}(M \cap \lambda) = \omega_1\}$.*

While these principles are substantial weakenings of ISP, they are still sufficient to obtain some of the combinatorial consequences of ISP. For example, we obtain the following results.

Theorem C. *Let μ be a regular uncountable cardinal.*

(1) If $\mathsf{wAGP}(\mu, \mu^+, \mu^+)$ holds, then there are no weak μ-Kurepa trees.
(2) Suppose that $\chi < \chi^+ < \kappa \leq \lambda$ are infinite cardinals, with κ regular and $\mathrm{cf}(\lambda) \geq \kappa$, and let $\mathcal{Y} := \{M \in \mathscr{P}_\kappa H(\lambda^+) \mid \mathrm{cf}(\sup(M \cap \lambda)) > \chi\}$. If $\mathsf{wAGP}_{\mathcal{Y}}(\kappa, \kappa, \lambda^+)$ holds, then there are no subadditive, strongly unbounded functions $c : [\lambda]^2 \to \chi$. In particular, $\square(\lambda)$ fails.

We remark that the proofs of Theorems A and B will show that the hypothesis of clause (2) of Theorem C holds in the strongly compact Mitchell model with $\chi = \omega$, $\mu = \omega_1$, $\kappa = \omega_2$, and any value of $\lambda \geq \omega_2$.

Finally, we prove a consistency result indicating that increasing the first parameter in the principles under consideration does lead to strictly weaker statements. In particular, we show that, unlike ISP_{ω_2} (or even $\mathsf{wAGP}(\omega_1, \omega_2, H(\omega_2))$, by clause (1) of Theorem C), the existence of a Kurepa tree is compatible with $\mathsf{ISP}(\omega_2, \omega_2, \lambda)$ for all $\lambda \geq \omega_2$.

Theorem D. *Let κ be a supercompact cardinal. Then there is a generic extension in which there is an ω_1-Kurepa tree and $\mathsf{ISP}(\omega_2, \omega_2, \lambda)$ holds for every $\lambda \geq \omega_2$. In particular, $\mathsf{SP}(\omega_1, \omega_2, |H(\omega_2)|)$ fails.*

Theorem D can be seen as a variation on a related result of Cummings [3], who proved, using a similar argument, that the tree property at ω_2 is compatible with the existence of an ω_1-Kurepa tree. This also addresses a question of Viale from [24] by providing a model in which, for all $\theta \geq \omega_2$, there are ω_2-guessing models in $\mathscr{P}_{\omega_2} H(\theta)$ that are not ω_1-guessing models.

The structure of the paper is as follows. In Section 2, we introduce the notion of Λ-tree for an arbitrary directed partial order Λ, providing a very general setting in which many of our results naturally sit. In Section 3, we review some basic facts about $\mathscr{P}_\kappa \lambda$ combinatorics. In Section 4, we recall the strong tree properties and guessing model properties introduced in [26] and [25] and establish some basic facts thereon. We also answer a question of Fontanella and Matet from [7] by showing that two tree properties considered there are equivalent. At the end of the section, we prove a separation result between two strong tree properties by showing that one of them entails a failure of approachability while the other does not.

In Section 5, we introduce our "almost guessing properties $\mathsf{(w)AGP}(\ldots)$ and prove Theorem A. We also consider a variation of $\mathsf{GMP}(\ldots)$ in which only subsets of "small" sets are guessed and show that this variation does not require the full strength of $\mathsf{ISP}(\ldots)$ but already follows from the relevant instance of $\mathsf{SP}(\ldots)$. In Section 6, we prove that an instance of $\mathsf{wAGP}(\ldots)$ implies the nonexistence of certain strongly unbounded subadditive functions and hence the failure of square, thus establishing clause (2) of Theorem C. In Section 7, we present a variety of preservation lemmas that will be useful in our consistency results in the remainder of the paper. In Section 8, we analyze various principles in the Mitchell extension obtained from a strongly compact cardinal, proving Theorem B, among other results. As a corollary, for instance, we obtain a model of $\mathsf{MA} + \mathsf{TP}(\omega_2, \geq \omega_2)$ starting just with a strongly compact cardinal. Finally, in Section 9, we investigate the effect of various principles on the existence of (weak) Kurepa trees, proving clause (1) of Theorem C and Theorem D.

Unless specifically noted, our notation and terminology is standard. We use [12] as our standard background reference for set theory.

2. Generalized trees

Although the primary setting for the results of this paper is that of (κ, λ)- lists, a number of our results hold in a more general setting. We therefore take the time here to establish this setting and introduce the notion of generalized trees indexed by an arbitrary partial order Λ.

Throughout the paper, unless otherwise specified, κ denotes an arbitrary regular uncountable cardinal. Also, for this section, fix a partial order $(\Lambda, <_\Lambda)$. In most cases of interest Λ will be κ-directed, but for now let it be arbitrary. We will sometimes abuse notation and simply use Λ to refer to the order $(\Lambda, <_\Lambda)$.

Definition 2.1. A Λ-*tree* is a pair $T = (\langle T_u \mid u \in \Lambda\rangle, <_T)$ such that the following conditions all hold.

(i) $\langle T_u \mid u \in \Lambda\rangle$ is a sequence of nonempty, pairwise disjoint sets.
(ii) $<_T$ is a transitive partial ordering on $\bigcup_{u\in\Lambda} T_u$.
(iii) For all $u, v \in \Lambda$, all $s \in T_u$, and all $t \in T_v$, if $s <_T t$, then $u <_\Lambda v$.
(iv) $<_T$ is *tree-like*, i.e., for all $u <_\Lambda v <_\Lambda w$, all $r \in T_u$, all $s \in T_v$ and all $t \in T_w$, if $r, s <_T t$, then $r <_T s$.
(v) For all $u \leq_\Lambda v$ in Λ and all $t \in T_v$, there is a unique $s \in T_u$, denoted $t \restriction u$, such that $s \leq_T t$.

If $T = (\langle T_u \mid u \in \Lambda\rangle, <_T)$ is a Λ-tree, we let $\mathrm{width}(T)$ denote the least cardinal μ such that $|T_u| < \mu$ for all $u \in \Lambda$. For a cardinal κ, we say that T is a κ-Λ-tree if, in addition to the above requirements, we have $\mathrm{width}(T) \leq \kappa$.

Given a Λ-tree $T = (\langle T_u \mid u \in \Lambda\rangle, <_T)$, we will sometimes abuse notation and use T to refer to the underlying set of the tree, $\bigcup_{u\in\Lambda} T_u$. The set T_u is called the u^{th} *level* of T. If a Λ-tree T is given, it should be understood unless otherwise specified that its u^{th} level is denoted by T_u for each $u \in \Lambda$, and its ordering is denoted by $<_T$. Given $t \in T$, we let $\mathrm{lev}_T(t)$ be the unique $u \in \Lambda$ such that $t \in T_u$. A *subtree* of T is a Λ-tree $T' = (\langle T'_u \mid u \in \Lambda\rangle, <_{T'})$ such that $T'_u \subseteq T_u$ for all $u \in \Lambda$ and $<_{T'}$ is the restriction of $<_T$ to $\bigcup_{u\in\Lambda} T'_u$.

Definition 2.2. Suppose that T is a Λ-tree. A *cofinal branch* through T is a function $b \in \prod_{u\in\Lambda} T_u$ such that, for all $u <_\Lambda v$ in Λ, we have $b(u) <_T b(v)$. The (κ, Λ)-*tree property*, denoted $\mathsf{TP}_\kappa(\Lambda)$, is the assertion that every κ-Λ-tree has a cofinal branch.

Remark 2.3. Note that, if $\Lambda = \kappa$ (with the ordinal ordering), then a κ-Λ-tree is simply a classical κ-tree, and $\mathsf{TP}_\kappa(\Lambda)$ is the classical tree property $\mathsf{TP}(\kappa)$. Another important special case, which will be the setting for many of the results of this paper, is $\Lambda = \mathscr{P}_\kappa\lambda$ for some cardinal $\lambda \geq \kappa$, ordered by $\subseteq$.

Many of the standard facts about κ-trees remain true about κ-Λ-trees when Λ is κ-directed. We establish some of these facts now, after some preliminary definitions.

Definition 2.4. Suppose that T is a Λ-tree.

(i) We say that T is *well-pruned* if, for all $u <_\Lambda v$ in Λ and all $s \in T_u$, there is $t \in T_v$ such that $s <_T t$.
(ii) For a fixed $\mathcal{Y} \subseteq \Lambda$, we say that T is κ-$\mathcal{Y}$-*thin* if $|T_u| < \kappa$ for all $u \in \mathcal{Y}$.
(iii) We say that T is *very thin* if Λ is $\mathrm{width}(T)^+$-directed.

Proposition 2.5. *Suppose that Λ is κ-directed, $X \subseteq \Lambda$ is $<_\Lambda$-cofinal, and T is a κ-$\mathcal{Y}$-thin Λ-tree. Then T has a well-pruned subtree.*

Proof. For each $u \in \Lambda$, simply let T'_u be the set of all $s \in T_u$ such that, for all $v \in \Lambda$ with $u <_\Lambda v$, there is $t \in T_v$ such that $s <_T t$. Let $<_{T'}$ be the restriction of $<_T$ to $\bigcup_{u\in\Lambda} T'_u$. We claim that $T' = (\langle T'_u \mid u \in \Lambda\rangle, <_{T'})$ is the desired well-pruned subtree.

First note that, for all $u <_\Lambda v$ and all $t \in T'_v$, the definition of T' ensures that $t \restriction u \in T'_u$. We next argue that $T'_u \neq \emptyset$ for all $u \in \Lambda$. Suppose for sake of contradiction that there is $u \in \Lambda$ such that $T'_u = \emptyset$. Note that, if $v \in \Lambda$ and $u <_\lambda v$, then the observation at the beginning of the paragraph implies that we must have $T'_v = \emptyset$ as well. Therefore, by increasing u if necessary, we may assume that $u \in \mathcal{Y}$.

Since $T'_u = \emptyset$, we know that, for each $s \in T_u$, there is $v_s \in \Lambda$ such that $u <_\Lambda v_s$ and, for all $t \in T_{v_s}$, we have $s \not<_T t$. Since Λ is κ-directed and $|T_u| < \kappa$, we can find $v \in \Lambda$ such that $v_s \leq_\Lambda v$ for all $s \in T_u$. Choose an arbitrary $t \in T_v$, and let $s := t \restriction u$. Then $t \restriction v_s \in T_{v_s}$ and $s <_T t \restriction v_s$, contradicting our choice of v_s. Therefore, $T'_u \neq \emptyset$ for all $u \in \Lambda$. It follows that T' is indeed a subtree of T.

It remains to show that T' is well-pruned. Suppose for sake of contradiction that $u <_\Lambda v$ and $s \in T'_u$, but there is no $t \in T'_v$ such that $s <_{T'} t$. Note that, if $v <_\Lambda w$, then there is still no $t \in T'_w$ such that $s <_{T'} t$, as otherwise $t \restriction v$ would contradict the previous sentence. Therefore, by increasing v if necessary, we may assume that $v \in \mathcal{Y}$. Let $S := \{t \in T_v \mid s <_T t\}$. Since $s \in T'_u$, we know that S is nonempty. For each $t \in S$, since $t \notin T'_v$, we can find $w_t \in \Lambda$ such that $v <_\Lambda w_t$ and there is no $r \in T_{w_t}$ such that $t <_T r$. Since Λ is κ-directed and $|S| \leq |T_v| < \kappa$, we can find $w \in \Lambda$ such that $w_t <_\Lambda w$ for all $t \in S$. Since $s \in T'_u$, we can find $r \in T_w$ such that $s <_T r$. Let $t := r \restriction v$. Then $t \in S$, $r \restriction w_t \in T_{w_t}$, and $t <_T (r \restriction w_t)$, contradicting our choice of w_t. Therefore, T' is indeed well-pruned. □

The following definition will be useful.

Definition 2.6. Suppose that T is a Λ-tree and $u <_\Lambda w$. We say that T *splits between u and v* if there are distinct $t_0, t_1 \in T_v$ such that $t_0 \restriction u = t_1 \restriction u$.

Note that, if T is a well-pruned Λ-tree, $u <_\Lambda v <_\Lambda w$, and T splits between u and v, then T also splits between u and w. The following lemma generalizes a result of Kurepa [14] from the setting of κ-trees to our general setting.

Lemma 2.7. *Suppose that T is a very thin Λ-tree. Then T has a cofinal branch.*

Proof. Since T is very thin, by Proposition 2.5 we can assume that it is well-pruned. Let $\mu := \mathrm{width}(T)$; then Λ is μ^+-directed and $|T_u| < \mu$ for all $u \in \Lambda$. We first show that μ must be a successor cardinal.

Claim 2.8. *There is $\nu < \mu$ such that $|T_u| \leq \nu$ for all $u \in \Lambda$.*

Proof. Let $X := \{\nu \mid \exists u \in \Lambda\ (|T_u| = \nu)\}$. Then X is a set of cardinals less than μ. For each $\nu \in X$, choose $u_\nu \in \Lambda$ such that $|T_{u_\nu}| = \nu$. Since Λ is μ^+-directed, we can find $v \in \Lambda$ such that $u_\nu <_\Lambda v$ for all $\nu \in X$. Since T is well-pruned, it follows $|T_v| \geq |T_{u_\nu}|$ for all $\nu \in X$. In particular, X has a maximal element, and $\max(X)$ is as desired in the statement of the claim. □

By minimality of μ, it must be the case that $\mu = \nu^+$, where ν is given by the preceding claim.

Let θ be a sufficiently large regular cardinal (in particular, we want $T \in H(\theta)$), and let $\mathcal{Z}$ be the set of all $M \prec (H(\theta), \in, T)$ such that

- $|M| = \mu$;
- there is a strictly $\subseteq$-increasing sequence $\langle M_\eta \mid \eta < \mu\rangle$ such that
  - $M = \bigcup_{\eta<\mu} M_\eta$;
  - for all $\eta < \mu$, $M_\eta \in M$.

Then $\mathcal{Z}$ is stationary in $\mathscr{P}_{\mu^+} H(\theta)$. For each $M \in \mathscr{P}_{\mu^+} H(\theta)$, use the fact that Λ is μ^+-directed and $|M| = \mu$ to find $v_M \in \Lambda$ such that $u <_\Lambda v_M$ for all $u \in M \cap \Lambda$.

Temporarily fix an $M \in \mathcal{Z}$. Since $|T_{v_M}| < \mu$ and μ is regular, we can find an $\eta_M < \mu$ such that, for all distinct $t_0, t_1 \in T_{v_M}$, if there is $u \in M \cap \Lambda$ such that $t_0 \restriction u \neq t_1 \restriction u$, then there is such a u in $M_{\eta_M} \cap \Lambda$. Since $M_{\eta_M} \in M$ and $|M_{\eta_M}| \leq \mu$, we can find $u_M \in M \cap \Lambda$ such that $u <_\Lambda u_M$ for all $u \in M_{\eta_M} \cap \Lambda$.

Claim 2.9. *For all $v \in M \cap \Lambda$ such that $u_M <_\Lambda v$, T does not split between u_M and v.*

Proof. Suppose for sake of contradiction that $v \in M$, s_0 and s_1 are distinct elements of T_v, and $s_0 \restriction u_M = s_1 \restriction u_M$. Since T is well-pruned, we can find $t_0, t_1 \in T_{v_M}$ such that $t_0 \restriction v = s_0$ and $t_1 \restriction v = s_1$. Then, by our choice of η_M, there is $u \in M_{\eta_M} \cap \Lambda$ such that $t_0 \restriction u \neq t_1 \restriction u$. But $u < u_M$ and $t_0 \restriction u_M = s_0 \restriction u_M = s_1 \restriction u_M = t_1 \restriction u_M$, so $t_0 \restriction u = (t_0 \restriction u_M) \restriction u = (t_1 \restriction u_M) \restriction u = t_1 \restriction u$, which is a contradiction. □

The function that takes each $M \in \mathcal{Z}$ to u_M is a regressive map, so, since $\mathcal{Z}$ is stationary in $\mathscr{P}_{\mu^+} H(\theta)$, we can find a stationary $\mathcal{Z}^* \subseteq \mathcal{Z}$ and a single $u^* \in \Lambda$ such that $u_M = u^*$ for all $M \in \mathcal{Z}^*$. Choose an arbitrary $s^* \in T_{u^*}$, and define a function $b \in \prod_{u \in \Lambda} T_u$ as follows. For each $u \in \Lambda$, find $v \in \Lambda$ such that $u, u^* <_\Lambda v$, use the fact that T is well-pruned to find $t \in T_v$ such that $s^* <_T t$, and let $b(u) = t \restriction u$.

We claim that b is a cofinal branch through T, which will complete the proof of the lemma. Suppose for sake of contradiction that there are $u_0 <_\Lambda u_1$ such that $b(u_0) \not<_T b(u_1)$. Recalling the definition of b, for each $i < 2$ find $v_i \in \Lambda$ and $t_i \in T_{v_i}$ such that $u_i, u^* <_\Lambda v_i$, $s^* <_T t_i$, and $b(u_i) = t_i \restriction u_i$. Then find $v \in \Lambda$ such that $v_0, v_1 <_\Lambda v$ and $t_0^*, t_1^* \in T_v$ such that $t_0 <_T t_0^*$ and $t_1 <_T t_1^*$. Then $b(u_0) <_T t_0^*$ and $b(u_1) <_T t_1^*$, so, since $b(u_0) \not<_T b(u_1)$, it follows that $t_0^* \neq t_1^*$. Moreover, we know that $t_0^* \restriction u^* = s^* = t_1^* \restriction u^*$. Since $u^* <_T v$, it follows that T splits between u^* and v. Now fix $M \in \mathcal{Z}^*$ such that $v \in M$. Since $u_M = u^*$, Claim 2.9 implies that T does not split between u^* and v, which is the desired contradiction. □

Proposition 2.10. *Suppose that κ is strongly compact and Λ is κ-directed. Then $\mathsf{TP}_\kappa(\Lambda)$ holds.*

Proof. Let T be a κ-Λ-tree, and let $j : V \to M$ be an elementary embedding with critical point κ such that $j(\kappa) > |\Lambda|$ and there is $X \subseteq j(\Lambda)$ in M such that $M \models$ "$|X| < j(\kappa)$" and $j``\Lambda \subseteq X$. In $M$, let $j(T) = (\langle \bar{T}_w \mid w \in j(\Lambda) \rangle, <_{j(T)})$. Note that, for all $u \in \Lambda$, since $|T_u| < \kappa = \mathrm{crit}(j)$, we have $\bar{T}_{j(u)} = j``T_u$. In M, $j(\Lambda)$ is $j(\kappa)$-directed, so we can find $w \in j(\Lambda)$ such that $v <_{j(\Lambda)} w$ for all $v \in X$. Fix an arbitrary $t \in \bar{T}_w$. Now define a function $b \in \prod_{u \in \Lambda} T_u$ by letting $b(u)$ be the unique $s \in T_u$ such that $j(s) = t \restriction j(u)$ for all $u \in \Lambda$. Then whenever $u_0 <_\Lambda u_1$, we know that, in M, we have $j(b(u_0)) = t \restriction j(u_0)$ and $j(b(u_1)) = t \restriction j(u_1)$. Therefore, $j(b(u_0)) <_{j(T)} j(b(u_1))$, so, by elementarity, $b(u_0) <_T b(u_1)$. Therefore, b is a cofinal branch through T. □

3. Background on two-cardinal combinatorics

As noted in the previous section, many of our results in this paper are in the specific context of the partial order $(\mathscr{P}_\kappa \lambda, \subseteq)$. We therefore briefly review some of the relevant combinatorial definitions and facts about $\mathscr{P}_\kappa \lambda$. For this section, let X denote an arbitrary set with $\kappa \leq |X|$.

Definition 3.1. Suppose that $\mathcal{C} \subseteq \mathscr{P}_\kappa X$.

(1) C is *closed* if whenever $D \subseteq C$ is such that $|D| < \kappa$ and D is linearly ordered by $\subseteq$, we have $\bigcup D \in C$;
(2) C is *strongly closed* if whenever $D \subseteq C$ and $|D| < \kappa$, we have $\bigcup D \in C$;
(3) C is *cofinal* if for all $x \in \mathscr{P}_\kappa \lambda$, there is $y \in C$ such that $x \subseteq y$;
(4) C is a *club* in $\mathscr{P}_\kappa X$ if it is closed and cofinal;
(5) C is a *strong club* in $\mathscr{P}_\kappa X$ if it is strongly closed and cofinal.

Definition 3.2. The *club filter* on $\mathscr{P}_\kappa X$, denoted $\mathsf{CF}_{\kappa,X}$, is the set of all $B \subseteq \mathscr{P}_\kappa X$ for which there is a club $C \subseteq \mathscr{P}_\kappa X$ such that $C \subseteq B$. Similarly, the *strong club filter* on $\mathscr{P}_\kappa X$, denoted $\mathsf{SCF}_{\kappa,X}$, is the set of all $B \subseteq \mathscr{P}_\kappa X$ for which there is a strong club $C \subseteq \mathscr{P}_\kappa X$ such that $C \subseteq B$. The dual ideals to the club filter and the strong club filter are denoted $\mathsf{NS}_{\kappa,X}$ and $\mathsf{SNS}_{\kappa,X}$, respectively. Elements of $\mathsf{NS}^+_{\kappa,X}$ and $\mathsf{SNS}^+_{\kappa,X}$ are called *stationary* and *weakly stationary* subsets of $\mathscr{P}_\kappa X$, respectively.

Given a set $x \subseteq \mathscr{P}_\kappa X$ and a function $f : X \to \mathscr{P}_\kappa X$, we say that x is *closed under* f if $f(a) \subseteq x$ for all $a \in x$. Similarly, if $g : [X]^2 \to \mathscr{P}_\kappa X$, then x is closed under g if $g(a) \subseteq x$ for all $a \in [x]^2$. The following proposition is immediate.

Proposition 3.3. *Suppose that $f : X \to \mathscr{P}_\kappa X$ is a function. Then the set $\{x \in \mathscr{P}_\kappa X \mid x$ is closed under $f\}$ is a strong club in $\mathscr{P}_\kappa X$. In particular, if $\mathcal{Y} \subseteq \mathscr{P}_\kappa X$ is weakly stationary, then there is $x \in \mathcal{Y}$ such that x is closed under f.*

The following characterization of $\mathsf{CF}_{\kappa,X}$ is due to Menas [21].

Proposition 3.4. *If $g : [X]^2 \to \mathscr{P}_\kappa X$ is a function, then the set*

$$C_g := \{x \in \mathscr{P}_\kappa X \mid x \text{ is infinite and closed under } g\}$$

is a club in $\mathscr{P}_\kappa X$. Moreover, for any club C in $\mathscr{P}_\kappa X$, there is $g : [X]^2 \to \mathscr{P}_\kappa X$ such that $C_g \subseteq C$.

Given sets $X \subseteq X'$ and a subset $S \subseteq \mathscr{P}_\kappa X$, let $S \uparrow \mathscr{P}_\kappa X' := \{y \in \mathscr{P}_\kappa S' \mid y \cap X \in S\}$. Dually, if $S' \subseteq \mathscr{P}_\kappa X'$, let $S \downarrow \mathscr{P}_\kappa X := \{y' \cap X \mid y' \in S'\}$. The following facts are standard.

Proposition 3.5. *Suppose that $X \subseteq X'$ are sets such that $\kappa \leq |X|$, $S \subseteq \mathscr{P}_\kappa X$, and $S' \subseteq \mathscr{P}_\kappa X'$.*

(1) If S is a (strong) club in $\mathscr{P}_\kappa X$, then $S \uparrow \mathscr{P}_\kappa X'$ is a (strong) club in $\mathscr{P}_\kappa X'$.
(2) If S' is a strong club in $\mathscr{P}_\kappa X'$, then $S \downarrow \mathscr{P}_\kappa X$ is a strong club in $\mathscr{P}_\kappa X$.
(3) If S is a club in $\mathscr{P}_\kappa X'$, then $S \downarrow \mathscr{P}_\kappa X$ contains a club in $\mathscr{P}_\kappa X$.
(4) If S is (weakly) stationary in $\mathscr{P}_\kappa X$, then $S \uparrow \mathscr{P}_\kappa X'$ is (weakly) stationary in $\mathscr{P}_\kappa X'$.
(5) If S' is (weakly) stationary in $\mathscr{P}_\kappa X'$, then $S' \downarrow \mathscr{P}_\kappa X$ is (weakly) stationary in $\mathscr{P}_\kappa X$.

4. Strong tree properties and guessing models

In this section, we review the basic definitions and facts about two-cardinal tree properties and guessing models. These definitions and facts are largely generalizations of definitions and results from [26] and [25]. The end of the section contains some new results, separating certain two-cardinal tree properties and answering a question of Fontanella and Matet.

Let κ be a regular uncountable cardinal, and let X be a set with $|X| \geq \kappa$. We say that a sequence $\langle d_x \mid x \in \mathscr{P}_\kappa X\rangle$ is a (κ, X)-list if $d_x \subseteq x$ for all $x \in \mathscr{P}_\kappa X$.

Definition 4.1. Assume that $D = \langle d_x \mid x \in \mathscr{P}_\kappa X\rangle$ is a (κ, X)-list and $\mathcal{Y} \subseteq \mathscr{P}_\kappa X$.

(i) We say that D is $\mathcal{Y}$-*thin* if there is a closed unbounded set $C \subseteq \mathscr{P}_\kappa X$ such that $|\{d_x \cap y \mid y \subseteq x \in \mathscr{P}_\kappa X\}| < \kappa$ for every $y \in C \cap \mathcal{Y}$.
(ii) Let $\mu \leq \kappa$ be an uncountable cardinal. We say that D is μ-$\mathcal{Y}$-*slender* if for all sufficiently large θ there is a club $C \subseteq \mathscr{P}_\kappa H(\theta)$ such that for all $M \in C$ and all $y \in M \cap \mathscr{P}_\mu X$, if $M \cap X \in \mathcal{Y}$, then $d_{M\cap X} \cap y \in M$.

Here and in all similar later situations, if $\mathcal{Y} = \mathscr{P}_\kappa X$, then we will typically omit mention of $\mathcal{Y}$ and simply refer to D as being *thin* or *μ-slender*.

The following fact is easily established (cf. [26, Proposition 2.2]).

Fact 4.2. *Assume that D is a (κ, X)-list and $\mathcal{Y} \subseteq \mathscr{P}_\kappa X$. If D is $\mathcal{Y}$-thin, then it is κ-$\mathcal{Y}$-slender.*

The following proposition follows almost immediately from the definitions but is often useful, showing that the value of θ in Clause (ii) of Definition 4.1 can always be taken to be any cardinal θ for which $\mathscr{P}_\mu X \subseteq H(\theta)$.

Proposition 4.3. *Suppose that $\mu \leq \kappa$ is an infinite cardinal, $D = \langle d_x \mid x \in \mathscr{P}_\kappa X\rangle$ is a (κ, X)-list, $\mathcal{Y} \subseteq \mathscr{P}_\kappa X$, and θ is a cardinal such that $\mathscr{P}_\mu X \subseteq H(\theta)$. Then the following are equivalent.*

(1) D is μ-$\mathcal{Y}$-slender.
(2) There is a club $C \subseteq \mathscr{P}_\kappa H(\theta)$ such that for all $M \in C$ and all $y \in M \cap \mathscr{P}_\mu X$, if $M \cap X \in \mathcal{Y}$, then $d_{M\cap X} \cap y \in M$.

Proof. Suppose that $\theta_0 \leq \theta_1 \leq \theta_2$ are cardinals for which $\mathscr{P}_\mu X \subseteq H(\theta_0)$, and suppose that $C \subseteq \mathscr{P}_\kappa H(\theta_1)$ is a club witnessing the μ-$\mathcal{Y}$-slenderness of D (i.e., C is as in Clause (ii) of Definition 4.1). Then it is easily verified that $C \downarrow H(\theta_0)$ and $C \uparrow H(\theta_2)$ also witness the μ-$\mathcal{Y}$-slenderness of D (recall Proposition 3.5). The proposition is then immediate. □

Definition 4.4. Assume that $D = \langle d_x \,|\, x \in \mathscr{P}_\kappa X\rangle$ is a (κ, X)-list, $\mathcal{Y} \subseteq \mathscr{P}_\kappa X$ is stationary, and $d \subseteq X$.

(i) We say that d is a *cofinal branch* of D if for all $x \in \mathscr{P}_\kappa X$ there is $z_x \supseteq x$ such that $d \cap x = d_{z_x} \cap x$.
(ii) We say that d is a *$\mathcal{Y}$-ineffable branch* of D if the set $\{x \in \mathcal{Y} \,|\, d \cap x = d_x\}$ is stationary. (Again, we will omit mention of $\mathcal{Y}$ if $\mathcal{Y} = \mathscr{P}_\kappa X$.)

Remark 4.5. Given a (κ, X)-list, there is a canonical way of generating a $\mathscr{P}_\kappa X$-tree (in the sense of Section 2) from it, where $\mathscr{P}_\kappa X$ is seen as a poset ordered by $\subseteq$. Namely, fix a (κ, X)-list $D = \langle d_x \mid x \in \mathscr{P}_\kappa X\rangle$, and define a $\mathscr{P}_\kappa X$-tree $T = (\langle T_x \mid x \in \mathscr{P}_\kappa X\rangle, <_T)$ as follows. First, for each $x \in \mathscr{P}_\kappa X$, let $T_x := \{d_y \cap x \mid y \in \mathscr{P}_\kappa X \text{ and } x \subseteq y\}$. Given $x \subsetneq y$ in $\mathscr{P}_\kappa X$, $s \in T_x$, and $t \in T_y$, we set $s <_T t$ if and only if $s = t \cap x$. The following are then immediate:

- T is a $\mathscr{P}_\kappa X$-tree;
- for every $\mathcal{Y} \subseteq \mathscr{P}_\kappa X$, D is $\mathcal{Y}$-thin if and only if T is $\mathcal{Y}$-κ-thin;
- T has a cofinal branch if and only if D has a cofinal branch.

Definition 4.6. If $D = \langle d_x \mid x \in \mathscr{P}_\kappa X\rangle$ is a (κ, X)-list, then we let width(D) denote width(T), where T is the $\mathscr{P}_\kappa X$-tree generated from D as in Remark 4.5.

Definition 4.7. Assume that $\mu \leq \kappa$ is regular and $\mathcal{Y} \subseteq \mathscr{P}_\kappa X$ is stationary. We say that

(i) the (κ, X)-tree property holds on $\mathcal{Y}$, denoted $\mathsf{TP}_{\mathcal{Y}}(\kappa, X)$, if every $\mathcal{Y}$-thin (κ, X)-list has a cofinal branch.
(ii) the (κ, X)-ineffable tree property holds on $\mathcal{Y}$, denoted $\mathsf{ITP}_{\mathcal{Y}}(\kappa, X)$, if every $\mathcal{Y}$-thin (κ, X)-list has a $\mathcal{Y}$-ineffable branch.
(iii) the (μ, κ, X)-slender tree property holds on $\mathcal{Y}$, denoted $\mathsf{SP}_{\mathcal{Y}}(\mu, \kappa, X)$, if every μ-$\mathcal{Y}$-slender (κ, X)-list has a cofinal branch.

(iv) the (μ, κ, X)-ineffable slender tree property holds on $\mathcal{Y}$, denoted $\mathsf{ISP}_{\mathcal{Y}}(\mu, \kappa, X)$, if every μ-$\mathcal{Y}$-slender (κ, X)-list has a $\mathcal{Y}$-ineffable branch.

Remark 4.8. In order to ease notation, we introduce a couple of conventions. As before, if mention of $\mathcal{Y}$ is omitted in any of the principles from Definition 4.7, then it should be understood that $\mathcal{Y} = \mathscr{P}_\kappa X$. We will use notations such as $(\mathsf{I})\mathsf{TP}(\kappa, \geq \kappa)$ (resp. $(\mathsf{I})\mathsf{SP}(\mu, \kappa, \geq \kappa)$) to assert that $(\mathsf{I})\mathsf{TP}(\kappa, \lambda)$ (resp. $(\mathsf{I})\mathsf{SP}(\mu, \kappa, \lambda)$) holds for all $\lambda \geq \kappa$. Finally, in the principles $(\mathsf{I})\mathsf{SP}_{\mathcal{Y}}(\mu, \kappa, \lambda)$, the value of μ that has most often been considered in the literature is ω_1; we will therefore use $(\mathsf{I})\mathsf{SP}_\kappa$ to denote $(\mathsf{I})\mathsf{SP}(\omega_1, \kappa, \geq \kappa)$.

If $f : X_0 \to X_1$ is a bijection, $\mathcal{Y}_0 \subseteq \mathscr{P}_\kappa X_0$ is stationary, and $\mathcal{Y}_1 := \{f[x] \mid x \in \mathcal{Y}_0\}$, then $\mathsf{TP}_{\mathcal{Y}_0}(\kappa, X_0)$ is equivalent to $\mathsf{TP}_{\mathcal{Y}_1}(\kappa, X_1)$, $\mathsf{ITP}_{\mathcal{Y}_0}(\mu, \kappa, X_0)$ is equivalent to $\mathsf{ITP}_{\mathcal{Y}_1}(\mu, \kappa, X_1)$, and similarly for SP and ISP. We will therefore not lose any generality by stating our results in the context in which X is an infinite cardinal, which is what is typically seen in the literature. There are instances, though, in which it is more convenient to work with, e.g., $(\kappa, H(\theta))$-lists, so we have opted for the more general terminology and notation. Also, if $C \subseteq \mathscr{P}_\kappa X_0$ is a club, then $\mathsf{TP}_{\mathcal{Y}_0}(\kappa, X_0)$ is equivalent to $\mathsf{TP}_{\mathcal{Y}_0 \cap C}(\kappa, X_0)$, and similarly for the other principles.

Note that $(\mathsf{I})\mathsf{SP}_{\mathcal{Y}}(\mu, \kappa, \lambda)$ implies $(\mathsf{I})\mathsf{SP}_{\mathcal{Y}}(\nu, \kappa, \lambda)$ for $\kappa \geq \nu \geq \mu$. The converse does not hold: In Theorem 9.3 we show that $\mathsf{SP}(\omega_1, \omega_2, \omega_2)$ implies that there are no weak ω_1-Kurepa trees, while in Theorem 9.9 we show that $\mathsf{ISP}(\omega_2, \omega_2, \lambda)$, and hence also the weaker $\mathsf{SP}(\omega_2, \omega_2, \lambda)$, is consistent with the existence of a (thin) ω_1-Kurepa tree.[1] Also, by Fact 4.2, $(\mathsf{I})\mathsf{SP}_{\mathcal{Y}}(\kappa, \kappa, \lambda)$ implies $(\mathsf{I})\mathsf{TP}(\kappa, \lambda)$. We will see in Subsection 4.2 below that these implications are also in general not reversible.

Note additionally that there is monotonicity in the last coordinate of these principles: if $\lambda' \geq \lambda \geq \kappa$ and $\mathcal{Y}' = \mathcal{Y} \uparrow \mathscr{P}_\kappa \lambda'$, then $(\mathsf{I})\mathsf{SP}_{\mathcal{Y}'}(\mu, \kappa, \lambda')$ implies $(\mathsf{I})\mathsf{SP}_{\mathcal{Y}}(\mu, \kappa, \lambda)$, and $(\mathsf{I})\mathsf{TP}_{\mathcal{Y}'}(\kappa, \lambda')$ implies $(\mathsf{I})\mathsf{TP}_{\mathcal{Y}}(\kappa, \lambda)$.

We now recall a useful reformulation of instances of $\mathsf{ISP}(\ldots)$ in terms of *guessing models*. We first introduce some basic definitions. Terminology regarding guessing models is slightly inconsistent across sources; we will primarily be following the terminology and notation from [24]. We note, however, that our definition of a *(μ, M)-approximated set* is formally weaker than the standard definition. It is easily seen to be equivalent if M is closed under pairwise intersections of its elements (in particular, if $M \prec H(\theta)$), but since we will sometimes want to apply the definition to sets M that are not elementary submodels of $H(\theta)$, our weaker definition seems more appropriate.

Definition 4.9. Suppose that θ is a sufficiently large regular cardinal and $M \subseteq H(\theta)$.

(1) Given a set $x \in M$, a subset $d \subseteq x$, and an uncountable cardinal μ, we say that
   (a) d is *(μ, M)-approximated* if, for every $z \in M \cap \mathscr{P}_\mu(x)$, there is $e \in M$ such that $d \cap z = e \cap z$;
   (b) d is *M-guessed* if there is $e \in M$ such that $d \cap M = e \cap M$.
(2) M is a *μ-guessing model for x* if $M \prec H(\theta)$ and every (μ, M)-approximated subset of x is M-guessed.
(3) M is a *μ-guessing model* if, for every $x \in M$, it is a μ-guessing model for x.

[1] For concreteness, we formulate the result of ω_2, but it can be easily generalized to an arbitrary double successor of a regular cardinal.

(4) Given uncountable cardinals $\mu \leq \kappa \leq \theta$ with κ and θ regular, and given $x \in H(\theta)$, let $\mathcal{G}^x_{\mu,\kappa}H(\theta)$ denote the set of $M \in \mathscr{P}_\kappa H(\theta)$ such that M is a μ-guessing model for x. Let $\mathcal{G}_{\mu,\kappa}H(\theta)$ denote the set of $M \in \mathscr{P}_\kappa H(\theta)$ such that M is a μ-guessing model.

(5) Suppose that $\mu \leq \kappa \leq \theta$ are uncountable cardinals with κ and θ regular, and that $\mathcal{Y} \subseteq \mathscr{P}_\kappa H(\theta)$ is stationary. Then $\mathsf{GMP}_{\mathcal{Y}}(\mu, \kappa, \theta)$ is the assertion that $\mathcal{G}_{\mu,\kappa}H(\theta) \cap \mathcal{Y}$ is stationary in $\mathscr{P}_\kappa H(\theta)$, i.e., $\mathcal{Y}$ contains stationarily many μ-guessing models.

Remark 4.10. As was the case with Definition 4.7, we introduce some notational conveniences. If $\mathcal{Y}$ is omitted from $\mathsf{GMP}_{\mathcal{Y}}(\mu, \kappa, \theta)$, then it should be understood to be $\mathscr{P}_\kappa H(\theta)$. We let $\mathsf{GMP}(\mu, \kappa, \geq \kappa)$ denote the assertion that $\mathsf{GMP}(\mu, \kappa, \theta)$ holds for every regular $\theta \geq \kappa$. Again, the case $\mu = \omega_1$ is the most prominent in the literature; we will simply say that a model M is a *guessing model* to mean that it is an ω_1-guessing model, and we will write $\mathcal{G}^x_\kappa H(\theta)$ and $\mathcal{G}_\kappa H(\theta)$ instead of $\mathcal{G}^x_{\omega_1,\kappa}H(\theta)$ or $\mathcal{G}_{\omega_1,\kappa}H(\theta)$.

The following proposition is immediate from the definitions.

Proposition 4.11. *Suppose that $\mu \leq \kappa \leq \theta \leq \theta'$ are uncountable cardinals, with κ, θ, and θ' regular.*

(1) Suppose that $x \in M \prec H(\theta)$, $M' \prec H(\theta')$, and $M' \cap H(\theta) = M$. Then M is a μ-guessing model for x if and only if M' is a μ-guessing model for x. In particular, $\mathcal{G}^x_{\mu,\kappa}H(\theta)$ is stationary in $\mathscr{P}_\kappa H(\theta)$ if and only if $\mathcal{G}^x_{\mu,\kappa}H(\theta')$ is stationary in $\mathscr{P}_\kappa H(\theta')$.

(2) Suppose that $M \prec H(\theta)$, $M' \prec H(\theta')$, and $M' \cap H(\theta) = M$. If M' is a μ-guessing model, then M is also a μ-guessing model. In particular, $\mathsf{GMP}(\mu, \kappa, \theta')$ implies $\mathsf{GMP}(\mu, \kappa, \theta)$.

The proofs of the following propositions are essentially the same as those of [25, Propositions 3.2 and 3.3]; we include them for completeness.

Proposition 4.12. *Let $\mu \leq \kappa \leq \theta$ be regular uncountable cardinals, and let $\mathcal{Y} \subseteq \mathscr{P}_\kappa H(\theta)$ be stationary. If $\mathsf{ISP}_{\mathcal{Y}}(\mu, \kappa, H(\theta))$ holds, then $\mathsf{GMP}_{\mathcal{Y}}(\mu, \kappa, \theta)$ holds.*

Proof. Suppose for sake of contradiction that there is a club $C \subseteq \mathscr{P}_\kappa H(\theta)$ such that every element of $C \cap \mathcal{Y}$ is not a μ-guessing model. Then for every $M \in C \cap \mathcal{Y}$, we can fix a set $z_M \in M$ and $d_M \subseteq z_M$ such that d_M is (μ, M)-approximated but not M-guessed. The same is easily seen to also be true for $d_M \cap M$, so we can assume that $d_M \subseteq M$. For $M \in \mathscr{P}_\kappa H(\theta) \setminus C \cap \mathcal{Y}$, let d_M be an arbitrary subset of M. This defines a $(\kappa, H(\theta))$-list $D := \langle d_M \mid M \in \mathscr{P}_\kappa H(\theta) \rangle$.

We claim that D is μ-$\mathcal{Y}$-slender. Let $\theta' > |H(\theta)|$ be a regular cardinal, and let $C' := \{M' \in C \uparrow \mathscr{P}_\kappa H(\theta') \mid M' \prec H(\theta')\}$. Then C' is a club in $\mathscr{P}_\kappa H(\theta')$, and the fact that C' witnesses the μ-$\mathcal{Y}$-slenderness of D follows immediately from the fact that d_M is (μ, M)-approximated for all $M \in C \cap \mathcal{Y}$.

We can therefore apply $\mathsf{ISP}_{\mathcal{Y}}(\mu, \kappa, H(\theta))$ to find a $\mathcal{Y}$-ineffable branch d of D. Let $S_0 := \{M \in C \cap \mathcal{Y} \mid d \cap M = d_M\}$. Then S_0 is stationary in $\mathscr{P}_\kappa H(\theta)$, so, by an application of Fodor's lemma, we can fix a stationary $S_1 \subseteq S_0$ and a fixed set z such that $z_M = z$ for all $M \in S_1$. Since S_1 is $\subseteq$-cofinal in $\mathscr{P}_\kappa H(\theta)$, it follows that $d \subseteq z$, and hence $d \in H(\theta)$. We can therefore find $M \in S_1$ such that $d \in M$. But then $d_M \cap M = d \cap M$, contradicting the fact that d_M is not M-guessed. □

Proposition 4.13. *Let $\mu \leq \kappa \leq \lambda$ be regular uncountable cardinals, and let $\mathcal{Y} \subseteq \mathscr{P}_\kappa \lambda$ be stationary. Suppose that there is a regular cardinal $\theta > \lambda^{<\kappa}$ such that $\mathsf{GMP}_{\mathcal{Y}'}(\mu, \kappa, \theta)$ holds, where $\mathcal{Y}' := \mathcal{Y} \uparrow \mathscr{P}_\kappa H(\theta)$. Then $\mathsf{ISP}_{\mathcal{Y}}(\mu, \kappa, \lambda)$ holds.*

Proof. Let $D = \langle d_x \mid x \in \mathscr{P}_\kappa \lambda \rangle$ be a μ-$\mathcal{Y}$-slender (κ, λ)-list. We will find a $\mathcal{Y}$-ineffable branch for D. Let $C \subseteq \mathscr{P}_\kappa H(\lambda^+)$ be a club witnessing that D is μ-$\mathcal{Y}$-slender. By our hypothesis, we can find an $M \in \mathscr{P}_\kappa H(\theta)$ such that $D, \mathcal{Y} \in M$, $M \cap H(\lambda^+) \in C$, $M \cap \lambda \in \mathcal{Y}$, and M is a μ-guessing model for λ. By the fact that $M \cap H(\lambda^+) \in C$, we have $d_{M \cap \lambda} \cap y \in M$ for all $y \in M \cap \mathscr{P}_\mu \lambda$, which directly implies that $d_{M \cap \lambda}$ is (μ, M)-approximated. Since M is a μ-guessing model, there is $e \in M$ such that $e \cap M = d_{M \cap \lambda} \cap M = d_{M \cap \lambda}$. Note that, since $d_{M \cap \lambda} \subseteq \lambda$ and $\lambda \in M$, it follows that $e \subseteq \lambda$.

We claim that e is a $\mathcal{Y}$-ineffable branch for D. Let $S := \{x \in \mathcal{Y} \mid e \cap x = d_x\}$, and note that $M \cap \lambda \in S$. If S were not stationary, then there would be a club $E \subseteq \mathscr{P}_\kappa \lambda$ such that $E \cap S = \emptyset$. Since everything needed to define S is in M, we can assume by elementarity that $E \in M$. But then, since E is a club, we have $M \cap \lambda \in E$, and we already saw that $M \cap \lambda \in S$, contradicting the assumption that $E \cap S = \emptyset$. Therefore, e is indeed a $\mathcal{Y}$-ineffable branch for D. □

Corollary 4.14. *Suppose that $\mu \leq \kappa$ are regular uncountable cardinals. Then the following are equivalent:*

(1) $\mathsf{ISP}(\mu, \kappa, \geq \kappa)$;
(2) $\mathsf{GMP}(\mu, \kappa, \geq \kappa)$.

Proof. This is immediate from Propositions 4.12 and 4.13. □

Note that there is a local asymmetry between Propositions 4.12 and 4.13: by Proposition 4.12, $\mathsf{ISP}(\mu, \kappa, |H(\theta)|)$ implies $\mathsf{GMP}(\mu, \kappa, H(\theta))$, but Proposition 4.13 does not provide an exact converse to this fact. Instead, we must assume that $\mathsf{GMP}(\mu, \kappa, H(\theta'))$ holds for some $\theta' > |H(\theta)|$ to conclude that $\mathsf{ISP}(\mu, \kappa, |H(\theta)|)$ holds. We now show that this is necessary; in fact, $\mathsf{GMP}(\mu, \kappa, H(\theta))$ does not in general imply even $\mathsf{ITP}(\kappa, |H(\theta)|)$. We first need the following standard proposition.

Proposition 4.15. *Suppose that $\kappa \leq \theta$ are regular uncountable cardinals, $S \subseteq \mathscr{P}_\kappa H(\theta)$ is stationary, and $\mathbb{P}$ is a $|H(\theta)|$-strategically closed forcing notion. Then S remains a stationary subset of $\mathscr{P}_\kappa H(\theta)$ in $V^{\mathbb{P}}$.*

Proof. Note first that, since $\mathbb{P}$ is $|H(\theta)|$-strategically closed and thus certainly $<\theta$-distributive, we have $H(\theta)^V = H(\theta)^{V[G]}$ and $(\mathscr{P}_\kappa H(\theta))^V = (\mathscr{P}_\kappa H(\theta))^{V[G]}$. By Proposition 3.4, it suffices to show that, for every $p \in \mathbb{P}$ and every $\mathbb{P}$-name $\dot{g}$ for a function from $[H(\theta)]^2$ to $\mathscr{P}_\kappa H(\theta)$, there is $q \leq_{\mathbb{P}} p$ and $M \in S$ such that $q \Vdash_{\mathbb{P}}$ "M is closed under $\dot{g}$". To this end, fix such a p and $\dot{g}$, and enumerate $H(\theta)$ as $\langle x_\alpha \mid \alpha < |H(\theta)| \rangle$. For all $\beta < |H(\theta)|$, let $X_\beta := \{x_\alpha \mid \alpha < \beta\}$. Using the fact that $\mathbb{P}$ is $|H(\theta)|$-strategically closed, recursively construct a $\leq_{\mathbb{P}}$-decreasing sequence $\langle p_\beta \mid \beta < |H(\theta)| \rangle$ such that $p_0 = p$ and, for all $\beta < |H(\theta)|$, $p_{\beta+1}$ decides the value of $\dot{g} \restriction [X_\beta]^2$, say as $g_\beta : [X_\beta]^2 \to \mathscr{P}_\kappa H(\theta)$. Let $g^* = \bigcup_{\beta < |H(\theta)|} g_\beta$. Then $g^* : [H(\theta)]^2 \to \mathscr{P}_\kappa H(\theta)$, so we can find $M \in S$ such that M is closed under g^*. Let $\beta < |H(\theta)|$ be large enough so that $M \subseteq X_\beta$ (this is possible, since $\mathrm{cf}(|H(\theta)|) \geq \theta \geq \kappa$). Then $p_\beta \Vdash_{\mathbb{P}}$ "$\dot{g} \restriction [M]^2 = g_\beta \restriction [M]^2$", so p_β forces that M is closed under $\dot{g}$. □

Theorem 4.16. *Suppose that $\mu \leq \kappa \leq \theta$ are regular uncountable cardinals, $|H(\theta)|$ is a regular cardinal, and $\mathsf{GMP}(\mu, \kappa, \theta)$ holds. Then there is a forcing extension, preserving all cofinalities $\leq |H(\theta)|$, in which $\mathsf{GMP}(\mu, \kappa, \theta) + \square(|H(\theta)|)$ holds; in particular, $\mathsf{ITP}(\kappa, |H(\theta)|)$ (and hence $\mathsf{ISP}(\mu, \kappa, |H(\theta)|)$) fails in this model.*

Proof. Let $\lambda := |H(\theta)|$. Let $\mathbb{P}$ be the standard forcing poset to add a $\square(\lambda)$-sequence by closed initial segments (cf. [15, §3] for a precise definition of $\mathbb{P}$ and proofs of relevant facts thereon). $\mathbb{P}$ is λ-strategically closed, so $(\mathscr{P}_\kappa H(\theta))^V = (\mathscr{P}_\kappa H(\theta))^{V[G]}$. It follows that every μ-guessing model $M \in \mathscr{P}_\kappa H(\theta)$ in V remains a μ-guessing model in $V[G]$, since no new subsets of M are added by forcing with $\mathbb{P}$. In addition, Proposition 4.15 implies that the set of such μ-guessing models remains stationary in $V^{\mathbb{P}}$. Therefore, $V^{\mathbb{P}}$ satisfies $\mathsf{GMP}(\mu, \kappa, \theta) + \square(\lambda)$.

To see that $\mathsf{ITP}(\kappa, \lambda)$ fails in $V^{\mathbb{P}}$, simply appeal to [26, Theorem 4.2], which states that, if $\mathrm{cf}(\lambda) \geq \kappa$ and $\square(\lambda)$ (or even a substantial weakening of $\square(\lambda)$) holds, then $\mathsf{ITP}(\kappa, \lambda)$ fails. □

4.1. **A question of Fontanella and Matet.** Let us take a brief detour to show that the results of Section 2 provide an answer to a question of Fontanella and Matet from [7]. In that paper, the authors consider the tree property $\mathsf{TP}(\kappa, \lambda)$ as well as an apparent weakening, which they denote $\mathsf{TP}^-(\kappa, \lambda)$.

Definition 4.17. $\mathsf{TP}^-(\kappa, \lambda)$ is the following assertion: If $D = \langle d_x \mid x \in \mathscr{P}_\kappa\lambda\rangle$ is a (κ, λ)-list for which there is a strongly closed, cofinal $C \subseteq \mathscr{P}_\kappa\lambda$ such that $|\{d_x \cap y \mid y \subseteq x \in \mathscr{P}_\kappa\lambda\}| < \kappa$ for every $y \in C$, then D has a cofinal branch.

The authors isolate a certain partition relation, denoted $\mathrm{PS}(\kappa, \lambda)$, and prove that $\mathrm{PS}(\kappa, \lambda)$ implies $\mathsf{TP}^-(\kappa, \lambda)$. They then ask whether $\mathrm{PS}(\kappa, \lambda)$ implies $\mathsf{TP}(\kappa, \lambda)$ and, in particular, whether $\mathsf{TP}^-(\kappa, \lambda)$ and $\mathsf{TP}(\kappa, \lambda)$ are equivalent. The following corollary gives a positive answer to both questions.

Corollary 4.18. *Suppose that $\mathsf{TP}(\kappa, \lambda)$ holds, and suppose that $D = \langle d_x \mid x \in \mathscr{P}_\kappa\lambda\rangle$ is a (κ, λ)-list for which there is a cofinal $\mathcal{Y} \subseteq \mathscr{P}_\kappa\lambda$ such that $|\{d_x \cap y \mid y \subseteq x \in \mathscr{P}_\kappa\lambda\}| < \kappa$ for every $y \in \mathcal{Y}$. Then D has a cofinal branch.*

Proof. Define a $\mathscr{P}_\kappa\lambda$-tree $T = \langle T_x \mid x \in \mathscr{P}_\kappa\lambda\rangle$ from D as in Remark 4.5. In particular, $T_x := \{d_x \cap y \mid y \subseteq x \in \mathscr{P}_\kappa\lambda\}$ for all $x \in \mathscr{P}_\kappa\lambda$. Then T is κ-$\mathcal{Y}$-thin, so, by Proposition 2.5, we can find a well-pruned subtree $T' = \langle T'_x \mid x \in \mathscr{P}_\kappa\lambda\rangle$ of T. The fact that T' is well-pruned implies that $|T'_x| \leq |T'_y|$ for all $x \subseteq y \in \mathscr{P}_\kappa\lambda$. In particular, since T is κ-$\mathcal{Y}$-thin and $\mathcal{Y}$ is cofinal in $\mathscr{P}_\kappa\lambda$, we have $|T'_x| < \kappa$ for all $x \in \mathscr{P}_\kappa\lambda$. Now let $D' = \langle d'_x \mid x \in \mathscr{P}_\kappa\lambda\rangle$ be a (κ, λ)-list such that $d'_x \in T'_x$ for all $x \in \mathscr{P}_\kappa\lambda$. D' is thin, so, by $\mathsf{TP}(\kappa, \lambda)$, it has a cofinal branch, d. By construction, for all $x \in \mathscr{P}_\kappa\lambda$, there is $y \in \mathscr{P}_\kappa\lambda$ such that $y \supseteq x$ and $d'_x = d_y \cap x$; it follows that d is a cofinal branch through D, as well. □

Corollary 4.19. *If $\mathcal{Y} \subseteq \mathscr{P}_\kappa\lambda$ is cofinal, then $\mathsf{TP}(\kappa, \lambda)$ is equivalent to $\mathsf{TP}_{\mathcal{Y}}(\kappa, \lambda)$.*

4.2. **Approachability and separating ISP from ITP.** It follows immediately from the definitions that, for $\lambda \geq \omega_2$, we have

$$\mathsf{ISP}(\omega_1, \omega_2, \lambda) \Rightarrow \mathsf{ISP}(\omega_2, \omega_2, \lambda) \Rightarrow \mathsf{ITP}(\omega_2, \lambda),$$

and, as mentioned above, we will see in Theorem 9.9 that $\mathsf{ISP}(\omega_2, \omega_2, \lambda)$ does not imply $\mathsf{ISP}(\omega_1, \omega_2, \lambda)$. This raises the natural question of whether $\mathsf{ITP}(\omega_2, \lambda)$ implies $\mathsf{ISP}(\omega_2, \omega_2, \lambda)$. In this subsection, we answer this question negatively.[2]

We first note that $\mathsf{ISP}(\omega_2, \omega_2, \omega_2)$ implies a failure of approachability.

Proposition 4.20. *Suppose that* $\mathsf{ISP}(\omega_2, \omega_2, \omega_2)$ *holds. Then* $\neg\mathsf{AP}_{\omega_1}$ *holds.*

Proof. In [25, Corollary 4.9], Viale and Weiss show that $\mathsf{ISP}(\omega_1, \omega_2, \omega_2)$ implies $\neg\mathsf{AP}_{\omega_1}$ (for a more detailed proof, using guessing models, see [2, Proposition 2.6]). An examination of the proofs in [25] and [2], though, shows that they only really need $\mathsf{ISP}(\omega_2, \omega_2, \omega_2)$. □

We next show that $\mathsf{ITP}(\omega_2, \lambda)$ is consistent with AP_{ω_1}.

Proposition 4.21. *Suppose that* κ *is a supercompact cardinal. Then there is a forcing extension in which* $\kappa = \omega_2$ *and* $\mathsf{ITP}(\omega_2, \lambda)$ *holds for all* $\lambda \geq \omega_2$.

Proof. We use the Mitchell forcing variation $\mathbb{M}_0$ from [4, §3.2], with the parameter "κ" from that paper set to be ω. As shown in [4, §3.5], in $V^{\mathbb{M}_0}$ we have $\omega_2 \in I[\omega_2]$, i.e., AP_{ω_1} holds.

In [6], Fontanella proves that, in the extension by a slightly different variant of Mitchell forcing, $\mathsf{ITP}(\omega_2, \lambda)$ holds for all $\lambda \geq \omega_2$. The same argument works for $\mathbb{M}_0$; the key point is that $\mathbb{M}_0$, as well as all of its quotients over initial segments of inaccessible length, have the property that there are projections onto them from products $\mathbb{P}\times\mathbb{Q}$, where $\mathbb{P}$ is the forcing to add κ-many Cohen reals and $\mathbb{Q}$ is countably closed (see Section 8 below for more about Mitchell forcing and this property in particular). □

The following is now immediate.

Corollary 4.22. *Suppose that* κ *is a supercompact cardinal. Then there is a forcing extension in which* $\mathsf{ISP}(\omega_2, \omega_2, \omega_2)$ *fails but* $\mathsf{ITP}(\omega_2, \lambda)$ *holds for all* $\lambda \geq \omega_2$.

5. Slender trees and the almost guessing property

In this section, we isolate a guessing principle in the style of $\mathsf{GMP}(\ldots)$ that provides an alternative formulation of $\mathsf{SP}(\ldots)$ in the same way that $\mathsf{GMP}(\ldots)$ provides an alternative formulation of $\mathsf{ISP}(\ldots)$. We begin with a rough heuristic that the reader may or may not find helpful. Note that the guessing models witnessing principles of the form $\mathsf{GMP}(\ldots)$ are in effect performing two roles. First, they are approximating a given set of interest. Second, they are guessing this set, and providing the setting for subsequent elementarity arguments. The fact that $\mathsf{ISP}(\ldots)$ produces *ineffable* branches for slender lists, i.e., branches whose small pieces are precisely approximated by the elements of the list at stationarily many entries, is what allows us to find guessing models that fulfill these two roles simultaneously. When we only have $\mathsf{SP}(\ldots)$, though, and are not necessarily able to find ineffable branches, it is possible that these two roles must be pulled apart and fulfilled by two different sets, the second a possibly proper subset of the first. We now make this heuristic more precise.

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[2]For concreteness, we focus here on ω_2, but analogous arguments work at other double successors of regular cardinals.

Definition 5.1. Let $\mu \leq \kappa \leq \theta$ be uncountable cardinals, with κ and θ regular, and suppose that $x \in H(\theta)$, $S \subseteq \mathscr{P}_\kappa H(\theta)$ is cofinal, and $M \subseteq H(\theta)$. We say that (M, x) is *almost μ-guessed by S* if $x \in M$ and, for every (μ, M)-approximated subset $d \subseteq x$, there is an $N \in S$ such that

- $x \in N \subseteq M$; and
- d is N-guessed.

Definition 5.2. Let $\mu \leq \kappa \leq \theta$ be uncountable cardinals, with κ and θ regular, and let $\mathcal{Y} \subseteq \mathscr{P}_\kappa H(\theta)$ be stationary. We say that $\mathsf{AGP}_{\mathcal{Y}}(\mu, \kappa, \theta)$ holds if for every cofinal $S \subseteq \mathscr{P}_\kappa H(\theta)$ and every $x \in H(\theta)$, the set of $M \in \mathcal{Y}$ such that (M, x) is almost μ-guessed by S is stationary in $\mathscr{P}_\kappa H(\theta)$.

All of the notational conventions regarding $\mathsf{GMP}_{\mathcal{Y}}(\ldots)$ from Remark 4.10 apply, *mutatis mutandis*, to the setting of $\mathsf{AGP}_{\mathcal{Y}}(\ldots)$.

Theorem 5.3. *Let $\mu \leq \kappa \leq \theta$ be uncountable cardinals, with κ and θ regular, and let $\mathcal{Y} \subseteq \mathscr{P}_\kappa H(\theta)$ be stationary. If $\mathsf{SP}_{\mathcal{Y}}(\mu, \kappa, H(\theta))$ holds, then $\mathsf{AGP}_{\mathcal{Y}}(\mu, \kappa, \theta)$ holds.*

Proof. Suppose that $\mathsf{SP}_{\mathcal{Y}}(\mu, \kappa, H(\theta))$ holds. Fix a cofinal $S \subseteq \mathscr{P}_\kappa H(\theta)$ and a set $x \in H(\theta)$, and assume for sake of contradiction that there is a club $C \subseteq \mathscr{P}_\kappa H(\theta)$ such that, for all $M \in C \cap \mathcal{Y}$, (M, x) is not almost μ-guessed by S. By thinning out C if necessary, we can assume that $x \in M$ for all $M \in C$. Therefore, for each $M \in C \cap \mathcal{Y}$, we can fix a set $d_M \subseteq x$ such that d_M is (μ, M)-approximated but there is no set $N \in S$ such that $x \in N$, $N \subseteq M$, and d_M is N-guessed. As in the proof of Proposition 4.12, by replacing d_M by $d_M \cap M$ if necessary, we can assume that $d_M \subseteq M$. For $M \in \mathscr{P}_\kappa H(\theta) \setminus C \cap \mathcal{Y}$, choose an arbitrary $M^* \in C \cap \mathcal{Y}$ such that $M \subseteq M^*$, and let $d_M := d_{M^*} \cap M$.

Let $D := \langle d_M \mid M \in \mathscr{P}_\kappa H(\theta) \rangle$. Again as in the proof of Proposition 4.12, it follows that D is a μ-$\mathcal{Y}$-slender $(\kappa, H(\theta))$-list. Therefore, by $\mathsf{SP}_{\mathcal{Y}}(\mu, \kappa, H(\theta))$, we can find a cofinal branch $d \subseteq H(\theta)$ through D. Since $d_M \subseteq x$ for all $M \in \mathscr{P}_\kappa H(\theta)$, it follows that $d \subseteq x$, and hence $d \in H(\theta)$. We can therefore fix an $N \in S$ such that $d, x \in N$ and then fix an $M \in \mathscr{P}_\kappa H(\theta)$ such that $M \supseteq N$ and $d \cap N = d_M \cap N$. We can assume that $M \in C \cap \mathcal{Y}_1$ (if not, we can replace it with the M^* used in the definition of d_M, since $d_M = d_{M^*} \cap M$). But then d_M is N-guessed, as witnessed by d, contradicting the fact that there is no $N \in S$ such that $x \in N$, $N \subseteq M$, and d_M is N-guessed. □

Theorem 5.4. *Let $\mu \leq \kappa \leq \lambda$ be uncountable cardinals, with κ and θ regular, and let $\mathcal{Y} \subseteq \mathscr{P}_\kappa \lambda$ be stationary. Suppose that there is a regular cardinal $\theta > \lambda^{<\kappa}$ such that $\mathsf{AGP}_{\mathcal{Y}'}(\mu, \kappa, \theta)$ holds, where $\mathcal{Y}' := \mathcal{Y} \uparrow \mathscr{P}_\kappa H(\theta)$. Then $\mathsf{SP}_{\mathcal{Y}}(\mu, \kappa, \lambda)$ holds.*

Proof. Let $D = \langle d_x \mid x \in \mathscr{P}_\kappa \lambda \rangle$ be a μ-$\mathcal{Y}$-slender (κ, λ)-list. We will find a cofinal branch for D. Let $C \subseteq \mathscr{P}_\kappa H(\lambda^+)$ be a club witnessing that D is μ-$\mathcal{Y}$-slender. Let S be the set of $N \prec H(\theta)$ such that $|N| < \kappa$, $D \in N$, and $N \cap \kappa \in \kappa$. By hypothesis, we can find $M \in \mathscr{P}_\kappa H(\theta) \cap \mathcal{Y}'$ such that $M \cap H(\lambda^+) \in C$ and (M, λ) is almost μ-guessed by S. Since $M \cap H(\lambda^+) \in C$ and $M \cap H(\lambda) \in \mathcal{Y}$, the fact that C witnesses that D is μ-$\mathcal{Y}$-slender implies that $d_{M \cap \lambda}$ is a (μ, M)-approximated subset of λ. Therefore, we can find $N \in S$ such that $N \subseteq M$ and $d_{M \cap \lambda}$ is N-guessed, i.e., there is $e \in N$ such that $e \cap N = d_{M \cap \lambda} \cap N$.

Since $d_{M \cap \lambda} \subseteq \lambda$, it follows from the elementarity of N that $e \subseteq \lambda$. We claim that e is a cofinal branch of D, which will finish the proof. To verify this claim, first fix an arbitrary $x \in \mathscr{P}_\kappa\lambda \cap N$. Since $N \cap \kappa \in \kappa$, it follows that $|x| \subseteq N$ and hence, by elementarity, $x \subseteq N$. In particular, we have $e \cap x = d_{M\cap\lambda} \cap x$. Now

$$H(\theta) \models \text{“}\exists z \in \mathscr{P}_\kappa\lambda(z \supseteq x \text{ and } e \cap x = d_z \cap x)\text{”},$$

as witnessed by $M \cap \lambda$. By elementarity, N satisfies this statement as well. Since this holds for all $x \in \mathscr{P}_\kappa\lambda \cap N$, we have

$$N \models \text{“}\forall x \in \mathscr{P}_\kappa\lambda\ \exists z \in \mathscr{P}_\kappa\lambda(z \supseteq x \text{ and } e \cap x = d_z \cap x)\text{”}.$$

Again by elementarity, this statement holds in $H(\theta)$. But this is precisely the assertion that e is a cofinal branch of D, as desired. □

We therefore obtain the following corollary, proving half of Theorem A.

Corollary 5.5. *Suppose that $\mu \leq \kappa$ are regular uncountable cardinals. Then the following are equivalent:*

(i) $\mathsf{SP}(\mu, \kappa, \geq \kappa)$;
(ii) $\mathsf{AGP}(\mu, \kappa, \geq \kappa)$.

In analogy with the principle $\mathsf{TP}^-(\kappa, \lambda)$ from subsection 4.1, we will also be interested in weakenings of the slender tree property and almost guessing property in which “club” and “stationary” are replaced by “strong club” and “weakly stationary”. Unlike the situation with TP^- and TP however, we do not know whether these weak versions are equivalent to their seemingly stronger relatives.

Definition 5.6. Let $\mu \leq \kappa \leq \lambda$ be uncountable cardinals, with κ regular, and let $\mathcal{Y} \subseteq \mathscr{P}_\kappa\lambda$ be weakly stationary.

(i) We say that a (κ, λ)-list $D = \langle d_x \mid x \in \mathscr{P}_\kappa\lambda\rangle$ is *strongly μ-$\mathcal{Y}$-slender* if for a sufficiently large regular cardinal θ there is a strong club $C \subseteq \mathscr{P}_\kappa H(\theta)$ such that, for all $M \in C$ and all $y \in M \cap \mathscr{P}_\mu\lambda$, if $M \cap \lambda \in \mathcal{Y}$, then there is $e \in M$ such that we have $d_{M\cap\lambda} \cap y = e \cap y$.
(ii) The *weak (μ, κ, λ)-slender tree property on $\mathcal{Y}$*, denoted $\mathsf{wSP}_{\mathcal{Y}}(\mu, \kappa, \lambda)$, is the assertion that every strongly μ-$\mathcal{Y}$-slender (κ, λ)-list has a cofinal branch.
(iii) For a regular cardinal $\theta \geq \kappa$ and a weakly stationary $\mathcal{Y}' \subseteq \mathscr{P}_\kappa H(\theta)$, the principle $\mathsf{wAGP}_{\mathcal{Y}'}(\mu, \kappa, \theta)$ is the assertion that for every cofinal $S \subseteq \mathscr{P}_\kappa H(\theta)$ and every $x \in H(\theta)$, the set of $M \in \mathscr{P}_\kappa H(\theta) \cap \mathcal{Y}'$ such that (M, x) is almost μ-guessed by S is weakly stationary in $\mathscr{P}_\kappa H(\theta)$.

Again, the notational conventions from Remarks 4.8 and 4.10 apply to the principles $\mathsf{wSP}_{\mathcal{Y}}(\ldots)$ and $\mathsf{wAGP}_{\mathcal{Y}}(\ldots)$, respectively.

Remark 5.7. Note that our formulation of strongly μ-slender differs from the formulation of μ-slender in that the conclusion does not require $d_{M\cap\lambda} \cap y \in M$ but rather the existence of $e \in M$ such that $d_{M\cap\lambda} \cap y = e \cap y$. These are clearly equivalent if M is closed under finite intersections, but since we cannot assume that all elements of a strong club are closed under intersections (unlike the situation with clubs), this seems like the more appropriate definition.

Remark 5.8. Just as in Fact 4.2, it can be shown that a thin (κ, λ)-list is in fact *strongly* κ-slender, so $\mathsf{wSP}(\kappa, \kappa, \lambda)$ is enough to imply $\mathsf{TP}(\kappa, \lambda)$.

The straightforward analogues of Theorems 5.3 and 5.4 hold for wSP and wAGP. The proofs are essentially identical, so we omit them, noting only that, in the proof of the analogue of Theorem 5.3, when verifying that the constructed $(\kappa, H(\theta))$-list is strongly μ-$\mathcal{Y}_1$-slender, it is important that the conclusion in the definition of strongly μ-slender is weakened from the conclusion of μ-slender, as discussed in Remark 5.7. We therefore obtain the following corollary, completing the proof of Theorem A.

Corollary 5.9. *Suppose that $\mu \leq \kappa$ are regular uncountable cardinals. Then the following are equivalent:*
(i) $\mathsf{wSP}(\mu, \kappa, \geq \kappa)$*;*
(ii) $\mathsf{wAGP}(\mu, \kappa, \geq \kappa)$.

5.1. **Guessing models for small sets.** In this subsection, we examine weakenings of GMP in which we only require models to be guessing for sets of some fixed small cardinality. We will see that such principles do not require the full power of $\mathsf{ISP}(\ldots)$ but in fact follow already from an appropriate instance of $\mathsf{SP}(\ldots)$. At the same time, these principles are strong enough to imply statements such as the failure of the weak Kurepa Hypothesis. For concreteness, we focus on guessing models for sets of size ω_1, but it will be evident how to adjust the results for other values of the relevant parameters.

Let us say that M is a *guessing model for sets of size* ω_1 if for every $z \in M$ with $|z| = \omega_1$, if $d \subseteq z$ is (ω_1, M)-approximated, then it is M-guessed. We first show that if M is a guessing model for z for some set $z \in M$, then M is a guessing model for y for all sets y in M which have the same size as z. Therefore to show that

$$\{M \prec H(\theta) \mid |M| < \omega_2 \text{ and } M \text{ is a guessing model for sets of size } \omega_1\}$$

is stationary, it is enough to show that $\mathcal{G}^z_{\omega_2} H(\theta)$ is stationary for some $z \in H(\theta)$ of size ω_1.

Lemma 5.10. *Let $M \prec H(\theta)$ and $z \in M$. If M is a guessing model for z, then M is a guessing model for y for every $y \in M$ such that $|z| = |y|$.*

Proof. Let $M \prec H(\theta)$ be a guessing model for z and let $y \in M$ have the same size as z. This means that there is in M a bijection $f : y \to z$. Let $d \subseteq y$ be M-approximated. Then $f''d$ is a subset of z, which is also M-approximated. To see this, let $a \subseteq z$ be a countable and in M. Then $a \cap f''d \in M$, since $f^{-1}{}''(a \cap f''d) = f^{-1}{}''a \cap d$ is in M. Therefore there is $e \in M$, $e \subseteq z$ such that $e \cap M = f''d \cap M$ and hence $f^{-1}{}''e \cap M = d \cap M$. □

Lemma 5.11. *Assume that $\mathsf{SP}(\omega_1, \omega_2, H(\theta))$ holds. Then for all $z \in H(\theta)$ with $|z| = \omega_1$, the set $\mathcal{G}^z_{\omega_2} H(\theta)$ is stationary in $\mathscr{P}_{\omega_2} H(\theta)$.*

Proof. Assume for sake of contradiction that $z \in H(\theta)$, $|z| = \omega_1$, and $\mathcal{G}^z_{\omega_2} H(\theta)$ is nonstationary. This means that there is a club $C \subseteq \mathscr{P}_{\omega_2} H(\theta)$ such that for all $M \in C$ there is $d_M \subseteq z \in M$ such that d_M is (ω_1, M)-approximated but not M-guessed. Note that we can assume that z is also a subset of M for all $M \in C$ since z has size ω_1. For $N \in \mathscr{P}_{\omega_2} H(\theta) \setminus C$, choose $M \in C$ such that $N \subseteq M$ and let $d_N := d_M \cap N$.

Let $D = \langle d_M \mid M \in \mathscr{P}_\kappa H(\theta) \rangle$. Again as in the proof of Proposition 4.12, it follows that D is an ω_1-slender $(\omega_2, H(\theta))$-list. Therefore, by $\mathsf{SP}(\omega_1, \omega_2, H(\theta))$ there is $d \subseteq H(\theta)$ such that for all $M \in C$ there is $N \supseteq M$ such that

$$d \cap M = d_N \cap M. \tag{1}$$

By our definition of D, we can always choose such an N to be in C. By (1), d is a subset of z and in particular it is in $H(\theta)$. Let M in C be such that $d \in M$. Then there is $N \supseteq M$ such that $N \in C$ and $d \cap M = d_N \cap M$. We then have

$$d = d \cap M = d_N \cap M = d_N = d_N \cap N. \tag{2}$$

The first equality above holds because $d \subseteq z \subseteq M$, and the last two equalities hold because $d_N \subseteq z \subseteq M \subseteq N$. But then $d \in N$ and $d \cap N = d_N \cap N$, contradicting the assumption that d_N is not N-guessed. □

Putting together a couple of results, we can actually show that $\mathsf{SP}(\omega_1, \omega_2, H(\omega_2))$ is sufficient to obtain the conclusion of the previous lemma.

Corollary 5.12. *Assume that $\mathsf{SP}(\omega_1, \omega_2, H(\omega_2))$ holds. Then for every regular $\theta \geq \omega_2$, the set*

$$\{M \prec H(\theta) \mid |M| < \omega_2 \text{ and } M \text{ is a guessing model for sets of size } \omega_1\}$$

is stationary in $\mathscr{P}_{\omega_2} H(\theta)$.

Proof. By Lemma 5.10, it is enough to show that, for every regular $\theta \geq \omega_2$, $\mathcal{G}^{\omega_1}_{\omega_2} H(\theta)$ is stationary in $\mathscr{P}_{\omega_2} H(\theta)$. By Clause 1 of Proposition 4.11, this is equivalent to the assertion that $\mathcal{G}^{\omega_1}_{\omega_2} H(\omega_2)$ is stationary in $\mathscr{P}_{\omega_2} H(\omega_2)$. By Lemma 5.11, this follows from the hypothesis of $\mathsf{SP}(\omega_1, \omega_2, H(\omega_2))$. □

Since every element of $H(\omega_2)$ has cardinality at most ω_1, it follows that the principle $\mathsf{GMP}(\omega_1, \omega_2, H(\omega_2))$ is equivalent to the conclusion of the previous corollary. In particular, as a special case of Theorem 9.3 below, we see that the existence of guessing models for sets of size ω_1 is sufficient to imply the nonexistence of weak Kurepa trees.

6. Subadditive functions and weak guessing properties

Definition 6.1. Suppose that χ and λ are infinite cardinals and $c : [\lambda]^2 \to \chi$. We say that c is *subadditive* if, for all $\alpha < \beta < \gamma < \lambda$, the following two triangle inequalities hold:
(i) $c(\alpha, \gamma) \leq \max\{c(\alpha, \beta), c(\beta, \gamma)\}$;
(ii) $c(\alpha, \beta) \leq \max\{c(\alpha, \gamma), c(\beta, \gamma)\}$.
We say that c is *strongly unbounded* if, for every unbounded $A \subseteq \lambda$, $c``[A]^2$ is unbounded in χ.

In [16, Theorem 10.3 and Proposition 6.1], it is shown that $\mathsf{GMP}(\omega_1, \omega_2, \geq \omega_2)$ implies that, for every regular $\lambda \geq \omega_2$, there are no subadditive, strongly unbounded functions $c : [\lambda]^2 \to \omega$. We now prove a generalization and strengthening of this result by proving that the nonexistence of subadditive, strongly unbounded functions follows from relevant instances of $\mathsf{wAGP}(\ldots)$. Together with Corollary 6.4, this yields clause (2) of Theorem C. In Section 8, we will see that the hypotheses of the following theorem hold, for instance, after forcing with $\mathbb{M}(\mu, \kappa)$ when κ is strongly compact, where μ is a regular infinite cardinal and $\mathbb{M}(\mu, \kappa)$ is the Mitchell forcing that collapses κ to be μ^{++}.

Theorem 6.2. *Suppose that the following hypotheses hold:*
(1) $\chi < \chi^+ < \kappa \leq \lambda$ are infinite cardinals, with κ regular and $\mathrm{cf}(\lambda) \geq \kappa$;
(2) $\mathcal{Y} := \{M \in \mathscr{P}_\kappa H(\lambda^+) \mid \mathrm{cf}(\sup(M \cap \lambda)) > \chi\}$;

(3) $\mathsf{wAGP}_{\mathcal{Y}}(\kappa, \kappa, \lambda^+)$ *holds.*
Then there are no subadditive, strongly unbounded functions $c : [\lambda]^2 \to \chi$.

Proof. Fix a subadditive function $c : [\lambda]^2 \to \chi$. We will find an unbounded $A \subseteq \lambda$ such that $c``[A]^2$ is bounded below χ.

For each $\beta < \lambda$ and each $i < \chi$, let $c_{\beta,i} : \beta \to \chi$ be defined by setting

$$c_{\beta,i}(\alpha) := \begin{cases} c(\alpha, \beta) & \text{if } c(\alpha, \beta) \geq i \\ i & \text{if } c(\alpha, \beta) < i. \end{cases}$$

Note that the subadditivity of c implies that, for all $\alpha < \beta < \lambda$ and all $i < \chi$, if $c(\alpha, \beta) \leq i$, then $c_{\beta,i} \restriction \alpha = c_{\alpha,i}$. We will think of functions such as $c_{\beta,i}$ as subsets of $\lambda \times \chi$ in the natural way.

Let S be the set of $N \prec H(\lambda^+)$ such that $|N| < \kappa$, $\chi \subseteq N$, $c \in N$, and $\mathrm{cf}(\sup(N \cap \lambda)) > \chi$. Since $\mathsf{wAGP}_{\mathcal{Y}}(\kappa, \kappa, H(\lambda^+))$ holds, there are weakly stationarily many $M \in \mathcal{Y}$ such that $(M, \lambda \times \chi)$ is almost κ-guessed by S. In particular, we can find such an $M \in \mathcal{Y}$ with the following additional properties:

- $c, \lambda \times \omega \in M$;
- for all $\beta \in M \cap \lambda$ and all $i < \theta$, we have $c_{\beta,i} \in M$;
- for all $z \in M \cap \mathscr{P}_\kappa(\lambda \times \chi)$, we have $\sup\{\alpha < \lambda \mid \exists i < \chi[(\alpha, i) \in z]\} \in M$.

Let $\gamma := \sup(M \cap \lambda)$. Since $M \in \mathcal{Y}$, we have $\mathrm{cf}(\gamma) > \chi$. We can therefore fix an $i_0 < \chi$ and an unbounded $B \subseteq M \cap \lambda$ such that, for all $\beta \in B$, we have $c(\beta, \gamma) = i_0$.

Claim 6.3. c_{γ,i_0} *is* (κ, M)*-approximated.*

Proof. Fix a set $z \in M \cap \mathscr{P}_\kappa(\lambda \times \chi)$. We must find $e \in M$ such that $c_{\gamma,i_0} \cap z = e \cap z$. We have $\sup\{\alpha < \lambda \mid \exists i < \chi[(\alpha, i) \in z]\} \in M$, so we can find $\beta \in B$ such that $z \subseteq \beta \times \chi$. Since $c(\beta, \gamma) = i_0$, we have $c_{\gamma,i_0} \restriction \beta = c_{\beta,i_0}$, and hence $c_{\gamma,i_0} \cap z = c_{\beta,i_0} \cap z$. Moreover, we have $c_{\beta,i_0} \in M$, so c_{β,i_0} is as desired. $\square$

Since $(M, \lambda \times \chi)$ is almost κ-guessed by S, we can find $N \in S$ such that $N \subseteq M$ and c_{γ,i_0} is N-guessed, i.e., there is $e \in N$ such that $c_{\gamma,i_0} \cap N = e \cap N$. Note that $c_{\gamma,i_0} \cap N$ is a function from $N \cap \lambda$ to χ. By elementarity and the fact that $\chi + 1 \subseteq N$, it follows that e is a function from λ to χ. Let $\delta := \sup(N \cap \lambda)$. Since $N \in S$, we have $\mathrm{cf}(\delta) > \chi$. We can therefore find $i_1 \in [i_0, \chi)$ such that $A_0 := \{\alpha \in N \cap \lambda \mid e(\alpha) \leq i_1\}$ is unbounded in δ. Let $A := \{\alpha < \lambda \mid e(\alpha) \leq i_1\}$. All of the parameters needed to define A are in N, so $A \in N$. Moreover, $N \models$ "A is unbounded in λ", so, by elementarity, A is in fact unbounded in λ. We will therefore be finished if we show that $c(\alpha, \beta) \leq i_1$ for all $\alpha < \beta$ in A. By elementarity, it suffices to show that $c(\alpha, \beta) \leq i_1$ for all $\alpha < \beta$ in $N \cap A$.

To this end, fix $\alpha < \beta$ in $N \cap A$. By definition of A, we know that $e(\alpha), e(\beta) \leq i_1$. Since $N \subseteq M$, $e \cap N = c_{\gamma,i_0} \cap N$, and $i_1 \geq i_0$, it follows that $\max\{c(\alpha, \gamma), c(\beta, \gamma)\} \leq i_1$. By the subadditivity of c, we can conclude that $c(\alpha, \beta) \leq i_1$, finishing the proof. $\square$

Corollary 6.4. *Suppose that* $\omega_2 \leq \kappa \leq \lambda$ *are regular cardinals,*

$$\mathcal{Y} := \{M \in \mathscr{P}_\kappa H(\lambda^+) \mid \mathrm{cf}(\sup(M \cap \lambda)) > \omega\},$$

and $\mathsf{wAGP}_{\mathcal{Y}}(\kappa, \kappa, \lambda^+)$ *holds. Then* $\square(\lambda)$ *fails.*

Proof. By the combination of [16, Proposition 6.5] and [17, Theorem 3.4], $\square(\lambda)$ implies the existence of a subadditive, strongly unbounded function $c : [\lambda]^2 \to \omega$. The corollary now follows from Theorem 6.2. $\square$

7. Preservation lemmas

In this section, we prove a variety of preservation lemmas that will be used in our consistency results in the remaining sections of the paper.

Lemma 7.1. *Suppose that $\mu < \kappa$ are regular uncountable cardinals, Λ is a κ-directed poset, T is a Λ-tree, $\mathbb{P}$ is a μ-c.c. forcing notion, and*

$$\Vdash_{\mathbb{P}} \text{“there is a cofinal branch through } T\text{”}.$$

Then there is a cofinal branch through T in V.

Proof. Let $\dot{b}$ be a $\mathbb{P}$-name for a cofinal branch through T. For each $u \in \Lambda$, let

$$T'_u := \{s \in T_u \mid \exists p \in \mathbb{P}\ [p \Vdash_{\mathbb{P}} \text{“}\dot{b}(u) = s\text{”}]\}.$$

Note that, if $u <_\Lambda v$, $t \in T_v$, $s = t \restriction u$, $p \in \mathbb{P}$, and $p \Vdash_{\mathbb{P}}$ “$\dot{b}(v) = t$”, then we also have $p \Vdash_{\mathbb{P}}$ “$\dot{b}(u) = s$”. It follows that $T' := (\langle T'_u \mid u \in \Lambda\rangle, <_{T'})$ is a subtree of T, where $<_{T'}$ is the restriction of $<_T$ to $\bigcup_{u\in\Lambda} T'_u$. Also, since $\mathbb{P}$ has the μ-c.c., we know that $|T'_u| < \mu$ for all $u \in \Lambda$. In particular, since $\mu < \kappa$ and Λ is κ-directed, T' is very thin. By Lemma 2.7, T' has a cofinal branch, which is plainly a cofinal branch through T, as well. □

Lemma 7.2. *Let ξ be a cardinal and $\mu < \kappa$ be regular cardinals such that $2^\mu \geq \xi$ and $2^{<\mu} < \kappa$. Let $\mathbb{Q}$ be a μ^+-closed forcing and $\mathbb{P}$ be μ^+-cc. Assume that Λ is a κ-directed poset in V. If T is a Λ-tree with width at most ξ in $V^{\mathbb{P}}$, then forcing with Q over $V^{\mathbb{P}}$ does not add a cofinal branch through T.*

Proof. Let $p \in \mathbb{P}$ be a condition which forces that $\dot{T}$ is a Λ-tree of width at most ξ, and assume further that for some $q \in \mathbb{Q}$, (p, q) forces that $\dot{b}$ is a cofinal branch through $\dot{T}$ which is not in $V^{\mathbb{P}}$ (we view $\dot{b}$ as a $\mathbb{P} \times \mathbb{Q}$-name).

First we prove the following auxiliary claim.

Claim 7.3. *Assume that q'_0 and q'_1 are conditions in $\mathbb{Q}$ which extend q, and let $x' \in \Lambda$. Then there are a maximal antichain Y in $\mathbb{P}$ below p, conditions $q_0 \leq_{\mathbb{Q}} q'_0$, $q_1 \leq_{\mathbb{Q}} q'_1$, and $x >_\Lambda x'$ such that whenever $p' \in Y$, then (p', q_0) and (p', q_1) force contradictory information about $\dot{b}$ on level x; i.e. there are $\mathbb{P}$-names $\dot{t}_0$ and $\dot{t}_1$ such that $p' \Vdash_{\mathbb{P}} \dot{t}_0 \neq \dot{t}_1 \in \dot{T}_x$ and*

$$(p', q_0) \Vdash_{\mathbb{P}\times\mathbb{Q}} \dot{b}(x) = \dot{t}_0 \text{ and } (p', q_1) \Vdash_{\mathbb{P}\times\mathbb{Q}} \dot{b}(x) = \dot{t}_1.$$

Proof. Let $q'_0, q'_1 \leq_{\mathbb{Q}} q$ and $x' \in \Lambda$ be given. We will construct by induction a maximal antichain $Y = \{p'_i \in \mathbb{P} \,|\, i < \gamma\}$ below p, decreasing sequences $\langle q^i_0 \,|\, i < \gamma\rangle$ and $\langle q^i_1 \,|\, i < \gamma\rangle$ of conditions in $\mathbb{Q}$ and a $<_\Lambda$-increasing sequence $\langle x_i \,|\, i < \gamma\rangle$, for some $\gamma < \mu^+$.

Assume $\delta < \mu^+$ and we already constructed $\{p'_i \in \mathbb{P} \,|\, i < \delta\}$, $\langle q^i_0 \,|\, i < \delta\rangle$, $\langle q^i_1 \,|\, i < \delta\rangle$ and $\langle x_i \,|\, i < \delta\rangle$. Suppose that $\{p'_i \in \mathbb{P} \,|\, i < \delta\}$ is not a maximal antichain in $\mathbb{P}$ below p. We distinguish two cases: (A) δ is a successor ordinal, and (B) δ is a limit ordinal.

Case (A). Suppose $\delta = \delta' + 1$. Since $\{p'_i \,|\, i < \delta\}$ is not a maximal antichain in $\mathbb{P}$ below p, there is $p^* \in \mathbb{P}$ such that $p^* \leq_{\mathbb{P}} p$ and p^* is incompatible with p'_i for all $i < \delta$. Now, consider the condition $(p^*, q^0_{\delta'})$ and $x_{\delta'}$. Since $\dot{b}$ is forced by (p, q) to be a cofinal branch through $\dot{T}$ which is not in $V^{\mathbb{P}}$, there are $x' >_\Lambda x_{\delta'}$, $p' \leq_{\mathbb{P}} p^*$, and r_0 and r_1 both extending $q^0_{\delta'}$ in $\mathbb{Q}$, such that $(p', r_0) \Vdash_{\mathbb{P}\times\mathbb{Q}} \dot{b}(x') = \dot{t}_0$ and $(p', r_1) \Vdash_{\mathbb{P}\times\mathbb{Q}} \dot{b}(x') = \dot{t}_1$, where $\dot{t}_0$ and $\dot{t}_1$ are forced by p' to be distinct elements of

$\dot{T}_{x'}$. Moreover, let $(p'', r_2) \leq (p', q^1_{\delta'})$ be such that it decides $\dot{b}$ on level x'; i.e there is a P-name $\dot{t}$ such that $(p'', r_2) \Vdash_{\mathbb{P}\times\mathbb{Q}} \dot{b}(x') = \dot{t}$. Since $p'' \leq p'$ and $p' \Vdash_{\mathbb{P}} \dot{t}_0 \neq \dot{t}_1$, p'' has to force that $\dot{t} \neq \dot{t}_0$ or $\dot{t} \neq \dot{t}_1$. If p'' forces that $\dot{t} \neq \dot{t}_0$ then define $q^0_\delta = r_0$, if not then define $q^0_\delta = r_1$. Let $p'_\delta = p''$, $q^1_\delta = r_2$ and $x_\delta = x'$.

Case (B). If δ is a limit ordinal, then we first take q^*_0 and q^*_1 to be some lower bounds of $\langle q^i_0 \,|\, i < \delta\rangle$ and $\langle q^i_1 \,|\, i < \delta\rangle$ respectively and x^* be an upper bound of $\langle x_i \,|\, i < \delta\rangle$. Then we proceed as in the successor step, using q^*_0, q^*_1 and x^* instead of $q^0_{\delta'}$, $q^1_{\delta'}$ and $x_{\delta'}$, respectively.

By the κ^+-cc of $\mathbb{P}$, the inductive construction must stop at some $\delta < \kappa^+$ in the sense that $\{p'_i \,|\, i < \delta\}$ is a maximal antichain in $\mathbb{P}$ below p. Let $Y = \{p'_i \,|\, i < \delta\}$, let q_0 and q_1 be some lower bounds of $\langle q^0_i \,|\, i < \delta\rangle$ and $\langle q^1_i \,|\, i < \delta\rangle$, respectively, and finally let x be an upper bound of $\{x_i \,|\, i < \delta\}$.

It is easy to verify that for every $p' \in Y$, (p', q_0) and (p', q_1) force contradictory information about $\dot{b}$ on level x. This follows immediately from the construction of Y, q_0 and q_1. □

Iteratively using Claim 7.3, one can build a labeled full binary tree $\mathscr{T}$ in $2^{<\mu}$ such that for every $s \in 2^{<\mu}$ there are $q_s \in \mathbb{Q}$, $x_s \in \Lambda$ and $Y_s \subseteq \mathbb{P}$ such that the following hold:

(1) For all $s \in 2^{<\mu}$, $q_s \leq q$ and Y_s is a maximal antichain in $\mathbb{P}$ below p.
(2) For all $s \in 2^{<\mu}$ and $p' \in Y_s$, $(p', q_{s^\frown 0})$ and $(p', q_{s^\frown 1})$ force contradictory information about $\dot{b}$ on level x_s.
(3) For all $f \in 2^\mu$, the conditions $\langle q_{f\restriction\alpha} \,|\, \alpha < \mu\rangle$ are decreasing in $\leq_Q$.
(4) For all $f \in 2^\mu$, the sets $\langle x_{f\restriction\alpha} \,|\, \alpha < \mu\rangle$ are increasing in $<_\Lambda$.

Suppose for the moment that $\mathscr{T}$ has been constructed; then we can conclude the proof as follows. Since $2^{<\mu} < \kappa$ and Λ is κ-directed, we can fix an $x \in \Lambda$ such that $x_s \leq_\Lambda x$ for all $s \in 2^{<\mu}$. Let G be a $\mathbb{P}$-generic filter over V such that $p \in \mathbb{P}$. Next, still in V, let us fix lower bounds q_f of $\langle q_{f\restriction\alpha} \,|\, \alpha < \mu\rangle$ for all $f \in 2^\mu$ (they exist since $\mathbb{Q}$ is μ^+-closed in V). In $V[G]$, for each f, we find $q'_f \leq q_f$ such that q'_f decides $\dot{b}(x)$. We claim that, for each $f \neq g \in (2^\mu)^V$, q'_f and q'_g force contradictory information about $\dot{b}$ on level x. To see this, assume that $f \neq g \in (2^\mu)^V$. Let $s = f \cap g$, and without loss of generality let f extend $s^\frown 0$ and g extend $s^\frown 1$. By the properties of the tree $\mathscr{T}$, there is $p' \in Y_s$ such that $p' \in G$ and $(p', q_{s^\frown 0})$ and $(p', q_{s^\frown 1})$ force contradictory information about $\dot{b}$ on level x_s. Since $q'_f \leq q_{s^\frown 0}$ and $q'_g \leq q_{s^\frown 1}$, (p', q'_f) and (p', q'_g) force contradictory information about $\dot{b}$ on level x_s, and hence also on level $x >_\Lambda x_s$. However, this is a contradiction since $2^\mu \geq \xi$ and there are only $< \xi$ many nodes on level x in T; i.e., $< \xi$ many possible values of $\dot{b}(x)$.

To finish the argument it is enough to construct the tree $\mathscr{T}$. The construction will proceed by induction on the length of s; during the construction we will also construct an auxiliary increasing sequence $\langle x_\alpha \,|\, \alpha < \mu\rangle$ of elements of Λ.

For $\alpha = 0$, let $q_\emptyset = q$ and let $x_0 = x_\emptyset$ be an arbitrary element of Λ.

Assume now that we have constructed $\mathscr{T} \restriction \alpha$. If α is limit, then for $s \in 2^\alpha$, let q_s be a lower bound of $\langle q_{s\restriction\beta} \,|\, \beta < \alpha\rangle$ and x_α an upper bound of $\{x_t \in \Lambda \,|\, t \in 2^{<\alpha}\}$. Note that x_α exists since we assume that Λ is κ-directed and $2^{<\mu} < \kappa$.

If α is equal to $\beta + 1$, and $s \in 2^\beta$, then the existence of $q_{s^\frown 0}, q_{s^\frown 1} \in \mathbb{Q}$, $x_s \in \Lambda$ and Y_s follows from Claim 7.3 starting with $q'_0 = q'_1 = q_s$ and $x' = x_\alpha$, and defining $Y_s = Y$, $q_{s^\frown 0} = q_0$, $q_{s^\frown 1} = q_1$ and $x_s = x$. The properties (1)–(4) of the labeled tree $\mathscr{T}$ now follow immediately from the construction. □

Corollary 7.4. *Let ξ be a cardinal and $\mu < \kappa \leq \lambda$ be regular cardinals such that $2^\mu \geq \xi$ and $2^{<\mu} < \kappa$. Let $\mathbb{Q}$ be μ^+-closed forcing and $\mathbb{P}$ be μ^+-cc. If T is a $\mathscr{P}_\kappa\lambda$-tree with width at most ξ in $V^{\mathbb{P}}$, then forcing with $\mathbb{Q}$ over $V^{\mathbb{P}}$ does not add a cofinal branch through T.*

Proof. Note that, since $\mathbb{P}$ is μ^+-cc, $(\mathscr{P}_\kappa\lambda)^V$ is $\subseteq$-cofinal in $(\mathscr{P}_\kappa\lambda)^{V^{\mathbb{P}}}$. Hence the proof follows immediately from the previous theorem. If T is a $(\mathscr{P}_\kappa\lambda)^{V^{\mathbb{P}}}$-tree in $V^{\mathbb{P}}$ with a new cofinal branch in $V^{\mathbb{P}\times\mathbb{Q}}$, then $T' = \{T_x \mid x \in (\mathscr{P}_\kappa\lambda)^V\}$ also has a new cofinal branch in $V^{\mathbb{P}\times\mathbb{Q}}$, which is a contradiction to the previous theorem. □

We next recall the *covering* and *approximation* properties, introduced by Hamkins (cf. [9]).

Definition 7.5. Suppose that $V \subseteq W$ are transitive models of ZFC and μ is a regular uncountable cardinal.

(1) (V, W) satisfies the *μ-covering property* if, for every $x \in W$ such that $x \subseteq V$ and $|x|^W < \mu$, there is $y \in V$ such that $|y|^V < \mu$ and $x \subseteq y$.
(2) (V, W) satisfies the *μ-approximation property* if, for all $x \in W$ such that $x \subseteq V$ and $x \cap z \in V$ for all $z \in V$ with $|z| < \mu$, we in fact have $x \in V$.

A poset $\mathbb{P}$ has the *μ-covering property* (resp. *μ-approximation property*) if, for every V-generic filter $G \subseteq \mathbb{P}$, the pair $(V, V[G])$ has the μ-covering property (resp. μ-approximation property).

Lemma 7.6. *Suppose that $V \subseteq W$ are transitive models of ZFC, κ is a regular uncountable cardinal, and (V, W) has the κ-approximation property. Suppose also that, in V, Λ is a κ-directed partial order and T is a Λ-tree. Then every cofinal branch through T in W is already in V.*

Proof. Suppose that $b \in W$ is a cofinal branch through T. We will show that $b \in V$. Towards an application of the κ-approximation property, fix a set $z \in V$ with $|z| < \kappa$. We want to show that $b \cap z \in V$, so we may as well assume that $z \subseteq \bigcup\{\{u\} \times T_u \mid u \in \Lambda\}$, since b is also a subset of this set. Let

$$W := \{u \in \Lambda \mid \exists t \in T_u[(u, t) \in z]\}.$$

Then $|W| < \kappa$ and Λ is κ-directed, we can find $v \in \Lambda$ such that $u <_\Lambda v$ for all $u \in W$. Let $s := b(v)$. Then $b \cap z = \{(u, t) \in z \mid t <_T s\}$. All of the parameters in the right-hand side of this equation are in V, so $b \cap z \in V$. Since z was arbitrary, the κ-approximation property implies that $b \in V$. □

Lemma 7.7. *Suppose that $V \subseteq W$ are models of ZFC, κ is a regular uncountable cardinal in V, (V, W) satisfies the κ-covering property, Λ is a κ-directed partial order in V, and T is a Λ-tree in V. Suppose also that, in W, $\mathbb{P}$ is a κ-c.c. forcing that adds a new cofinal branch to T. Then there is a κ-Suslin tree $S \in W$ such that $\mathbb{P}$ adds a cofinal branch to S.*

Proof. Work in W, and let $\dot{b}$ be a $\mathbb{P}$-name for a new cofinal branch through T. For all $u \in \Lambda$, let $B_u := \{t \in T_u \mid \exists p \in \mathbb{P}\ (p \Vdash_{\mathbb{P}} \text{“}\dot{b}(u) = t\text{”})\}$. Since $\mathbb{P}$ has the κ-c.c., each B_u has cardinality less than κ.

We now recursively define a $<_\Lambda$-increasing sequence $\langle u_\eta \mid \eta < \kappa\rangle$ as follows. First, since $\Vdash_{\mathbb{P}}$ “$\dot{b} \notin W$”, we can find $u_0 \in \Lambda$ such that $|B_{u_0}| > 1$. Next, suppose that $\eta < \kappa$ and we have defined u_η. For each $t \in B_{u_\eta}$, again using the fact that $\Vdash_{\mathbb{P}}$ “$\dot{b} \notin W$”, we can find $u_{\eta,t} \in \Lambda$ such that $u_\eta <_\Lambda u_{\eta,t}$ and $|\{t' \in B_{u_{\eta,t}} \mid t' \restriction u_\eta = t\}| > 1$.

Since $|B_{u_\eta}| < \kappa$, (V, W) satisfies the κ-covering property and Λ is κ-directed in V, it follows that we can find $u_{\eta+1} \in \Lambda$ such that $u_{\eta,t} \leq_\Lambda u_{\eta+1}$ for all $t \in B_{u_\eta}$. Finally, if $\xi < \kappa$ is a limit ordinal and we have defined $\langle u_\eta \mid \eta < \xi \rangle$, again use the κ-covering property to find $u_\xi \in \Lambda$ such that $u_\eta \leq_\Lambda u_\xi$ for all $\eta < \xi$.

Now define a tree S by letting the underlying set of S be $\bigcup\{B_{u_\eta} \mid \eta < \kappa\}$ and letting the ordering $<_S$ be the restriction of $<_T$ to S. It is immediate that S is a tree of height κ and, for all $\eta < \kappa$, the η^{th} level of S is precisely B_{u_η}. Since $|B_{u_\eta}| < \kappa$, it follows that S is a κ-tree. Moreover, it follows from our construction that, for all $\eta < \xi < \kappa$ and all $t \in B_{u_\eta}$, there are distinct $t_0, t_1 \in B_{u_\xi}$ such that $t_0 \restriction u_\eta = t_1 \restriction u_\eta = t$. Therefore, S is splitting, so to show that S is κ-Suslin, it suffices to show that it has no antichain of cardinality κ. To this end, suppose that $\langle t_\alpha \mid \alpha < \kappa \rangle$ is a sequence of elements of S. For each $\alpha < \kappa$, fix $\eta_\alpha < \kappa$ such that $t_\alpha \in B_{u_{\eta_\alpha}}$. For each $\alpha < \kappa$, fix $p_\alpha \in \mathbb{P}$ such that $p_\alpha \Vdash_{\mathbb{P}}$ "$\dot{b}(u_{\eta_\alpha}) = t_\alpha$". Since $\mathbb{P}$ has the κ-c.c., we can find $\alpha < \beta < \kappa$ such that p_α and p_β are compatible in $\mathbb{P}$. Let $q \in \mathbb{P}$ be a common extension of p_α and p_β. Then $q \Vdash_{\mathbb{P}}$ "$(\dot{b}(u_{\eta_\alpha}) = t_\alpha) \wedge (\dot{b}(u_{\eta_\beta}) = t_\beta)$". In particular, it follows that t_α and t_β are comparable in S, so $\langle t_\alpha \mid \alpha < \kappa \rangle$ is not an antichain in S.

Finally, the interpretation of $\dot{b}$ in any forcing extension by $\mathbb{P}$ defines a cofinal branch through S, namely $\{\dot{b}(u_\eta) \mid \eta < \kappa\}$. Therefore, $\mathbb{P}$ necessarily adds a cofinal branch to S. □

8. Mitchell forcing

A variant of Mitchell forcing will be one of our primary tools for proving consistency results. We begin this section by briefly reviewing its definition and some of its important properties. Throughout this section, let μ be an infinite cardinal such that $\mu^{<\mu} = \mu$, and let δ be an ordinal with $\mathrm{cf}(\delta) > \mu$. In practice, δ will typically be (at least) a Mahlo cardinal, but will sometimes need to consider δ of the form $j(\kappa)$, where $j : V \to M$ is an elementary embedding with critical point κ, in which case δ is inaccessible in M but may not even be a cardinal in V.

For this section, let $\mathbb{P}$ denote the forcing $\mathrm{Add}(\mu, \delta)$ for adding δ-many Cohen subsets to μ. More precisely, $\mathbb{P}$ consists of all partial functions of size $<\mu$ from δ to ${}^{<\mu}2$, where, for $q, p \in \mathbb{P}$, we have $q \leq_{\mathbb{P}} p$ if and only if $\mathrm{dom}(q) \supseteq \mathrm{dom}(p)$ and, for all $\alpha \in \mathrm{dom}(p)$, $q(\alpha)$ end-extends $p(\alpha)$. For $\beta < \delta$, let $\mathbb{P}_\beta$ denote the suborder $\mathrm{Add}(\mu, \beta)$. We can now define the variant of the Mitchell forcing that we will use, $\mathbb{M} = \mathbb{M}(\mu, \delta)$, as follows. First, let A be some unbounded subset of δ with $\min(A) > \mu$. Recall that $\mathrm{acc}(A)$, the set of *accumulation points* of A, is defined to be $\{\alpha \in A \mid \sup(A \cap \alpha) = \alpha\}$, and $\mathrm{nacc}(A) := A \setminus \mathrm{acc}(A)$. Also, $\mathrm{acc}^+(A) := \{\alpha < \sup(A) \mid \sup(A \cap \alpha) = \alpha\}$. Though our definition of $\mathbb{M}$ will depend on our choice of A, we suppress mention of A in the notation; if we need to make the choice of A explicit, we will speak of "the Mitchell forcing $\mathbb{M}(\mu, \delta)$ defined using the set A". Unless otherwise specified, we will always let A be the set of all inaccessible cardinals in the interval (μ, δ), but the definition of $\mathbb{M}$ and the properties listed below work for any choice of A. Conditions in $\mathbb{M}$ are all pairs (p^0, p^1) such that

- $p^0 \in \mathbb{P}$;
- p^1 is a function and $\mathrm{dom}(p^1) \in [\mathrm{nacc}(A)]^{\leq\mu}$;
- for all $\alpha \in \mathrm{dom}(p^1)$, $p^0 \restriction \alpha \Vdash_{\mathbb{P}_\alpha}$ "$p^1(\alpha) \in \mathrm{Add}(\mu^+, 1)^{V^{\mathbb{P}_\alpha}}$".

If $(p^0, p^1), (q^0, q^1) \in \mathbb{M}$, then $(q^0, q^1) \leq_{\mathbb{M}} (p^0, p^1)$ if and only if

- $q^0 \leq_{\mathbb{P}} p^0$;
- $\mathrm{dom}(q^1) \supseteq \mathrm{dom}(p^1)$;
- for all $\alpha \in \mathrm{dom}(p^1)$, $q^0 \restriction \alpha \Vdash_{\mathbb{P}_\alpha}$ "$q^1(\alpha) \leq p^1(\alpha)$".

For $\beta < \kappa$, we let $\mathbb{M}_\beta$ denote the suborder of $\mathbb{M}$ consisting of all conditions (p^0, p^1) such that the domains of both p^0 and p^1 are contained in β. We will sometimes denote $\mathbb{M}$ by $\mathbb{M}_\delta$.

Remark 8.1. The following are some of the key properties of $\mathbb{M}$ (cf. [1], [22] for further details and proofs):

(1) $\mathbb{M}$ is μ-closed and, if δ is inaccessible, it is δ-Knaster.
(2) $\mathbb{M}$ has the μ^+-covering and μ^+-approximation properties. Together with the previous item, this implies that, if δ is inaccessible, then forcing with $\mathbb{M}$ preserves all cardinals less than or equal to μ^+ and all cardinals greater than or equal to κ.
(3) If δ is inaccessible, then $\Vdash_{\mathbb{M}}$ "$2^\mu = \delta = \mu^{++}$".
(4) There is a projection onto $\mathbb{M}$ from a forcing of the form $\mathrm{Add}(\mu, \delta) \times \mathbb{Q}$, where $\mathbb{Q}$ is μ^+-closed.
(5) For all inaccessible cardinals $\alpha \in \mathrm{acc}^+(A)$, there is a projection from $\mathbb{M}$ to $\mathbb{M}_\alpha$ and, in $V^{\mathbb{M}_\alpha}$, the quotient forcing $\mathbb{M}/\mathbb{M}_\alpha$ has the μ^+-approximation property.
(6) For all inaccessible cardinals $\alpha \in \mathrm{acc}^+(A)$, let $\alpha^\dagger$ denote $\min(A \setminus \alpha + 1)$. Then, in $V^{\mathbb{M}_\alpha}$, the quotient forcing $\mathbb{M}/\mathbb{M}_\alpha$ is of the form $\mathrm{Add}(\mu, \alpha^\dagger - \alpha) * \dot{\mathbb{M}}^\alpha$, where, in $V^{\mathbb{M} * \mathrm{Add}(\mu, \alpha^\dagger - \alpha)}$, there is a projection onto $\mathbb{M}^\alpha$ from a forcing of the form $\mathrm{Add}(\mu, \delta - \alpha^\dagger) \times \mathbb{Q}_\alpha$, where $\mathbb{Q}_\alpha$ is μ^+-closed.

In [26], Weiß presents a proof of the fact that, if κ is supercompact and $\mathbb{M} = \mathbb{M}(\omega, \kappa)$, then, in $V^{\mathbb{M}}$, ISP_{ω_2} holds. In [10], Holy, Lücke, and Njegomir point out a mistake in Weiß's proof but give an alternate proof of the theorem. In [25], Viale and Weiß prove the following theorem.[3]

Theorem 8.2 (Viale–Weiß [25, Proposition 6.8 and Theorem 6.9])**.** *Suppose that $V \subseteq W$ are transitive models of* ZFC *such that*

- *κ is an inaccessible cardinal in V;*
- *(V, W) satisfies the κ-covering and κ-approximation properties;*
- *for every $\gamma < \kappa$ and $S \subseteq (S^\gamma_\omega)^V$ in V, if S is stationary in V, then S remains stationary in W;*
- *in W, $\mathsf{ITP}(\kappa, \geq \kappa)$ holds.*

Then κ is supercompact in V.

Since for inaccessible κ the Mitchell forcing $\mathbb{M}(\omega, \kappa)$ is κ-Knaster, it follows that it satisfies the κ-covering and κ-approximation properties. Moreover, since there is a projection onto $\mathbb{M}(\omega, \kappa)$ from a forcing of the form $\mathrm{Add}(\omega, \kappa) \times \mathbb{Q}$, where $\mathbb{Q}$ is countably closed, it follows that $\mathbb{M}(\omega, \kappa)$ preserves the stationarity of all stationary subsets of S^γ_ω for every ordinal γ of uncountable cofinality. Therefore, $V \subseteq V^{\mathbb{M}(\omega, \kappa)}$ satisfy the hypothesis of Theorem 8.2. In particular, if κ is a strongly compact but not supercompact cardinal, then in $V^{\mathbb{M}(\omega, \kappa)}$, there is $\lambda \geq \omega_2$ for which $\mathsf{ITP}(\omega_2, \lambda)$ fails.

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[3]The theorem is not stated in this form in [25], but it follows immediately from the two cited results.

In light of this, and as part of a broader set-theoretic project to understand the relationship between strongly compact and supercompact cardinals, it is natural to ask precisely which strong tree properties necessarily hold in the strongly compact Mitchell model. In [26], Weiß claims without providing a proof that if κ is strongly compact, then, in $V^{\mathbb{M}(\omega,\kappa)}$, $\mathsf{SP}(\omega_1, \omega_2, \geq \omega_2)$ holds. We have been unable to verify the truth of this claim; nontrivial problems arise when attempting to use standard techniques to prove it. The next two results record the strongest results in this direction we were able to obtain. The first is a result about generalized tree properties, indicating that $\mathsf{TP}_{\omega_2}\Lambda$ holds for every ω_2-directed partial order Λ. The second indicates that a strong version of $\mathsf{wSP}(\omega_1, \omega_2, \geq \omega_2)$ holds, and yields Theorem B from the Introduction. We present our results in more generality, though, in terms of $\mathbb{M}(\mu, \kappa)$ for arbitrary μ (recall our assumption for this section that $\mu^{<\mu} = \mu$).

Theorem 8.3. *Suppose that κ is a strongly compact cardinal and $\mathbb{M} = \mathbb{M}(\mu, \kappa)$. Then, in $V^{\mathbb{M}}$, $\mathsf{TP}_{\mu^{++}}(\Lambda)$ holds for every μ^{++}-directed partial order Λ.*

Proof. Recall that $\Vdash_{\mathbb{M}}$ "$\kappa = \mu^{++}$". Let $\dot{\Lambda}$ be an $\mathbb{M}$-name for a κ-directed partial order, and let $\dot{T}$ be an $\mathbb{M}$-name for a κ-$\dot{\Lambda}$-tree. Also fix an arbitrary condition $(p^0, p^1) \in \mathbb{M}$. Without loss of generality, we can assume that the underlying set of $\dot{\Lambda}$ is forced to be a cardinal, and, by strengthening (p^0, p^1) if necessary, we can assume that there is a cardinal λ that is forced by (p^0, p^1) to be the underlying set of $\dot{\Lambda}$.

Let $j : V \to M$ be an elementary embedding with critical point κ given by a fine κ-complete ultrafilter U on $\mathscr{P}_\kappa\lambda$, i.e. $M \cong \mathrm{Ult}(V, U)$. In particular, we can find a set $x \in (\mathscr{P}_{j(\kappa)}(j(\lambda)))^M$ such that $j``\lambda \subseteq x$. Note that $j(\mathbb{M}) = \mathbb{M}(\mu, j(\kappa))$ (defined using the set $j(A)$), $j \restriction \mathbb{M}$ is the identity function, and, by clause (5) of Remark 8.1 applied to $\mathbb{M}(\mu, j(\kappa))$, there is a projection from $j(\mathbb{M})$ to $\mathbb{M}$ such that the quotient forcing is forced to have the μ^+-approximation property. Let G be $\mathbb{M}$-generic over V, and let g be $j(\mathbb{M})/\mathbb{M}$ generic over $V[G]$. Then, in $V[G][g]$, we can lift j to $j : V[G] \to M[G][g]$.

In $V[G]$, let Λ and T be the realizations of $\dot{\Lambda}$ and $\dot{T}$, respectively. In $M[G][g]$, $j(\Lambda)$ is a $j(\kappa)$-directed partial order and $j(T)$ is a $j(\kappa)$-$j(\Lambda)$-tree. Moreover, for all $u \in \Lambda$, since $|T_u| < \kappa$ and $\mathrm{crit}(j) = \kappa$, we have $j(T)_{j(u)} = j``T_u$. Since $|x|^{M[G][g]} < j(\kappa)$, we can therefore find a $w \in j(\Lambda)$ such that $v <_{j(\Lambda)} w$ for all $v \in x$. Fix an arbitrary $s \in j(T)_w$, and define a function $b \in \prod_{u \in \Lambda} T_u$ by letting $b(u)$ be the unique $t \in T_u$ such that $j(t) = s \restriction j(u)$.

Note that, if $u <_\Lambda u'$, then $j(b(u)), j(b(u')) <_{j(T)} s$, so $j(b(u)) <_{j(T)} j(b(u'))$ and hence, by elementarity, $b(u) <_T b(u')$. It follows that b is a cofinal branch through T. Moreover, $b \in V[G][g]$, but g is generic over $V[G]$ for $j(\mathbb{M})/\mathbb{M}$, which has the μ^+-approximation property and hence, *a fortiori*, the κ-approximation property. Therefore, by Lemma 7.6, we have $b \in V[G]$, and hence $\mathsf{TP}_\kappa(\Lambda)$ holds in $V[G]$. □

Theorem 8.4. *Suppose that κ is a strongly compact cardinal, $\lambda \geq \kappa$ is regular, and $\Sigma \subseteq [\kappa, \lambda]$ is a set of regular cardinals such that $|\Sigma| < \kappa$. Then, in $V^{\mathbb{M}}$, $\mathsf{wSP}_{\mathcal{Y}}(\mu^+, \kappa, \lambda)$ holds, where $\mathcal{Y}$ denotes the set of $x \in \mathscr{P}_\kappa\lambda$ such that $\mathrm{cf}(\sup(x \cap \chi)) = \mu^+$ for all $\chi \in \Sigma$.*

Proof. By enlarging Σ if necessary, we can assume that $\kappa \in \Sigma$. Let $\dot{\mathcal{Y}}$ be an $\mathbb{M}$-name for $\mathcal{Y}$, and let $\dot{D} = \langle \dot{d}_{\dot{x}} \mid \dot{x} \in \mathscr{P}_\kappa\lambda \rangle$ be an $\mathbb{M}$-name for a strongly μ^+-$\dot{\mathcal{Y}}$-slender

(κ, λ)-list. Let G be $\mathbb{M}$-generic over V and, for each inaccessible $\alpha < \kappa$, let G_α be the $\mathbb{M}_\alpha$-generic filter induced by G. Let $\mathcal{Y}$ and D be the realizations of $\dot{\mathcal{Y}}$ and $\dot{D}$, respectively, in $V[G]$.

The first part of our proof largely follows the beginning of the proof of [26, Theorem 5.4]. Work for now in $V[G]$. By the κ-c.c. of $\mathbb{M}$, we know that, for all $x \in \mathscr{P}_\kappa \lambda$, there is an inaccessible $\alpha_x < \kappa$ such that $d_x \in V[G_{\alpha_x}]$. Let $\theta > \lambda$ be a sufficiently large regular cardinal, and let $C \subseteq \mathscr{P}_\kappa H(\theta)$ be a strong club witnessing that D is strongly μ^+-$\mathcal{Y}$-slender. We can assume that, for all $N \in C$, we have $\kappa_N := \kappa \cap N \in \kappa$ and, for all $x \in \mathscr{P}_\kappa \lambda \cap N$, we have $\alpha_x \in N$. It follows that, whenever $N \in C$, κ_N is inaccessible in V, and $x \in \mathscr{P}_\kappa \lambda \cap N$, we have $x \in V[G_{\kappa_N}]$.

In V, let $\sigma := \lambda^{<\kappa}$ and find $\bar{N} \prec H(\theta)$ such that $\mathscr{P}_\kappa \lambda \subseteq \bar{N}$ and $|\bar{N}| = \sigma$. In $V[G]$, let $C \restriction \bar{N} := \{N \cap \bar{N} \mid N \in C\}$. Then $C \restriction \bar{N}$ is a strong club in $\mathscr{P}_\kappa \bar{N}$. Now move back to V, and let $\dot{C}$ be an $\mathbb{M}$-name for C. Since $\mathbb{M}$ has the κ-c.c., we can find a strong club $E \subseteq \mathscr{P}_\kappa \bar{N}$ such that $\Vdash_{\mathbb{M}}$ "$E \subseteq \dot{C} \restriction \bar{N}$".

Let $j : V \to M$ be an elementary embedding witnessing that κ is σ-strongly compact. Therefore, in M, we can find a set $y \in \mathscr{P}_{j(\kappa)} j(\bar{N})$ such that $j``\bar{N} \subseteq y$. For each $\chi \in \Sigma$, let $\gamma_\chi := \sup j``\chi$. Note that, since $|\Sigma| < \kappa$, we have $\{\gamma_\chi \mid \chi \in \Sigma\} \in M$. Let

$$Z := \{z \subseteq y \mid z \in j(E) \text{ and, for all } \chi \in \Sigma,\ z \cap j(\chi) \subseteq \gamma_\chi\}.$$

Then $Z \in M$ and, in M, it is a subset of $j(E)$ of cardinality less than $j(\kappa)$. Therefore, $\bigcup Z \in j(E)$. Let $w := j(\lambda) \cap \bigcup Z$. The definition of Z implies that $j``E \subseteq Z$ and therefore $j``\lambda \subseteq w$. It follows that, for all $\chi \in \Sigma$, we have $\sup(w \cap j(\chi)) = \gamma_\chi$.

Note that $j(\mathbb{M}) = \mathbb{M}(\mu, j(\kappa))$, defined using the set $j(A)$. Let g be $j(\mathbb{M})/\mathbb{M}$-generic over $V[G]$, and lift j to $j : V[G] \to M[G][g]$. In $M[G][g]$, for every V-regular cardinal $\chi \in [\kappa, j(\kappa))$, we have $\mathrm{cf}(\chi) = \mu^+$. In particular, for every $\chi \in \Sigma$, we have $\mathrm{cf}(\gamma_\chi) = \mathrm{cf}(j``\chi) = \mathrm{cf}(\chi) = \mu^+$, and hence $w \in j(\mathcal{Y})$.

Let $j(D) = \langle d'_x \mid x \in \mathscr{P}_{j(\kappa)} j(\lambda) \rangle$. Since $\bigcup Z \in j(E)$, we can find $N \in j(C)$ such that $N \cap j(\bar{N}) = \bigcup Z$. In particular, $N \cap j(\lambda) = w$. Therefore, since $j(D)$ is a μ^+-$j(\mathcal{Y})$-slender $(j(\kappa), j(\lambda))$-list, as witnessed by the strong club $j(C)$, it follows that, for every $u \in N \cap \mathscr{P}_{\mu^+} j(\lambda)$, we have $d'_w \cap u \in N$. Note also that $j(\kappa) \cap N = \kappa$ is inaccessible in V, and hence we have $d'_w \cap u \in V[j(G)_\kappa] = V[G]$ for all $u \in N \cap \mathscr{P}_{\mu^+} j(\lambda)$.

Claim 8.5. *For all $u \in (\mathscr{P}_{\mu^+} \lambda)^V$, we have $j(u) \in N$.*

Proof. Fix $u \in (\mathscr{P}_{\mu^+} \lambda)^V$. Since E is cofinal in $\mathscr{P}_\kappa \bar{N}$ and $(\mathscr{P}_{\mu^+} \lambda)^V \subseteq \bar{N}$, we can find $z \in E$ such that $u \in z$. Then $j(z) \in Z$, and hence $j(u) \in \bigcup Z \subseteq N$. □

Let $b := \{\alpha < \lambda \mid j(\alpha) \in d'_w\}$.

Claim 8.6. *For all $u \in (\mathscr{P}_{\mu^+} \lambda)^{V[G]}$, we have $b \cap u \in V[G]$.*

Proof. Fix $u \in (\mathscr{P}_{\mu^+} \lambda)^{V[G]}$. Since $\mathbb{M}$ satisfies the μ^+-covering property, we can find $u' \in (\mathscr{P}_{\mu^+} \lambda)^V$ such that $u \subseteq u'$. By Claim 8.5, we have $j(u') \in N$. Then, by the sentence preceding Claim 8.5, we have $d'_w \cap j(u') \in V[G]$. But then $b \cap u = \{\alpha \in u \mid j(\alpha) \in d'_w \cap j(u')\}$ is definable in $V[G]$, and hence is in $V[G]$. □

In $V[G]$, the quotient forcing $j(\mathbb{M})/\mathbb{M}$ has the μ^+-approximation property. Therefore, Claim 8.6 implies that $b \in V[G]$. We will therefore be done if we can show that, in $V[G]$, b is a cofinal branch through D. To this end, fix $x \in (\mathscr{P}_\kappa \lambda)^{V[G]}$, and note that $j(x) = j``x$ and $j(b \cap x) = j``(b \cap x)$. Now

$$M[G][g] \models \exists z \in \mathscr{P}_{j(\kappa)} j(\lambda)\ [j(x) \subseteq z \text{ and } d'_z \cap j(x) = j(b \cap x)],$$

as witnessed by w. Therefore, by elementarity of j, we have

$$V[G] \models \exists z \in \mathscr{P}_\kappa \lambda\ [x \subseteq z \text{ and } d_z \cap x = b \cap x].$$

Since x was arbitrary, this implies that b is in fact a cofinal branch of D, thus completing the proof. □

Remark 8.7. Suppose that $\theta \geq \kappa$ is a regular uncountable cardinal and $f : |H(\theta)| \to H(\theta)$ is a bijection. Then

$$C := \{x \in \mathscr{P}_\kappa |H(\theta)| \mid f[x] \cap \theta = x \cap \theta\}$$

is a strong club in $\mathscr{P}_\kappa |H(\theta)|$ (and hence $\{f[X] \mid X \in C\}$ is a strong club in $\mathscr{P}_\kappa H(\theta)$). Therefore, by the discussion following Remark 4.8, we see that, in the forcing extension of the previous theorem, for every regular $\theta \geq \kappa$ and every collection $\Sigma \subseteq [\kappa, \theta]$ of regular cardinals with $|\Sigma| < \kappa$, we have $\mathsf{wSP}_{\mathcal{Y}}(\mu^+, \kappa, H(\theta))$, where $\mathcal{Y}$ is the set of $M \in \mathscr{P}_\kappa H(\theta)$ such that $\mathrm{cf}(\sup(M \cap \chi)) = \mu^+$ for all $\chi \in \Sigma$. In particular, by the analogue of Theorem 5.3 for wSP and wAGP, Theorem 6.2, and Corollary 6.4, the version of wSP obtained in the preceding theorem is enough to obtain the nonexistence of subadditive, strongly unbounded functions $c : [\lambda]^2 \to \chi$ for every regular $\lambda \geq \kappa$ and every χ with $\chi^+ < \kappa$, and hence the failure of $\square(\lambda)$ for every regular $\lambda \geq \kappa$.

We now present a result indicating that, if κ is strongly compact, then, in the extension by $\mathbb{M}(\omega, \kappa)$, an instance of $\mathsf{wSP}(\dots)$ is indestructible under certain iterations of small c.c.c. forcings. This is a variation on a result of Todorčević from [23].

Theorem 8.8. *Suppose that κ is strongly compact, $\mathbb{M} = \mathbb{M}(\omega, \kappa)$, and, in $V^{\mathbb{M}}$, $\mathbb{R} := \langle \mathbb{R}_\alpha, \dot{\mathbb{S}}_\beta \mid \alpha \leq \kappa,\ \beta < \kappa \rangle$ is a finite-support forcing iteration such that, for all $\beta < \kappa$, $\dot{\mathbb{S}}_\beta$ is forced to be a c.c.c. forcing of size at most $\aleph_1$ that does not add cofinal branches to any Suslin tree. Suppose also that $\lambda \geq \kappa$ is regular and $\Sigma \subseteq [\kappa, \lambda]$ is a set of regular cardinals such that $|\Sigma| < \kappa$. Then, in $V^{\mathbb{M} * \dot{\mathbb{R}}}$, $\mathsf{wSP}_{\mathcal{Y}}(\omega_1, \omega_2, \lambda)$ holds, where $\mathcal{Y}$ denotes the set of $x \in \mathscr{P}_{\omega_2} \lambda$ such that $\mathrm{cf}(\sup(x \cap \chi)) = \omega_1$ for all $\chi \in \Sigma$.*

Proof. The proof repeats many of the steps of the proof of Theorem 8.4, so we will leave out some details. We may again assume that κ is in Σ. Let $\dot{\mathcal{Y}}$ be an $\mathbb{M} * \dot{\mathbb{R}}$-name for $\mathcal{Y}$, and let $\dot{D} = \langle d_{\dot{x}} \mid \dot{x} \in \mathscr{P}_\kappa \lambda \rangle$ be an $\mathbb{M} * \dot{\mathbb{R}}$-name for a strongly ω_1-$\dot{\mathcal{Y}}$-slender (ω_2, λ)-list. Let $G * H$ be $\mathbb{M} * \dot{\mathbb{R}}$-generic over V and, for each inaccessible $\alpha < \kappa$, let G_α and H_α be the $\mathbb{M}_\alpha$ and $\mathbb{R}_\alpha$-generic filters induced by G and H, respectively. We can assume that the underlying set of each $\dot{\mathbb{S}}_\beta$ is forced to be ω_1 and $\dot{\mathbb{S}}_\beta$ is a nice name for a subset of ω_1^2. By the κ-c.c. of $\mathbb{M}$, there is a club $B \subseteq \kappa$ such that, for every $\alpha \in B$, $\dot{\mathbb{R}}_\alpha$ is an $\mathbb{M}_\alpha$-name. Therefore, for each $\alpha \in B$, it makes sense to speak about the forcing extension $V[G_\alpha][H_\alpha]$.

Let $\theta > \lambda$ be a sufficiently large regular cardinal. As in the proof of Theorem 8.4, working in $V[G][H]$, we can find a strong club $C \subseteq \mathscr{P}_\kappa H(\theta)$ such that, for every $N \in C$, we have $\kappa_N := \kappa \cap N \in \kappa$ and, whenever $N \in C$, κ_N is inaccessible in V, and $x \in \mathscr{P}_\kappa \lambda \cap N$, we have $x \in V[G_{\kappa_N}][H_{\kappa_N}]$.

Work now in V, and let σ, $\bar{N}$, $C \restriction \bar{N}$, E, $j : V \to M$, and w be defined as in the proof of Theorem 8.4. Note that $j(\mathbb{M}) = \mathbb{M}(\omega, j(\kappa))$ and $j(\dot{\mathbb{R}})$ is of the form

$\dot{\mathbb{R}} * \dot{\mathbb{U}}$, where $\dot{\mathbb{U}}$ is forced to be a finite-support iteration of length $j(\kappa)$ such that each iterand is forced to be a c.c.c. forcing of size at most $\aleph_1$ that does not add cofinal branches to any Suslin tree. By standard arguments, $\mathbb{M} * \dot{\mathbb{R}}$ is a complete suborder of $j(\mathbb{M} * \dot{\mathbb{R}})$; let $g * h$ be $j(\mathbb{M} * \dot{\mathbb{R}})/G * H$-generic over $V[G][H]$, and lift j to $j : V[G][H] \to M[G][H][g][h]$. In $V[G][H][g][h]$, let $j(D) = \langle d'_x \mid x \in \mathscr{P}_{j(\kappa)} j(\lambda)\rangle$, and define $b := \{\alpha < \lambda \mid j(\alpha) \in d'_w\}$.

As in the proof of Theorem 8.4, one can show that, for all $u \in (\mathscr{P}_{\omega_1}\lambda)^{V[G][H]}$, we have $b \cap u \in V[G][H]$ (this uses the fact that $\mathbb{M} * \dot{\mathbb{R}}$ has the ω_1-covering property, which follows from the fact that $\mathbb{M}$ has the ω_1-covering property and $\dot{\mathbb{R}}$ is forced to have the c.c.c.).

Claim 8.9. *For all $v \in (\mathscr{P}_\kappa\lambda)^{V[G][H]}$, we have $b \cap v \in V[G][H]$.*

Proof. Since $\mathbb{R}$ has the κ-covering property in $V[G]$, it suffices to prove the claim for all $v \in (\mathscr{P}_\kappa\lambda)^{V[G]}$. Let $\kappa^\dagger$ be the least inaccessible cardinal greater than κ. In $V[G]$, by Clause 6 of Remark 8.1 applied to $j(\mathbb{M})$, the quotient forcing $j(\mathbb{M})/\mathbb{M}$ is of the form $\mathrm{Add}(\omega, \kappa^\dagger - \kappa) * \dot{\mathbb{M}}^\kappa$. Let g_0 be the $\mathrm{Add}(\omega, \kappa^\dagger - \kappa)$-generic filter induced by g. In $V[G][g_0]$, there is a projection onto $\mathbb{M}^\kappa$ from a forcing of the form $\mathrm{Add}(\omega, j(\kappa) - \kappa^\dagger) \times \mathbb{Q}_\kappa$, where $\mathbb{Q}_\kappa$ is ω_1-closed. Let $g_1 \times g_2$ be $(\mathrm{Add}(\omega, j(\kappa) - \kappa^\dagger) \times \mathbb{Q}_\kappa)$-generic over $V[G][g_0]$ such that $V[G][g_0][g_1][g_2]$ is an extension of $V[G][g]$ by $(\mathrm{Add}(\omega, j(\kappa) - \kappa^\dagger) \times \mathbb{Q}_\kappa)/\mathbb{M}^\kappa$.

Fix $v \in (\mathscr{P}_\kappa\lambda)^{V[G]}$, and let $\Lambda := (\mathscr{P}_{\omega_1} v)^{V[G]}$. Since all forcing notions used in this proof have the ω_1-covering property, Λ is ω_1-directed in all of the forcing extensions of $V[G]$ appearing here. Now consider the Λ-tree $T := \langle\langle T_u \mid u \in \Lambda\rangle, \subseteq\rangle$, where $T_u := (\mathscr{P}(u))^{V[G][H]}$ for every $u \in \Lambda$. Note that $|T_u| = (2^{\omega_1})^{V[G][H]} = \kappa < \kappa^\dagger$.

By the paragraph immediately preceding the claim, $b \cap v$ is a branch through T, and we know that $b \cap v \in V[G][H][g][h]$.[4] First note that, in $V[G][H][g]$, $j(\mathbb{R})/H$ is equivalent to a tail of the iteration $j(\mathbb{R})$, which is a finite-support iteration of c.c.c. posets that do not add branches to any Suslin tree. By [5, Lemma 3.7], it follows that the entire iteration $j(\mathbb{R})/H$ cannot add branches to any Suslin tree. Therefore, by Lemma 7.7, $b \cap v$ could not have been added by h, so we have $b \cap v \in V[G][H][g]$, and, *a fortiori*, $b \cap v \in V[G][H][g_0][g_1][g_2]$.

We now work in the model $V[G][g_0]$. Since g_0 is generic for an ω_1-Knaster poset, we know that $\mathbb{R}$ remains c.c.c. in $V[G][g_0]$. Also, $\mathbb{Q}_\kappa$ is ω_1-closed. In $V[G][g_0]$, $2^\omega = \kappa^\dagger > \kappa$, and, in $V[G][g_0][H][g_1]$, T is a Λ-tree of width κ^+. We can therefore apply Lemma 7.2 to conclude that $b \cap v$ cannot have been added by g_2 and therefore lies in $V[G][g_0][H][g_1]$. Finally, $g_0 * g_1$ is generic over $V[G][H]$ for $\mathrm{Add}(\omega, \kappa^\dagger - \kappa) * \mathrm{Add}(\omega, j(\kappa) - \kappa^\dagger) \cong \mathrm{Add}(\omega, j(\kappa) - \kappa)$, so, again by Lemma 7.7, $b \cap v$ cannot have been added by $g_0 * g_1$ and therefore lies in $V[G][H]$, as desired. □

Claim 8.10. *b is a cofinal branch through D.*

Proof. Fix $v \in (\mathscr{P}_\kappa\lambda)^{V[G][H]}$. We must find $x \in (\mathscr{P}_\kappa\lambda)^{V[G][H]}$ such that $v \subseteq x$ and $d_x \cap v = b \cap v$. By Claim 8.9, we have $b \cap v \in V[G][H]$. Moreover, by the definition of b, we have

$$j(b \cap v) = j``(b \cap v) = d'_w \cap j(v).$$

[4]More precisely, the function $u \mapsto b \cap u$ defined on Λ is a branch through T, but it is clear that this function and b are definable from one another in all models of interest.

Therefore,

$$M[G][H][g][h] \models \exists x \in \mathscr{P}_{j(\kappa)} j(\lambda)\ [d'_x \cap j(v) = j(b \cap v)],$$

as witnessed by $x = w$. Therefore, by elementarity,

$$V[G][H] \models \exists x \in \mathscr{P}_\kappa \lambda\ [d_x \cap v = b \cap v],$$

as desired. $\square$

Now the verification that b is in $V[G][H]$ follows reasoning as in the proof of Claim 8.9. Let Λ' be the partial order $((\mathscr{P}_\kappa\lambda)^{V[G][H]}, \subseteq)$, and let T' be the Λ'-tree generated from D as in Remark 4.5. Then Λ' is κ-directed in all extensions of $V[G][H]$ appearing in this proof, and, in $V[G][H]$, T' is a Λ'-tree of width $((\kappa^{<\kappa})^+)^{V[G][H]} = (\kappa^+)^{V[G][H]} < \kappa^\dagger$.

Just as in the proof of Claim 8.9, Lemma 7.7 implies that b cannot have been added by h, so we have $b \in V[G][H][g]$, and hence also $b \in V[G][H][g_0][g_1][g_2]$. An application of Lemma 7.2 in $V[G][g_0]$ shows that b must be in $V[G][H][g_0][g_1]$, and the fact that $g_0 * g_1$ is generic for $\mathrm{Add}(\omega, j(\kappa) - \kappa)$ over $V[G][H]$ implies that b must be in $V[G][H]$. Therefore, D has a cofinal branch in $V[G][H]$, so we have verified this instance of $\mathsf{wSP}_{\mathcal{Y}}(\omega_1, \omega_2, \lambda)$ in $V[G][H]$. $\square$

As a corollary, we show that Martin's Axiom together with strong tree properties at ω_2 can be obtained starting only from a strongly compact cardinal (both are consequences of PFA, which can be forced from a supercompact cardinal).

Corollary 8.11. *Suppose that κ is a strongly compact cardinal. Then there is a forcing extension in which $\kappa = \omega_2 = 2^\omega$ and $\mathsf{MA} + \mathsf{wSP}(\omega_1, \omega_2, \geq \omega_2)$ (and hence $\mathsf{MA} + \mathsf{TP}(\omega_2, \geq \omega_2)$) holds.*

Proof. In V, let $\mathbb{M} := \mathbb{M}(\omega, \kappa)$. Now work in $V^{\mathbb{M}}$. In [5], Devlin constructed a finite-support iteration $\mathbb{R}$ of ccc forcings of size at most $\aleph_1$ that do not add cofinal branches to any Suslin tree and such that $\Vdash_{\mathbb{R}} \mathsf{MA}$. Intuitively speaking, $\mathbb{R}$ is the standard iteration to force MA, except that, when one encounters a ccc forcing of size $\aleph_1$ that adds a branch to a Suslin tree, one instead forces to specialize the Suslin tree; we refer the reader to [5, Section 3] (and also [23]) for details. Then MA holds in $V^{\mathbb{M} * \dot{\mathbb{R}}}$. By Theorem 8.8, $\mathsf{wSP}(\omega_1, \omega_2, \geq \omega_2)$, and hence $\mathsf{TP}(\omega_2, \geq \omega_2)$, holds in $V^{\mathbb{M} * \dot{\mathbb{R}}}$ as well. (Note that Theorem 8.8 in fact gives us a strengthening of $\mathsf{wSP}(\omega_1, \omega_2, \geq \omega_2)$ that is enough to yields, for instance, the failure of $\square(\lambda)$ for all regular $\lambda \geq \omega_2$.) $\square$

9. The slender tree property and Kurepa trees

In this section, we investigate the influence of various strong tree properties on the existence of (weak) Kurepa trees and prove Theorem D, in the process showing that, for instance, $\mathsf{ISP}(\omega_2, \omega_2, \geq \omega_2)$ does not imply $\mathsf{SP}(\omega_1, \omega_2, |H(\omega_2)|)$, i.e., the monotonicity in the first coordinate of these principles is in general strict. We first recall the notion of a *(weak) μ-Kurepa tree*.

Definition 9.1. Let μ be a regular uncountable cardinal.

(i) A *μ-Kurepa tree* is a μ-tree (i.e., a tree of height μ, all of whose levels have size less than μ) with at least μ^+-many cofinal branches.

(ii) A *weak μ-Kurepa tree* is a tree of height and cardinality μ with at least μ^+-many cofinal branches.

The following proposition is easily verified, so we leave its proof to the reader.

Proposition 9.2. *Let $\kappa \leq \theta$ be regular uncountable cardinals, and suppose that $S \subseteq \mathscr{P}_\kappa H(\theta)$ is cofinal. Then the set of elements of $\mathscr{P}_\kappa H(\theta)$ that can be written as unions of elements of S is a strong club in $\mathscr{P}_\kappa H(\theta)$.*

The following theorem is clause (1) of Theorem C.

Theorem 9.3. *Suppose that μ is a regular uncountable cardinal. If the principle $\mathsf{wAGP}(\mu, \mu^+, \mu^+)$ holds, then there are no weak μ-Kurepa trees.*

Proof. Let T be a tree of height and size μ. We will show that T has at most μ-many cofinal branches. Without loss of generality, assume that the underlying set of T is μ. Let S be the set of $N \in \mathscr{P}_{\mu^+} H(\mu^+)$ such that $N \prec H(\mu^+)$, $T \in N$, and $\mu \subseteq N$. By $\mathsf{wAGP}(\mu, \mu^+, \mu^+)$, and recalling Proposition 9.2, we can find $M \in \mathscr{P}_{\mu^+} H(\mu^+)$ such that $T \in M$, $\mu \subseteq M$, M can be written as a union of elements of S, and (M, μ) is almost μ-guessed by S.

Claim 9.4. *Let $b \subseteq \mu$ be a cofinal branch in T. Then b is (μ, M)-approximated.*

Proof. Fix a set $z \in M \cap \mathscr{P}_\mu \mu$, and consider the set $b \cap z$. Since M can be written as the union of elements of S, we can find $N \in S$ such that $N \subseteq M$ and $z \in N$. Since $|z| < \mu$, we can find $\gamma \in b$ such that $\alpha <_T \gamma$ for all $\alpha \in b \cap z$. Then $b \cap z$ is definable in N as $\{\alpha \in z \mid \alpha <_T \gamma\}$. Therefore, $b \cap z \in N$, and, since $N \subseteq M$, we have $b \cap z \in M$, as well. Therefore, b is (μ, M)-approximated. □

Fix a cofinal branch b in T. By Claim 9.4, b is (μ, M)-approximated. Therefore, since (M, μ) is almost μ-guessed by S, we can find $N \in S$ such that $N \subseteq M$ and b is N-guessed, i.e., there is $d \in N$ such that $d \cap N = b \cap N$. Since $b \subseteq \mu$, we can assume that $d \subseteq \mu$. But then, since $\mu \subseteq N$, we must have $d = b$. Therefore, $b \in N$ and, since $N \subseteq M$, we also have $b \in M$. Therefore, every cofinal branch of T is in M, so, since $|M| \leq \mu$, there are at most μ-many cofinal branches of T. □

We record the following immediate corollary to provide some additional context for Theorem 9.9.

Corollary 9.5. *If $\mathsf{SP}(\omega_1, \omega_2, |H(\omega_2)|)$ holds, then there is no weak ω_1-Kurepa tree.*

Proof. This is immediate from the definitions, from Theorem 5.3, and from Theorem 9.3. □

In preparation for the proof of Theorem 9.9, we review the forcing for adding an ω_1-Kurepa tree, which we denote $\mathbb{K}(\omega_1, \kappa)$. We use the definition from [3].

Definition 9.6. Let $\kappa > \omega_1$ be a cardinal. A condition p in $\mathbb{K} = \mathbb{K}(\omega_1, \kappa)$ is a pair (t_p, f_p) where the following hold:

(i) t_p is a countable normal tree of successor height $\alpha_p + 1$;
(ii) f_p is a countable partial function from κ to $(t_p)_{\alpha_p}$.

A condition q is stronger than p if the following hold:

(i) $\alpha_q \geq \alpha_p$ and $t_q \restriction (\alpha_p + 1) = t_p$;
(ii) $\mathrm{dom}(f_p) \subseteq \mathrm{dom}(f_q)$ and $f_p(\xi) \leq_{t_q} f_q(\xi)$ for all $\xi \in \mathrm{dom}(f_p)$.

Proofs of the following facts can be found in [3]:

Fact 9.7. *The forcing $\mathbb{K}$ is ω_1-closed and it adds an ω_1-Kurepa tree with κ-many branches.*

Fact 9.8. *Let $\mu > \omega$ be a regular cardinal such that $\gamma^\omega < \mu$ for all $\gamma < \mu$. Then the forcing $\mathbb{K}$ is μ-Knaster.*

If $\kappa < \kappa'$, we can define a projection $\pi : \mathbb{K}(\omega_1, \kappa) \to \mathbb{K}(\omega_1, \kappa')$ by $\pi(t, f) = (t, f \restriction \kappa)$. If H is $\mathbb{K}(\omega_1, \kappa)$-generic, π determines the quotient forcing $\mathbb{K}(\omega_1, \kappa')/H = \{p \in \mathbb{K}(\omega_1, \kappa') \,|\, \pi(p) \in \mathbb{K}(\omega_1, \kappa)\}$. Note that the tree $T = \bigcup\{t \,|\, (t, f) \in H\}$ is already added by H: the quotient $\mathbb{K}(\omega_1, \kappa')/H$ just adds new cofinal branches to T. Working in $V[H]$, it is easy to verify that the quotient $\mathbb{K}(\omega_1, \kappa')/H$ is forcing equivalent to the following forcing which we denote $\mathbb{K}(\omega_1, \kappa' \setminus \kappa)$. A condition in $\mathbb{K}(\omega_1, \kappa' \setminus \kappa)$ is a pair $p = (\alpha_p, f_p)$, where $\alpha_p < \omega_1$ and f_p is a countable partial function from $\kappa' \setminus \kappa$ to T_{α_p}. We say that q is stronger than p if $\alpha_p \leq \alpha_q$, $\mathrm{dom}(f_p) \subseteq \mathrm{dom}(f_q)$ and $f_p(\xi) \leq_T f_q(\xi)$ for all $\xi \in \mathrm{dom}(f_p)$.

Theorem 9.9. *Let κ be a supercompact cardinal, and assume that* GCH *holds above κ. Then there is a generic extension where the following hold:*
(i) $2^\omega = \kappa = \omega_2$*;*
(ii) there is an ω_1-Kurepa tree;
(iii) $\mathsf{ISP}(\omega_2, \omega_2, \geq \omega_2)$ *holds.*

Proof. The generic extension is obtained by forcing with a product of the Mitchell forcing and the forcing for adding an ω_1-Kurepa tree: $\mathbb{M}(\omega, \kappa) \times \mathbb{K}(\omega, \kappa)$, which we denote by $\mathbb{M} \times \mathbb{K}$ for simplicity. Let $G \times H$ be $\mathbb{M} \times \mathbb{K}$-generic over V. The Mitchell forcing preserves ω_1 and all cardinal greater or equal to κ and forces $2^\omega = \kappa = \omega_2$. Since $\mathbb{K}$ is κ-Knaster in V, $\mathbb{K}$ is still κ-Knaster in $V[G]$ and therefore it preserves all cardinals greater or equal to κ. It also preserves ω_1 over $V[G]$, since it is ω_1-closed in V and therefore it is ω_1-distributive in $V[G]$ by the standard product analysis of the Mitchell forcing $\mathbb{M}$. This finishes the proof of (i).

There is an ω_1-Kurepa tree T in $V[H]$ with κ-many cofinal branches by definition of $\mathbb{K}$. Since ω_1 is preserved by $\mathbb{K} \times \mathbb{M}$, T is still ω_1-tree with $\kappa = \omega_2$-many cofinal branches in $V[H][G] = V[G][H]$.

Now fix $\lambda > \kappa$; by our hypotheses, we can assume that $\lambda^{<\lambda} = \lambda$. To show that $\mathsf{ISP}(\kappa, \kappa, \lambda)$ holds in $V[G][H]$ we use the lifting argument and analysis of the quotient as in [3]. Let

$$j : V \to M \tag{3}$$

be a supercompact elementary embedding with critical point κ given by a normal ultrafilter U on $\mathscr{P}_\kappa(\lambda)$; i.e. $M \cong \mathrm{Ult}(V, U)$. We lift the elementary embedding j in two steps. Note that $j(\mathbb{M}(\omega, \kappa)) = \mathbb{M}(\omega, j(\kappa))$ and recall that there is a projection from $\mathbb{M}(\omega, j(\kappa))$ to $\mathbb{M}(\omega, \kappa)$. Let us denote the quotient $\mathbb{M}(\omega, j(\kappa))/G$ by $Q_\mathbb{M}$ and let g be $Q_\mathbb{M}$-generic over $V[G][H]$. Then we can lift in $V[G][g]$ the embedding to $j : V[G] \to \mathbb{M}[G][g]$. Now $j(\mathbb{K}(\omega_1, \kappa)) = \mathbb{K}(\omega_1, j(\kappa))$ and there is a projection from $\mathbb{K}(\omega_1, j(\kappa))$ to $\mathbb{K}(\omega_1, \kappa)$. Let us denote the quotient $\mathbb{K}(\omega_1, j(\kappa))/G$ by $Q_\mathbb{K}$ and let h be $Q_\mathbb{K}$-generic over $V[G][H][g]$. We can lift the embedding in $V[G][H][g][h]$ further to

$$j : V[G][H] \to M[G][H][g][h]. \tag{4}$$

Let $D = \langle d_x \,|\, x \in \mathscr{P}_\kappa(\lambda)^{V[G][H]} \rangle$ be a κ-slender list in $V[G][H]$. We want to show that there is an ineffable branch b through D in $V[G][H]$.

Let us consider the image of D under j:

$$j(\langle d_x \,|\, x \in \mathscr{P}_\kappa(\lambda)^{V[G][H]} \rangle) = \langle d'_y \,|\, y \in \mathscr{P}_{j(\kappa)}(j(\lambda))^{M[G][H][g][h]} \rangle.$$

The set $j``\lambda$ is a subset of $j(\lambda)$ of size $< j(\kappa)$. We define our ineffable branch $b : \lambda \to 2$ in $V[G][H][g][h]$ as follows:[5]

$$\text{for } \alpha < \lambda,\ b(\alpha) = d'_{j``\lambda}(j(\alpha)). \tag{5}$$

Before we prove the following claim, let us state some simple properties of the lifted embedding j. Since we assume $\lambda^{<\lambda} = \lambda$, $|H(\lambda)| = \lambda$, and this still holds in $V[G][H]$, i.e. $|H(\lambda)^{V[G][H]}| = \lambda$. In (4) we lifted an embedding generated by a normal ultrafilter U on $\mathscr{P}_\kappa(\lambda)$ and therefore $j``\lambda$ is an element of $j(C)$ for every club C in $\mathscr{P}_\kappa(\lambda)^{V[G][H]}$; since there is a bijection between λ and $H(\lambda)^{V[G][H]}$, there is a correspondence between $\mathscr{P}_\kappa\lambda^{V[G][H]}$ and $\mathscr{P}_\kappa H(\lambda)^{V[G][H]}$, and in particular it holds for every club C in $V[G][H]$ in $\mathscr{P}_\kappa H(\lambda)^{V[G][H]}$ that

$$j``H(\lambda)^{V[G][H]} \in j(C). \tag{6}$$

Claim 9.10. *For each $x \in \mathscr{P}_\kappa\lambda^{V[G][H]}$, $b \restriction x \in M[G][H]$.*

Proof. By slenderness of D, we can fix a club C in $\mathscr{P}_\kappa H(\lambda)^{V[G][H]}$ such that for every $N \in C$, $N \prec H(\lambda)^{V[G][H]}$, and for every $x \in N$ of size ω_1, we have $d_{N\cap\lambda} \cap x \in N$.

Let $x \in \mathscr{P}_\kappa\lambda^{V[G][H]}$ be arbitrary. By the κ-cc of the whole forcing, it holds that

$$x \in H(\lambda)^{V[G][H]} = H(\lambda)^{M[G][H]}$$

and therefore $j``x = j(x) \in j``H(\lambda)^{M[G][H]}$. Let us denote $H(\lambda)^{M[G][H]}$ by N. Notice that $j``N$ is an elementary submodel of $j(H(\lambda)^{V[G][H]})$ by a general model-theoretic argument, or by invoking (6). Since $j``H(\lambda)^{M[G][H]} \in j(C)$, we have $d'_{(j``N)\cap j(\lambda)} \restriction j``x = d'_{j``\lambda} \restriction j``x \in j``N$. It follows that there is a function f in $N \subseteq M[G][H]$ with domain x such that for every $\alpha \in x$,

$$f(\alpha) = d'_{j``\lambda}(j(\alpha)).$$

By the definition of b, $f = b \restriction x$, and the proof is finished. □

Claim 9.11. *b is in $M[G][H]$.*

Proof. We show that b – which is approximated in $M[G][H]$ on sets in $\mathscr{P}_\kappa\lambda^{V[G][H]} = \mathscr{P}_\kappa\lambda^{M[G][H]}$ by Claim 9.10 – cannot be added by $Q_{\mathbb{K}} \times Q_{\mathbb{M}}$ over $M[G][H]$ and therefore it is already in $M[G][H]$.

We first argue that $Q_{\mathbb{K}}$ is ω_1-distributive and ω_2-Knaster over $M[G][H]$. In M, $j(\mathbb{K}) \simeq \mathbb{K} * Q_{\mathbb{K}}$ is ω_1-closed and therefore by the product analysis of $\mathbb{M}$, it is ω_1-distributive in $M[G]$ and therefore $Q_{\mathbb{K}}$ is ω_1-distributive in $M[G][H]$. Regarding the ω_2-Knaster property, first note that $Q_{\mathbb{K}}$ is κ-Knaster in $M[H]$ since $Q_{\mathbb{K}}$ is forcing equivalent to $\mathbb{K}(\omega_1, j(\kappa) \setminus \kappa)$ and this forcing is κ-Knaster by a standard Δ-system argument (for more details see [3]). Since $\mathbb{M}$ is κ-Knaster in M it is still κ-Knaster in $M[H]$ and therefore $Q_{\mathbb{K}}$ is κ-Knaster in $M[H][G] = M[G][H]$. Since $Q_{\mathbb{K}}$ is κ-Knaster in $M[G][H]$ it has the $\kappa = \aleph_2$-approximation property and cannot add b over $M[G][H]$.

Now, we will show that over $M[G][H][h]$, $Q_{\mathbb{M}}$ also cannot add b. In $M[G][H][h]$, $\kappa = \aleph_2 = 2^\omega$ and $Q_{\mathbb{M}}$ is forcing equivalent to $\mathrm{Add}(\omega, \kappa^\dagger) * \mathbb{M}^\kappa$, where $\kappa^\dagger$ is the first inaccessible above κ. The branch b cannot be added by $\mathrm{Add}(\omega, \kappa^\dagger)$ since this forcing is ω_1-Knaster.

[5] We identify subsets of λ and $j``\lambda$ with their characteristic functions to make this formally correct.

In $M[G][H][h][\mathrm{Add}(\omega,\kappa^\dagger)]$, $2^\omega = \kappa^\dagger > (\omega_2)^{V[G][H]}$ and D is a list with width at most $((2^{\omega_1})^+)^{V[G][H]}) = (\omega_3)^{V[G][H]}$. By standard product analysis (see Section 8) $\mathbb{M}^\kappa$ is a projection of a product of the form $\mathrm{Add}(\omega, j(\kappa)) \times \mathbb{Q}_\kappa$, where $\mathbb{Q}_\kappa$ is ω_1-closed. Again the branch b cannot be added by $\mathrm{Add}(\omega, j(\kappa))$ since this forcing is ω_1-Knaster. Since D has width $(\omega_3)^{V[G][H]} < 2^\omega$ in $M[G][H][h][\mathrm{Add}(\omega,\kappa^\dagger)]$, we can apply Lemma 7.2 to $\mathrm{Add}(\omega, j(\kappa))$ as $\mathbb{P}$ and $\mathbb{Q}_\kappa$ as $\mathbb{Q}$ over the model $M[G][H][h][\mathrm{Add}(\omega,\kappa^\dagger)]$. Therefore the forcing $\mathbb{Q}_\kappa$ cannot add b over the model $M[G][H][h][\mathrm{Add}(\omega,\kappa^\dagger)][\mathrm{Add}(\omega, j(\kappa))]$, hence the cofinal branch b cannot be added by $\mathbb{M}^\kappa$ over $M[G][H][h][\mathrm{Add}(\omega,\kappa^\dagger)]$. □

Claim 9.12. *The function b is an ineffable branch in D.*

Proof. We need to show that the set $S = \{x \in \mathscr{P}_\kappa(\lambda) \mid b \restriction x = d_x\}$ is stationary, hence it is enough to show that $j``\lambda$ is in $j(S) = \{y \in \mathscr{P}_{j(\kappa)}(j(\lambda)) \mid j(b) \restriction y = d'_y\}$. However this follows from the definition of $b$: $j(b) \restriction j``\lambda = j``b = d'_{j``\lambda}$. □

This finishes the proof of Theorem 9.9. □

In [24, Question 2], Viale asks whether it is consistent that there exist μ-guessing models that are not ω_1-guessing for some $\mu > \omega_1$. A positive answer is given by Hachtman and Sinapova in [8, Corollary 4.6]; they prove that, if μ is a singular limit of ω-many supercompact cardinals, then, for every sufficiently large regular cardinal θ, there are stationarily many μ^+-guessing models in $\mathscr{P}_{\mu^+}H(\theta)$ that are not δ-guessing for any $\delta \leq \mu$. Theorem 9.9 gives a different path to a positive answer, at a smaller cardinal and from weaker large cardinal assumptions.

Corollary 9.13. *Let κ be a supercompact cardinal. There is a generic extension in which, for all regular $\theta \geq \omega_2$, there are stationarily many ω_2-guessing models $M \in \mathscr{P}_{\omega_2}H(\theta)$ that are not ω_1-guessing.*

Proof. Let W be the generic extension witnessing the conclusion of Theorem 9.9. Since $\mathsf{ISP}(\omega_2, \omega_2, \lambda)$ holds for all $\lambda \geq \omega_2$, Corollary 4.14 implies that there are stationarily many ω_2-guessing models in $\mathscr{P}_{\omega_2}H(\theta)$ for all regular $\theta \geq \omega_2$. On the other hand, there is an ω_1-Kurepa tree in W, so Theorem 9.3 implies that, for all regular $\theta \geq \omega_2$, the set of ω_1-guessing models is nonstationary in $\mathscr{P}_{\omega_2}H(\theta)$. The conclusion follows. □

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