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Abstract

We show that the set of “wild data”, meaning the initial data for which the barotropic Euler system admits infinitely many *admissible entropy* solutions, is dense in the L^p-topology of the phase space.

Keywords: compressible Euler system, wild data, convex integration.

1 Introduction

The concept of wild data/solution appeared recently in the context of the ill–posedness results obtained via the method of convex integration, see e.g. Buckmaster et al [3], [4] and the references cited therein. In contrast with the concept of wild solution that may be ambiguous, the wild data can be identified with those that give rise to infinitely many solutions of a given problem on any (short) time interval, see Definition 1.2 below. Székelyhidi and Wiedemann [8] showed that the set of wild data for the *incompressible* Euler system in the framework of weak solutions satisfying the global energy inequality is dense in the L^p-topology of the phase space. Our goal is to discuss the problem in the class of weak entropy solutions of the barotropic Euler system describing the motion of a compressible fluid.

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1.1 Barotropic Euler system

We consider the *barotropic Euler system*:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0 \tag{1.2}$$

describing the time evolution of the mass density $\varrho = \varrho(t,x)$ and the velocity $\mathbf{u} = \mathbf{u}(t,x)$ of a compressible inviscid fluid. For simplicity, we impose the space periodic boundary conditions identifying the fluid domain with the flat torus:

$$\mathbb{T}^d = \left([-1,1]|_{\{-1;1\}}\right)^d, \ d = 2,3. \tag{1.3}$$

The same method can be used to obatin similar results for fluids occupying a bounded domain $\Omega \subset R^d$, with impermeable boundary

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \mathbf{n} - \text{the outer normal vector to } \partial\Omega, \tag{1.4}$$

see Section 5.

Definition 1.1 (Admissible entropy solution). We say that $(\varrho, \mathbf{u})$ is *admissible entropy solution* to the Euler system (1.1)–(1.3) in $(0,T) \times \mathbb{T}^d$ with initial data

$$\varrho(0,\cdot) = \varrho_0, \ \mathbf{u}(0,\cdot) = \mathbf{u}_0$$

if the following holds:

$$\int_0^T \int_{\mathbb{T}^d} \left[\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \right] \ \mathrm{d}x \, \mathrm{d}t = - \int_{\mathbb{T}^d} \varrho_0 \varphi(0,\cdot) \ \mathrm{d}x, \tag{1.5}$$

$$\text{for any } \varphi \in C^1_c([0,T) \times \mathbb{T}^d),$$

$$\int_0^T \int_{\mathbb{T}^d} \left[\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x \, \mathrm{d}t = - \int_{\mathbb{T}^d} \varrho_0 \mathbf{u}_0 \cdot \boldsymbol{\varphi}(0,\cdot) \ \mathrm{d}x \tag{1.6}$$

$$\text{for any } \boldsymbol{\varphi} \in C^1_c([0,T) \times \mathbb{T}^d; R^d),$$

with the energy inequality:

$$\begin{aligned} &\int_0^T \int_{\mathbb{T}^d} \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right) \partial_t \varphi + \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) + p(\varrho) \right) \mathbf{u} \cdot \nabla_x \varphi \right] \ \mathrm{d}x \, \mathrm{d}t \\ &\quad \geq - \int_{\mathbb{T}^d} \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + P(\varrho_0) \right) \varphi(0,\cdot) \ \mathrm{d}x \end{aligned} \tag{1.7}$$

for any $\varphi \in C^1([0,T) \times \mathbb{T}^d), \ \varphi \geq 0$,

where we have introduced the pressure potential:

$$P'(\varrho)\varrho - P(\varrho) = p(\varrho).$$

1.2 Wild data, main result

Definition 1.2 (Wild data). We say that the initial data $\varrho_0, \mathbf{u}_0$ are *wild* if there exists $T_{\rm w} > 0$ such that the Euler system admits infinitely many admissible entropy solutions $(\varrho, \mathbf{u})$ on any interval $[0,T]$, $0 < T < T_{\rm w}$ such that

$$\varrho \in L^\infty((0,T) \times \mathbb{T}^d),\ \varrho > 0,\ \mathbf{u} \in L^\infty((0,T) \times \mathbb{T}^d; R^d).$$

We are ready to state our main result.

Theorem 1.3 (Density of wild data). *Suppose $p \in C^\infty(a,b)$, $p' > 0$ in (a,b), for some $0 \le a < b \le \infty$.*

Then for any

$$\varrho_0 \in W^{k,2}(\mathbb{T}^d),\ a < \inf_{\mathbb{T}^d} \varrho_0 \le \sup_{\mathbb{T}^d} \varrho_0 < b, \mathbf{u}_0 \in W^{k,2}(\mathbb{T}^d; R^d),\ k > \frac{d}{2} + 1,$$

any $\varepsilon > 0$, and any $1 \le p < \infty$, there exist wild data $\varrho_{0,\varepsilon}$, $\mathbf{u}_{0,\varepsilon}$ such that

$$\| \varrho_{0,\varepsilon} - \varrho_0 \|_{L^p(\mathbb{T}^d)} < \varepsilon,\ \| \mathbf{u}_{0,\varepsilon} - \mathbf{u}_0 \|_{L^p(\mathbb{T}^d;R^d)} < \varepsilon.$$

Recently, Chen, Vasseur, and You [5], established density of wild data for the isentropic Euler system in the class of weak solutions satisfying the total energy inequality

$$\int_{\mathbb{T}^d} \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right] (\tau, \cdot)\ {\rm d}x \le \int_{\mathbb{T}^d} \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + P(\varrho_0) \right]\ {\rm d}x \text{ for any } \tau > 0. \tag{1.8}$$

These solutions are global in time, however, the associated total energy profile

$$\int_{\mathbb{T}^d} \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right] (\tau, \cdot)\ {\rm d}x$$

may not be non–increasing in time.

The proof of Theorem 1.3 is based on a combination of the strong solution ansatz proposed by Chen, Vasseur, and You [5] with the abstract convex integration results concerning weak solutions with a given energy profile established in [6].

2 Convex integration ansatz

Similarly to Chen, Vasseur, and You [5], our convex integration ansazt is based on strong solutions to the Euler system.

2.1 Local in time smooth solutions

Proposition 2.1 (Local existence for smooth data). *Suppose $p \in C^\infty(a,b)$, $p' > 0$ in (a,b), for some $0 \leq a < b \leq \infty$.*

Then for any initial data

$$\varrho_0 \in W^{k,2}(\mathbb{T}^d),\ a < \inf_{\mathbb{T}^d} \varrho_0 \leq \sup_{\mathbb{T}^d} \varrho_0 < b, \mathbf{u}_0 \in W^{k,2}(\mathbb{T}^d; R^d),\ k > \frac{d}{2} + 1 \tag{2.1}$$

there exists $T_{\max} > 0$ such that the compressible Euler system admits a classical solution $(\varrho, \mathbf{u})$ unique in the class

$$\varrho \in C([0,T]; W^{k,2}(\mathbb{T}^d)),\ a < \varrho < b,\ \mathbf{u} \in C([0,T]; W^{k,2}(\mathbb{T}^d; R^d)) \tag{2.2}$$

for any $0 < T < T_{\max}$.

The proof of Proposition 2.1 is nowadays standard and essentially attributed to Kato [7], cf. also Benzoni-Gavage and Serre [2, Chapter 13, Theorem 13.1].

2.2 Basic convex integration ansatz

Consider the initial data $(\varrho_0, \mathbf{u}_0)$ in the regularity class (2.1) together with the associated smooth solution $(\tilde{\varrho}, \tilde{\mathbf{u}})$ in $[0,T] \times \mathbb{T}^d$, $T < T_{\max}$. In addition, denote $\widetilde{\mathbf{m}} = \tilde{\varrho}\tilde{\mathbf{u}}$. The Euler system written in the conservative variables $(\tilde{\varrho}, \widetilde{\mathbf{m}})$ reads

$$\partial_t \tilde{\varrho} + \mathrm{div}_x \widetilde{\mathbf{m}} = 0, \tag{2.3}$$

$$\partial_t \widetilde{\mathbf{m}} + \mathrm{div}_x \left(\frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\tilde{\varrho}} + p(\tilde{\varrho})\mathbb{I} \right) = 0. \tag{2.4}$$

We look for solutions in the form

$$\varrho = \tilde{\varrho},\ \mathbf{m} = \varrho \mathbf{u} = \widetilde{\mathbf{m}} + \mathbf{v},$$

where

$$\mathrm{div}_x \mathbf{v} = 0, \tag{2.5}$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \widetilde{\mathbf{m}}) \otimes (\mathbf{v} + \widetilde{\mathbf{m}})}{\tilde{\varrho}} - \frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\tilde{\varrho}} \right) = 0, , \tag{2.6}$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0. \tag{2.7}$$

To apply the abstract results of [6], we rewrite problem (2.5)–(2.7) in the form

$$\mathrm{div}_x \mathbf{v} = 0, \tag{2.8}$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v}+\widetilde{\mathbf{m}})\otimes(\mathbf{v}+\widetilde{\mathbf{m}})}{\tilde{\varrho}} - \frac{1}{d}\frac{|\mathbf{v}+\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}}\mathbb{I} - \frac{\widetilde{\mathbf{m}}\otimes\widetilde{\mathbf{m}}}{\tilde{\varrho}} + \frac{1}{d}\frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}}\mathbb{I} \right) = 0, \tag{2.9}$$

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0. \tag{2.10}$$

together with the prescribed “kinetic energy”

$$\frac{1}{2}\frac{|\mathbf{v}+\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} = \frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} + \Lambda, \tag{2.11}$$

with a suitable spatially homogeneous function $\Lambda = \Lambda(t)$ determined below.

3 Application of convex integration

Setting

$$\mathbb{H} = \frac{\widetilde{\mathbf{m}}\otimes\widetilde{\mathbf{m}}}{\tilde{\varrho}} - \frac{1}{d}\frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}}\mathbb{I} \in C^1([0,T]\times\mathbb{T}^d; R^{d\times d}_{0,\mathrm{sym}}),\ e = \frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} + \Lambda \in C([0,T]\times\mathbb{T}^d), \tag{3.1}$$

we may rewrite (2.8)–(2.11) in the form

$$\mathrm{div}_x \mathbf{v} = 0, \tag{3.2}$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v}+\widetilde{\mathbf{m}})\otimes(\mathbf{v}+\widetilde{\mathbf{m}})}{\tilde{\varrho}} - \frac{1}{d}\frac{|\mathbf{v}+\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}}\mathbb{I} - \mathbb{H} \right) = 0, \tag{3.3}$$

$$\frac{1}{2}\frac{|\mathbf{v}+\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} = e, \tag{3.4}$$

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0, \tag{3.5}$$

for fixed $\mathbb{H}$ and e given by (3.1).

3.1 Subsolutions

To apply the abstract results obtained in [6], we introduce the set of *subsolutions*

$$\begin{aligned} X_0 = \Big\{ & \mathbf{v} \in C_{\mathrm{weak}}([0,T]; L^2(\mathbb{T}^d;R^d)) \cap L^\infty((0,T)\times\mathbb{T}^d;R^d) \ \Big| \\ & \mathbf{v}(0,\cdot) = \mathbf{v}_0,\ \mathbf{v}(T;\cdot) = \mathbf{v}_T, \mathbf{v} \in C((0,T)\times\mathbb{T}^d;R^d), \\ & \mathrm{div}_x \mathbf{v} = 0,\ \partial_t \mathbf{v} + \mathrm{div}_x \mathbb{F} = 0 \text{ in } \mathcal{D}'((0,T)\times\mathbb{T}^d) \\ & \text{for some } \mathbb{F} \in L^\infty((0,T)\times\mathbb{T}^d; R^{d\times d}_{0,\mathrm{sym}}) \cap C((0,T)\times\mathbb{T}^d; R^{d\times d}_{0,\mathrm{sym}}), \\ & \sup_{0<\tau<t<T; x\in\mathbb{T}^d} \frac{d}{2}\lambda_{\max}\left[\frac{(\mathbf{v}+\widetilde{\mathbf{m}})\otimes(\mathbf{v}+\widetilde{\mathbf{m}})}{\tilde{\varrho}} - \mathbb{F} - \mathbb{H} \right] - e < 0, \\ & \text{for any } 0<\tau<T \Big\}. \end{aligned} \tag{3.6}$$

Here, the symbol $\lambda_{\max}[\mathbb{A}]$ denotes the maximal eigenvalue of a symmetric matrix $\mathbb{A}$.

3.2 First existence result

The following results is a special case of [6, Theorem 13.2.1].

Proposition 3.1. *Suppose that set of subsolutions* X_0 *is non–empty and bounded in* $L^\infty((0,T)\times\mathbb{T}^d;R^d)$, $d=2,3$.

Then problem (3.2)–(3.5) *admits infinitely many weak solutions.*

Fix $\mathbf{v}_0=\mathbf{v}_T=0$. Next, using the algebraic inequality

$$\frac{1}{2}\frac{|\mathbf{v}+\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} \leq \frac{d}{2}\lambda_{\max}\left[\frac{(\mathbf{v}+\widetilde{\mathbf{m}})\otimes(\mathbf{v}+\widetilde{\mathbf{m}})}{\tilde{\varrho}} - \mathbb{F} - \mathbb{H}\right] \tag{3.7}$$

we can see that the set X_0 is bounded in $L^\infty((0,T)\times\mathbb{T}^d;R^d)$ as long as $\Lambda\in C[0,T]$. Finally, we observe that $\mathbf{v}\equiv 0$ is a subsolution as soon as

$$\Lambda(t)>0 \text{ for any } t\in[0,T]. \tag{3.8}$$

Indeed we may consider $\mathbb{F}\equiv 0$ and compute

$$\frac{d}{2}\lambda_{\max}\left[\frac{(\mathbf{v}+\widetilde{\mathbf{m}})\otimes(\mathbf{v}+\widetilde{\mathbf{m}})}{\tilde{\varrho}} - \mathbb{F} - \mathbb{H}\right] = \frac{d}{2}\lambda_{\max}\left[\frac{\widetilde{\mathbf{m}}\otimes\widetilde{\mathbf{m}}}{\tilde{\varrho}} - \mathbb{H}\right] = \frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}}$$

while

$$e=\frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}}+\Lambda.$$

Thus a direct application of Proposition 3.1 yields the following result.

Theorem 3.2 (Existence with a small initial energy jump). *Let* $\Lambda\in C[0,T]$, $\inf_{t\in[0,T]}\Lambda(t)>0$ *be given. Let* $\mathbf{v}_0=0$.

Then problem (3.1)–(3.4) *admits infinitely many weak solutions* $\mathbf{v}$ *in* $(0,T)\times\mathbb{T}^d$.

As $\mathbf{v}_0=0$ and $\Lambda>0$, the solutions $\varrho=\tilde{\varrho}$, $\mathbf{m}=\widetilde{\mathbf{m}}+\mathbf{v}$ necessarily experience an initial energy jump therefore they are not physically admissible. This problem will be fixed in the next section.

3.3 Second existence result

The following results is a special case of [6, Theorem 13.6.1].

Proposition 3.3. *Suppose that set of subsolutions* X_0 *is non–empty and bounded in* $L^\infty((0,T)\times\mathbb{T}^d;R^d)$, $d=2,3$.

Then there exists a set of time $\mathfrak{T}\subset(0,T)$ *dense in* $(0,T)$ *with the following properties:*

For any $\tau\in\mathfrak{T}$ *there exists* $\mathbf{v}^\tau\in\overline{X_0}$ *satisfying:*

- $$\begin{aligned} \mathbf{v}^\tau \in C_{\rm weak}([0,T]; L^2(\mathbb{T}^d; R^d)) \cap L^\infty((0,T)\times \mathbb{T}^d; R^d) \cap C((\tau,T)\times\mathbb{T}^d; R^d), \\ \mathbf{v}^\tau(0,\cdot) = \mathbf{v}_0,\ \mathbf{v}^\tau(T,\cdot) = \mathbf{v}_T; \end{aligned} \tag{3.9}$$

- $$\partial_t \mathbf{v}^\tau + {\rm div}_x \mathbb{F} = 0 \ \mathit{in}\ \mathcal{D}'((\tau,T)\times\mathbb{T}^d) \tag{3.10}$$

  for some $\mathbb{F} \in L^\infty \cap C((\tau;T)\times\mathbb{T}^d; R^{d\times d}_{0,{\rm sym}})$*;*

- $$\sup_{\tau+s<t<T, x\in\mathbb{T}^d} \frac{d}{2}\lambda_{\max}\left[\frac{(\mathbf{v}^\tau + \widetilde{\mathbf{m}})\otimes(\mathbf{v}^\tau + \widetilde{\mathbf{m}})}{\tilde{\varrho}} - \mathbb{F} - \mathbb{H}\right] - e < 0 \tag{3.11}$$

  for any $0 < s < T - \tau$*;*

- $$\frac{1}{2}\int_{\mathbb{T}^d} \frac{|\mathbf{v}^\tau + \widetilde{\mathbf{m}}|^2}{\tilde{\varrho}}(\tau,\cdot)\ {\rm d}x = \int_{\mathbb{T}^d} e(\tau,\cdot)\ {\rm d}x. \tag{3.12}$$

In accordance with (3.9)–(3.12), the function $\mathbf{v}^\tau$ can be used as a subsolution on the time interval (τ, T). Then Proposition 3.1 yields the following result.

Theorem 3.4 (Existence without initial energy jump). *Let* $\Lambda \in C[0,T]$*,* $\inf_{t\in[0,T]}\Lambda(t) > 0$ *be given.*

Then there exists a sequence $\tau_n \to 0$ *and* $\mathbf{v}_{0,n}$*,*

$$\mathbf{v}_{0,n} \to 0 \ \textit{weakly-(*) in}\ L^\infty(\mathbb{T}^d; R^d)$$

such that problem (3.1)–(3.4) *admits infinitely many weak solutions in* $(\tau_n, T)\times\mathbb{T}^d$ *satisfying*

$$\mathbf{v}(\tau_n,\cdot) = \mathbf{v}_{0,n},\ \mathbf{v}(T,\cdot) = 0,\ \frac{1}{2}\frac{|\mathbf{v} + \widetilde{\mathbf{m}}|^2}{\tilde{\varrho}}(\tau_n,\cdot) = e(\tau_n). \tag{3.13}$$

Note carefully that continuity of the initial energy stated in (3.13) follows from (3.12) and weak continuity of $\mathbf{v}$.

4 Adjusting the energy profile

To complete the proof of Theorem 1.3, it remains to adjust the energy profile Λ so that:

- $$\limsup_{n\to 0} \|\mathbf{v}_{0,n}\|_{L^2(\mathbb{T}^d;R^d)} < \varepsilon; \tag{4.1}$$

- the energy inequality (1.7) holds for $\mathbf{u} = \mathbf{v} + \widetilde{\mathbf{m}}$, $\varrho = \tilde{\varrho}$, at least on a short time interval.

As for (4.1), it is enough to choose $\Lambda(0) > 0$ small enough. Indeed (3.1), (3.13) yield

$$\frac{1}{2} \frac{|\mathbf{v} + \widetilde{\mathbf{m}}|^2}{\tilde{\varrho}}(\tau_n, \cdot) = \frac{1}{2} \frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}}(\tau_n) + \Lambda(\tau_n).$$

Seeing that

$$\mathbf{v}(\tau_n, \cdot) = \mathbf{v}_{0,n} \to 0 \text{ weakly in } L^2(\mathbb{T}^d; R^d)$$

we easily conclude.

Finally, the total energy of the system reads

$$\frac{1}{2} \frac{|\mathbf{v} + \widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} + P(\tilde{\varrho}) = \frac{1}{2} \frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} + \Lambda + P(\tilde{\varrho}) \text{ a.a. in } (0,T) \times \mathbb{T}^d.$$

In particular, the energy is continuously differentiable as soon as $\Lambda \in C^1[0,T]$. The desired energy inequality reads

$$\partial_t \left(\frac{1}{2} \frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} + P(\tilde{\varrho}) \right) + \Lambda' + \mathrm{div}_x \left[\left(\frac{1}{2} \frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} + P(\tilde{\varrho}) + \Lambda + p(\tilde{\varrho}) \right) \frac{\widetilde{\mathbf{m}} + \mathbf{v}}{\tilde{\varrho}} \right] \leq 0. \tag{4.2}$$

Seeing that the smooth solution $(\tilde{\varrho}, \widetilde{\mathbf{m}})$ satisfies the energy equality we may simplify (4.2) to

$$\Lambda' + \Lambda \mathrm{div}_x \tilde{\mathbf{u}} + \mathrm{div}_x \left[\left(\frac{1}{2} \frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} + P(\tilde{\varrho}) + \Lambda + p(\tilde{\varrho}) \right) \frac{\mathbf{v}}{\tilde{\varrho}} \right] \leq 0. \tag{4.3}$$

Moreover, as $\mathrm{div}_x \mathbf{v} = 0$,

$$\mathrm{div}_x \left[\left(\frac{1}{2} \frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} + P(\tilde{\varrho}) + \Lambda + p(\tilde{\varrho}) \right) \frac{\mathbf{v}}{\tilde{\varrho}} \right] = \nabla_x \left[\frac{1}{\tilde{\varrho}} \left(\frac{1}{2} \frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} + P(\tilde{\varrho}) + \Lambda + p(\tilde{\varrho}) \right) \right] \cdot \mathbf{v},$$

and (4.3), reduces to

$$\Lambda' + \Lambda \mathrm{div}_x \tilde{\mathbf{u}} + \nabla_x \left[\frac{1}{\tilde{\varrho}} \left(\frac{1}{2} \frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} + P(\tilde{\varrho}) + \Lambda + p(\tilde{\varrho}) \right) \right] \cdot \mathbf{v} \leq 0. \tag{4.4}$$

As Λ is decreasing in t, we get

$$\Lambda(t) \leq \Lambda(0).$$

Similarly, we control $\|\mathbf{v}\|_{L^\infty((0,T) \times \mathbb{T}^d; R^d)}$ by means of $\Lambda(0)$ and certain norms of the strong solution $\tilde{\varrho}$, $\widetilde{\mathbf{m}}$.

Choosing

$$\Lambda(t) = \varepsilon \exp \left(-\frac{t}{\varepsilon^2} \right),$$

with $\varepsilon > 0$ small enough, we obtain the desired energy inequality at least on a short time interval $(0, T_{\mathrm{w}})$, $T_{\mathrm{w}} > 0$. We have proved Theorem 1.3 for $p = 2$. The same statement holds for a general $1 \leq p < \infty$ as all solutions in question are uniformly bounded.

5 Concluding remarks

A similar result can be shown for the more realistic complete slip boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

imposed on a bounded domain $\Omega \subset R^d$. Note that the result is local in time and that the smooth solutions of the Euler system enjoy the finite speed of propagation property. Consequently, the problem of compatibility conditions may be solved by considering the initial data in the form

$$\varrho_0 \in W^{k,2}(\mathbb{T}^d),\ a < \inf_{\mathbb{T}^d} \varrho_0 \leq \sup_{\mathbb{T}^d} \varrho_0 < b, \mathbf{u}_0 \in W^{k,2}(\mathbb{T}^d; R^d),\ k > \frac{d}{2} + 1,$$
$$\mathbf{u} = 0, \varrho = \overline{\varrho} - \text{a positive constant in a meighborhood of } \partial\Omega.$$

The relevant local existence result for strong solutions was proved by Beirão da Veiga [1].

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