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Abstract

We present new error estimates for the finite volume and finite difference methods applied to the compressible Navier–Stokes equations. The main innovative ingredients of the improved error estimates are a refined consistency analysis combined with a continuous version of the relative energy inequality. Consequently, we obtain better convergence rates than those available in the literature so far. Moreover, the error estimates hold in the whole physically relevant range of the adiabatic coefficient.

Keywords: compressible Navier–Stokes system, error estimates, relative energy, strong solution, upwind finite volume method, Marker-and-Cell finite difference method

1 Introduction

The Navier–Stokes equations governing the motion of viscous compressible fluids have numerous applications in engineering, physics, meteorology or biomedicine. In this paper we consider the viscous barotropic fluid endowed, for simplicity, with the isentropic pressure–density state equation $p = a\varrho^{\gamma}$, where $a > 0$ is a positive constant, and $\gamma > 1$ denotes the adiabatic coefficient. The global–in–time

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existence of weak solutions is known for any $\gamma > \frac{d}{2}$ in the d-dimensional setting, see Lions [20] and [7]. More recently, Plotnikov and Vaigant [26] extended the existence theory for any $\gamma \geq 1$ if $d = 2$. Unfortunately, the multilevel approach used in the existence proof is rather difficult to adapt directly to a numerical scheme; whence the numerical analysis of the problem remains rather incomplete.

In the last few decades, many efficient and robust numerical methods have been proposed to simulate the motion of viscous compressible fluid flows. We refer the reader to the monographs by Dolejší and Feistauer [2], Eymard, Gallouët and Herbin [5], Feistauer [3], Feistauer, Felcman and Straškraba [4], Toro [27], and the references therein. Despite a good agreement of the obtained results with experiments, a rigorous convergence analysis with the associated error estimates have been performed only in a few particular cases.

In his truly pioneering work, Karper [18], see also [9], showed convergence (up to a subsequence) of a mixed finite element-finite volume (or discontinuous Galerkin) approximation to a weak solution of the compressible multidimensional Navier–Stokes system under the technical restriction $\gamma > 3$. His proofs basically follows step by step the existence theory developed in [7] and as such is difficult to adapt to other numerical methods. Moreover, as the weak solutions are not known to be unique, the result holds up to a subsequence and no convergence rate is available.

Recently, see [10, 11, 12], we have developed a new approach based on the concept of more general dissipative weak (dissipative measure-valued) solution, which, combined with the weak–strong uniqueness and conditional regularity results, yields a rigorous proof of convergence for the mixed finite element-finite volume, finite volume and finite difference Marker-and-Cell (MAC) methods for any $\gamma > 1$ as long as the sequence of numerical solution remains uniformly bounded and/or if the strong solution exists. The aim of the present paper is to derive error estimates for the finite volume and the MAC methods for full range of the adiabatic coefficient $\gamma > 1$.

There are several results concerning error estimates for the compressible Navier–Stokes equations. Under the assumption of the L^2-bounds of the discrete derivatives of the numerical solutions, Jovanović [17] studied the convergence rate of a finite volume-finite difference method to the barotropic Navier–Stokes system. In [21, 22] Liu analyzed the errors for P^k conforming finite element method, $k \geq 2$, assuming the existence of a suitably regular smooth solution. However, the stability of the method with respect to the discrete energy was not investigated.

Furthermore, Gallouët et al. [14, 15] analyzed the *unconditional* convergence rates of the mixed finite volume-finite element method [9] and the MAC scheme for $\gamma > 3/2$ in the dimension $d = 3$. Similar results have been obtained by Mizerová and She [23]. All the above mentioned convergence results are based on a discrete version of the relative energy inequality estimating the error between the numerical and the strong solution. The obtained convergence error is $\mathcal{O}(h^A)$, where $h > 0$ is a mesh parameter and $A = \min\left\{\frac{2\gamma-3}{\gamma}, \frac{1}{2}\right\}$, cf. [14, 15, 23]. In particular, the convergence order tends to zero when $\gamma \to \frac{3}{2}$ and remains positive only if $\gamma > 3/2$. Moreover, if $\gamma \geq 2$, the convergence rate is only $\frac{1}{2}$ in the energy norm, though the numerical experiments indicate the second order convergence rate.

In view of the existing results, the main novelty of the present paper is two-fold:

- Extending the error analysis to the full range $\gamma > 1$.
- Improving the convergence rate via a detailed consistency and error analysis.

Following the strategy proposed in the monograph [12, Chapter 9], we combine the standard consistency errors with the "continuous" form of the relative energy inequality. In contrast with the existing methods based on ad hoc construction of an approximate relative energy inequality, the new approach is rather versatile and free of additional discretization errors. In particular, we can handle any consistent energy stable numerical method in the same fashion. We focus on the finite volume method proposed in [12] and the MAC method from [23]. The application to the mixed finite element-finite volume method of Karper [18] was studied independently and presented in the recent work by Novotný and Kwon [19]. Compared to the previous results of Gallouët et al. [14, 15], we employ the consistency formulation of the numerical solution where the test function is smooth. This new approach avoids the complicated integration by parts formulae on the discrete level and improves the convergence rates of the MAC method presented in [14, 23].

The paper is organized in the following way. After presenting the continuous model and the corresponding relative energy, we formulate the numerical schemes: the finite volume and the MAC method, see Section 2. Next, we discuss their energy stability and consistency. The main results on the error estimates are formulated and proved in Section 3.

1.1 Compressible Navier–Stokes system

We begin with formulating the compressible Navier–Stokes system

$$\begin{aligned} \partial_t \varrho + \mathrm{div}_x(\varrho \boldsymbol{u}) &= 0, \\ \partial_t(\varrho \boldsymbol{u}) + \mathrm{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_x p(\varrho) &= \mathrm{div}_x \mathbb{S} \end{aligned} \tag{1.1}$$

in the time–space cylinder $[0,T] \times \Omega$, $\Omega \subset R^d, d = 2,3$, where ϱ is the density, $\boldsymbol{u}$ is the velocity field, and $\mathbb{S}$ is the viscous stress tensor given by

$$\mathbb{S} = \mu(\nabla_x \boldsymbol{u} + \nabla_x^T \boldsymbol{u} - \frac{2}{d}\mathrm{div}_x \boldsymbol{u} \mathbb{I}) + \lambda \mathrm{div}_x \boldsymbol{u} \mathbb{I},\ \mu > 0,\ \lambda \geq 0.$$

The pressure is assumed to satisfy the *isentropic* law

$$p = a\varrho^\gamma,\ a > 0,\ \gamma > 1. \tag{1.2}$$

To avoid technical problems related to a proper numerical approximation of the physical boundary, we impose the periodic boundary conditions and identify the computational domain with the flat torus

$$\Omega = \mathbb{T}^d \equiv \left([0,1]|_{\{0,1\}}\right)^d.$$

The system (1.1) is supplemented with finite energy initial data $(\varrho_0, \boldsymbol{u}_0) : \mathbb{T}^d \to \mathbb{R}^+ \times \mathbb{R}^d$,

$$\varrho(0,x) = \varrho_0 > 0, (\varrho \boldsymbol{u})(0,x) = \varrho_0 \boldsymbol{u}_0, \quad \text{and } E_0 = \int_{\mathbb{T}^d} \left(\frac{1}{2}\varrho_0 |\boldsymbol{u}_0|^2 + P(\varrho_0) \right) \mathrm{d}x < \infty, \tag{1.3}$$

where P is the so-called pressure potential, $P(\varrho) = \frac{a\varrho^\gamma}{\gamma - 1}$ for the isentropic gas law (1.2).

1.2 Relative energy

The main tool to evaluate the distance between numerical and strong solutions is the relative energy functional, cf. [8]:

$$\mathfrak{E}(\varrho, \boldsymbol{u} | r, \boldsymbol{U}) = \int_{\mathbb{T}^d} \left(\frac{1}{2}\varrho \left| \boldsymbol{u} - \boldsymbol{U} \right|^2 + \mathbb{E}(\varrho|r) \right) \mathrm{d}x,\ \text{with } \mathbb{E}(\varrho|r) = P(\varrho) - P'(r)(\varrho - r) - P(r).$$

As pointed out, relative energy functionals are often used to estimate the distance between a suitable weak solution and the strong solution; whence yielding the weak-strong uniqueness property. Recently, a discrete version of the relative energy has been applied in the error analysis of numerical schemes, see [14, 15, 23].

1.3 Classical solutions

It will be useful to identify the regularity class of smooth (classical) solutions to the Navier–Stokes system (1.1) inherited from the initial data (1.3). The following result can be the deduced from [1, Theorem 3.3] and [6, Proposition 2.2].

Proposition 1.1. *Let the initial data belong to the class*

$$\varrho_0 \in C^3(\mathbb{T}^d),\ \varrho_0 > 0 \textit{ in } \mathbb{T}^d,\ \boldsymbol{u}_0 \in C^3(\mathbb{T}^d; R^d).$$

Let $(\varrho, \boldsymbol{u})$ *be a weak solution to problem* (1.1) *originating from the initial data* (1.3) *such that*

$$0 \le \varrho \le \bar{r} \quad \textit{and} \quad |\boldsymbol{u}| \le \bar{u} \textit{ a.e. in } (0,T) \times \mathbb{T}^d. \tag{1.4}$$

Then $(\varrho, \boldsymbol{u})$ *is a classical solution of* (1.1)-(1.3) *in* $[0,T] \times \mathbb{T}^d$.

If, in addition, ϱ_0, $\boldsymbol{u}_0$ *belong to the class*

$$\varrho_0 \in W^{k,2}(\mathbb{T}^d), \qquad \boldsymbol{u}_0 \in W^{k,2}(\mathbb{T}^d; \mathbb{R}^d), \quad k \ge 6, \tag{1.5}$$

then $\varrho \in C([0,T]; W^{k,2}(\mathbb{T}^d))$, $\boldsymbol{u} \in C([0,T]; W^{k,2}(\mathbb{T}^d;\mathbb{R}^d))$, *and the following estimate hold*

$$\begin{aligned}
&\|\partial_t^\ell \varrho\|_{C([0,T]\times\mathbb{T}^d)} + \|\varrho\|_{C^1([0,T]\times\mathbb{T}^d)} + \|1/\varrho\|_{C([0,T]\times\mathbb{T}^d)} + \|\varrho\|_{C([0,T];W^{k,2}(\mathbb{T}^d))} \le D,\ \ell = 1,2,\\
&\|\partial_t^\ell \boldsymbol{u}\|_{C([0,T]\times\mathbb{T}^d;\mathbb{R}^d)} + \|\boldsymbol{u}\|_{C^1([0,T]\times\mathbb{T}^d;\mathbb{R}^d)} + \|\boldsymbol{u}\|_{C([0,T];W^{k,2}(\mathbb{T}^d;\mathbb{R}^d))} \le D,\ \ell = 1,2,
\end{aligned} \tag{1.6}$$

where D *depends solely on* $T, \bar{r}, \bar{u}$ *and the initial data* $(\varrho_0, \boldsymbol{u}_0)$ *via the norm* $\|(\varrho_0, \boldsymbol{u}_0)\|_{W^{k,2}(\mathbb{T}^d;\mathbb{R}^{d+1})}$ *and* $\min_{x\in\mathbb{T}^d} \varrho_0(x)$.

Proof. The first part was proved in [6, Proposition 2.2] via the local existence theory by Valli and Zajaczkowski [25] combined the weak–strong uniqueness principle and the conditional regularity result by Sun, Wang and Zhang [24]. In particular, the bounds (1.6) were established for $k = 3$, $\ell = 1$.

Next, as shown in [1, Theorem 3.3], the solution inherit higher Sobolev regularity from the data as long as the norm $\|\boldsymbol{u}\|_{C([0,T];W^{2,\infty}(\mathbb{T}^d;\mathbb{R}^d))}$ is controlled. In particular, the estimates (1.6) can be established. Similarly to Gallagher [13], the proof in [1] is based on the particular isentropic form of the pressure that enables to transform the problem to a parabolic perturbation of a symmetric hyperbolic system. □

2 Numerical methods

First, we introduce suitable notation. By c we denote a positive constant independent of the discretization parameters Δt and h. We shall frequently write $A \lesssim B$ if $A \le cB$ and $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. We also write $c \in \mathrm{co}\{a,b\}$ if $\min(a,b) \le c \le \max(a,b)$. Moreover, we denote by $\|\cdot\|_{L^p}$, $\|\cdot\|_{L^pL^q}$, and $\|\cdot\|_{L^pW^{q,s}}$ the norms $\|\cdot\|_{L^p(\mathbb{T}^d)}$, $\|\cdot\|_{L^p(0,T;L^q(\mathbb{T}^d))}$, and $\|\cdot\|_{L^p(0,T;W^{q,s}(\mathbb{T}^d))}$, respectively.

2.1 Time discretization

We divide the time interval $[0,T]$ into N_t equidistant parts with a fixed time increment Δt $(= T/N_t)$. For a function f^n given at the discrete time instances $t_n = n\Delta t$, $n = 0, 1, \cdots, N_t$, we define a piecewise constant approximation $f(t)$ in the following way

$$f(t,\cdot) = f^0 \ \text{ for } \ t < \Delta t \text{ and } f(t) = f^n \ \text{ for } \ t \in [n\Delta t, (n+1)\Delta t),\ n \in \{1, \ldots, N_t\}.$$

The time derivative is approximated by the backward Euler method

$$D_t f = \frac{f(t,\cdot) - f(t-\Delta t,\cdot)}{\Delta t} \quad \text{for all } t \in [0,T].$$

2.2 Space discretization

To begin, we introduce a uniform structured mesh including primary, dual and bidual grids.

Primary grid
We call $\mathcal{T}$ the primary grid with the following properties and notations:

- The domain $\mathbb{T}^d$ is divided into compact uniform quadrilaterals $\mathbb{T}^d = \bigcup_{K\in\mathcal{T}} K$, where $\mathcal{T}$ is the set of all elements that forms the primary grid.
- $\mathcal{E}$ denotes the set of all faces of the primary grid $\mathcal{T}$. Given an element $K \in \mathcal{T}$, $\mathcal{E}(K)$ is the set of its faces; $\mathcal{E}_i$ is the set of all faces that are orthogonal to the unit basis vector $\boldsymbol{e}_i$; $\mathcal{E}_i(K) = \mathcal{E}(K) \cap \mathcal{E}_i$ for any $i \in \{1, \ldots, d\}$.
- h denotes the uniform size of the grid, meaning $|x_K - x_L| = h$ for any neighbouring elements K and L, where x_K and x_L are the centers of K and L, respectively.
- $\sigma_{K,i-}$ and $\sigma_{K,i+}$ denote the left and right face of an element K in the i^{th}-direction, respectively.
- $\mathcal{N}(K)$ denotes the set of all neighbouring elements of $K \in \mathcal{T}$.
- $\sigma = K|L$ denotes the face σ that separates the elements K and L. Moreover, $\sigma = \overrightarrow{K|L}$ means $\sigma = K|L$ and $x_L - x_K = h\boldsymbol{e}_i$ for some $i \in \{1, \ldots, d\}$.
- $\boldsymbol{n}$ denotes the outer normal of a generic face σ and $\boldsymbol{n}_{\sigma,K}$ denotes the outer normal vector to a face $\sigma \in \mathcal{E}(K)$.

Dual grid
The dual of the primary grid is determined as follows.

- For any face $\sigma = K|L \in \mathcal{E}_i$, a dual cell is defined as $D_\sigma = D_{\sigma,K} \cup D_{\sigma,L}$, where $D_{\sigma,K} = \{x \in K, x_i \in \text{co}\{(x_K)_i, (x_\sigma)_i\}\}$, see Figure 1(a) for a two dimensional graphic illustration.
- $\mathcal{D}_i = \{D_\sigma \mid \sigma \in \mathcal{E}_i\}$, $i \in \{1, \ldots, d\}$, represents the i^{th} dual grid of $\mathcal{T}$. Note that for each fixed $i \in \{1, \ldots, d\}$ it holds

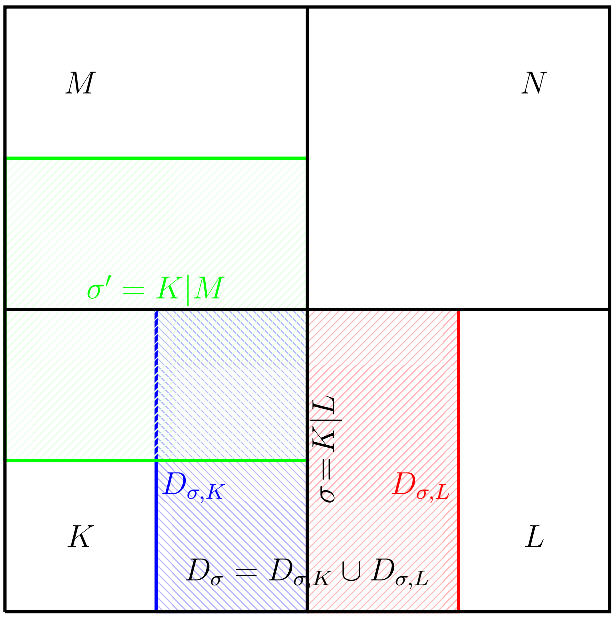
$$\mathbb{T}^d = \bigcup_{\sigma\in\mathcal{E}_i} D_\sigma, \;\; \text{int}(D_\sigma) \cap \text{int}(D_{\sigma'}) = \emptyset \text{ for } \sigma, \sigma' \in \mathcal{E}_i,\, \sigma \neq \sigma'.$$

- $\widetilde{\mathcal{E}}_i$ is the set of all faces of the i^{th} dual grid $\mathcal{D}_i$ and $\widetilde{\mathcal{E}}_{i,j} = \{\epsilon \in \widetilde{\mathcal{E}}_i | \epsilon \text{ is orthogonal to } \boldsymbol{e}_j\}$.
- A generic face of a dual cell D_σ is denoted as $\epsilon \in \widetilde{\mathcal{E}}(D_\sigma)$, where $\widetilde{\mathcal{E}}(D_\sigma)$ denotes the set of all faces of D_σ.
- $\epsilon = D_\sigma|D_{\sigma'}$ denotes a dual face that separates the dual cells D_σ and $D_{\sigma'}$. Moreover, $\epsilon = \overrightarrow{D_\sigma|D_{\sigma'}}$ means $\epsilon = D_\sigma|D_{\sigma'}$ and $x_{\sigma'} - x_\sigma = h\boldsymbol{e}_i$ for some $i \in \{1, \ldots, d\}$.
- $\mathcal{N}^\star(\sigma)$ denotes the set of all faces whose associated dual elements are the neighbours of D_σ, i.e.,

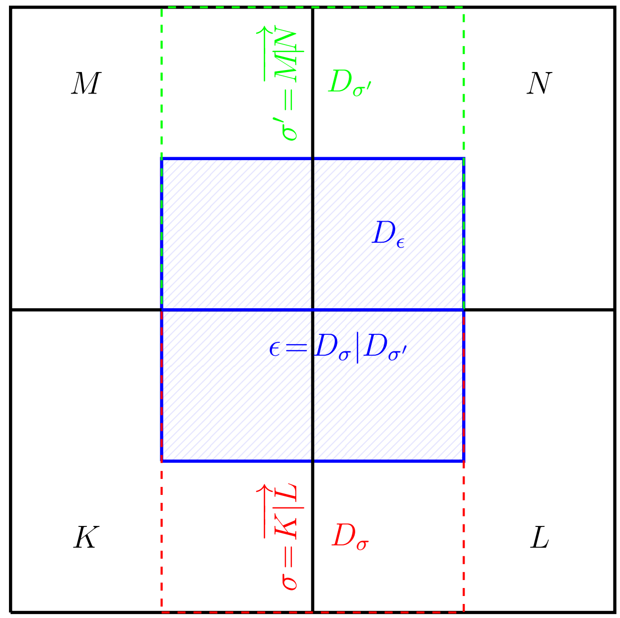
$$\mathcal{N}^\star(\sigma) = \{\sigma' \mid D_{\sigma'} \text{ is a neighbour of } D_\sigma\}.$$

Bidual grid

- Similarly to the definition of the dual cell, a bidual cell $D_\epsilon := D_{\epsilon,\sigma} \cap D_{\epsilon,\sigma'}$ associated to $\epsilon = D_\sigma | D_{\sigma'} \in \widetilde{\mathcal{E}}_{i,j}$ is defined as the union of adjacent halves of D_σ and $D_{\sigma'}$, where $D_{\epsilon,\sigma} = \{x \in D_\sigma | x_j \in \mathrm{co}\{(x_\sigma)_j, (x_\epsilon)_j\}\}$ see Figure 1(b) for a two dimensional graphic illustration.
- $\mathcal{B}_{i,j}$ denotes the j^{th} dual grid of $\mathcal{D}_i$, that is set of all bidual cells associated to the bidual faces of $\widetilde{\mathcal{E}}_{i,j}$. Note that $\mathcal{B}_{i,j} = \mathcal{T}$ in the case of $i = j$.



(a) Dual grid in two dimensions



(b) Bidual grid in two dimensions

Figure 1: MAC grid in two dimensions

Discrete function spaces. We introduce the following spaces of piecewise constant functions:

$$Q_h = \{\phi \mid \phi_h|_K = \text{ constant for all } K \in \mathcal{T}\}, \qquad \mathbf{Q}_h = Q_h^d,$$
$$\mathbf{W}_h = (W_{1,h}, \dots W_{d,h}), \quad W_{i,h} = \{\phi \mid \phi_h|_{D_\sigma} = \text{ constant for all } \sigma \in \mathcal{E}_i\}, \ i \in \{1, \dots, d\}.$$

The corresponding projections read

$$\Pi_Q : L^1(\mathbb{T}^d) \to Q_h, \qquad \Pi_Q \phi = \sum_{K \in \mathcal{T}} (\Pi_Q \phi)_K 1_K, \qquad (\Pi_Q \phi)_K = \frac{1}{|K|} \int_K \phi \, \mathrm{d}x,$$
$$\Pi_{\mathcal{E}}^{(i)} : W^{1,1}(\mathbb{T}^d) \to W_{i,h}, \qquad \Pi_{\mathcal{E}}^{(i)} \phi = \sum_{\sigma \in \mathcal{E}} (\Pi_{\mathcal{E}}^{(i)} \phi)_\sigma 1_{D_\sigma}, \qquad (\Pi_{\mathcal{E}}^{(i)} \phi)_\sigma = \frac{1}{|\sigma|} \int_\sigma \phi \, \mathrm{dS}(x),$$

where 1_K and 1_{D_σ} are the characteristic functions. Further, for any $\boldsymbol{\phi} = (\phi_1, \dots, \phi_d)$ we denote $\Pi_{\mathcal{E}} \boldsymbol{\phi} = \left(\Pi_{\mathcal{E}}^{(1)} \phi_1, \dots, \Pi_{\mathcal{E}}^{(d)} \phi_d\right)$. Moreover, for any bidual grid D_ϵ we define

$$\Pi_\epsilon \phi|_{D_\epsilon} = \frac{1}{|\epsilon|} \int_\epsilon \phi \, \mathrm{dS}(x). \tag{2.1}$$

2.3 Discrete operators

Average and jump. First, for an piecewise smooth function f_h, we define its trace

$$f_h^{\rm out}(x) = \lim_{\delta\to 0+} f_h(x+\delta\boldsymbol{n}) \ \text{ and } \ f_h^{\rm in}(x) = \lim_{\delta\to 0+} f_h(x-\delta\boldsymbol{n}).$$

Then for any $r_h \in Q_h$ we define the average operator

$$\{\!\{r_h\}\!\}_\sigma(x) = \frac{r_h^{\rm in}(x) + r_h^{\rm out}(x)}{2} \ \text{ for any } x \in \sigma \in \mathcal{E}.$$

If in addition, $\sigma \in \mathcal{E}_i$ for an $i \in \{1,\ldots,d\}$, we write $\{\!\{r_h\}\!\}_\sigma$ as $\{\!\{r_h\}\!\}_\sigma^{(i)}$ and denote

$$\{\!\{r_h\}\!\}^{(i)} = \sum_{\sigma\in\mathcal{E}_i} 1_{D_\sigma} \{\!\{r_h\}\!\}_\sigma^{(i)} \ \forall x \in \sigma \in \mathcal{E}.$$

Analogously to the average operator, we define the jump operator for $r_h \in Q_h$ as

$$[\![r_h]\!]_\sigma(x) = r_h^{\rm out}(x) - r_h^{\rm in}(x).$$

Further, for vector–valued functions $\boldsymbol{v}_h = (v_{1,h},\ldots,v_{d,h}) \in Q_h^d$ and $\boldsymbol{u}_h = (u_{1,h},\ldots,u_{d,h}) \in \mathbf{W}_h$, we define

$$\{\!\{\boldsymbol{v}_h\}\!\} = \left(\{\!\{v_{1,h}\}\!\}^{(1)},\ldots,\{\!\{v_{d,h}\}\!\}^{(d)}\right),$$

$$\overline{u_{i,h}}|_K = \frac{u_{i,h}|_{\sigma_{K,i+}} + u_{i,h}|_{\sigma_{K,i-}}}{2}, \quad \overline{u_{i,h}} = \sum_{K\in\mathcal{T}} 1_K \overline{u_{i,h}}|_K, \ \text{ and } \ \overline{\boldsymbol{u}_h} = \left(\overline{u_{1,h}},\ldots,\overline{u_{d,h}}\right).$$

Note that for any $\boldsymbol{u}_h \in \mathbf{W}_h$ we have $\overline{\boldsymbol{u}_h} = \Pi_Q \boldsymbol{u}_h$.

Gradient operator. For any $r_h \in Q_h$ and $\boldsymbol{u}_h \in \mathbf{W}_h$ we introduce the following gradient operators.

$$\nabla_\mathcal{D} r_h(x) = \left(\eth_{\mathcal{D}_1} r_h,\ldots,\eth_{\mathcal{D}_d} r_h\right)(x),$$
$$\nabla_\mathcal{B} \boldsymbol{u}_h(x) = \big(\nabla_\mathcal{B} u_{1,h}(x),\ldots,\nabla_\mathcal{B} u_{d,h}(x)\big) \quad \text{with} \quad \nabla_\mathcal{B} u_{i,h}(x) = \big(\eth_{\mathcal{B}_{i1}} u_{i,h}(x),\ldots,\eth_{\mathcal{B}_{id}} u_{i,h}(x)\big),$$

where

$$\eth_{\mathcal{D}_i} r_h(x) = \sum_{\sigma\in\mathcal{E}_i} 1_{D_\sigma} (\eth_{\mathcal{D}_i} r_h)_\sigma, \qquad (\eth_{\mathcal{D}_i} r_h)_\sigma = \frac{r_L - r_K}{h}, \ \sigma = \overrightarrow{K|L} \in \mathcal{E}_i,$$

$$\eth_{\mathcal{B}_{i,j}} u_{i,h}(x) = \sum_{\epsilon\in\widetilde{\mathcal{E}}_{i,j}} (\eth_{\mathcal{B}_{i,j}} u_{i,h})_{D_\epsilon} 1_{D_\epsilon}, \ (\eth_{\mathcal{B}_{i,j}} u_{i,h})_{D_\epsilon} = \frac{u_{\sigma'} - u_\sigma}{h}, \ \text{for } \epsilon = \overrightarrow{D_\sigma|D_{\sigma'}} \in \widetilde{\mathcal{E}}_{i,j}.$$

Furthermore, for any $\boldsymbol{v}_h \in \mathbf{Q}_h$ and $\phi \in W^{1,2}(\mathbb{T}^d)$ we set

$$\nabla_Q \boldsymbol{v}_h = \sum_{K\in\mathcal{T}} 1_K \nabla_Q \boldsymbol{v}_h|_K \quad \text{with} \quad \nabla_Q \boldsymbol{v}_h|_K = \sum_{\sigma\in\mathcal{E}(K)} \frac{|\sigma|}{|K|} \{\!\{\boldsymbol{v}_h\}\!\} \otimes \boldsymbol{n},$$
$$\nabla_\mathcal{T}^{\Pi_\mathcal{E}} \phi = \left(\partial_\mathcal{T}^{(1)} \Pi_\mathcal{E}^{(1)} \phi, \cdots, \partial_\mathcal{T}^{(d)} \Pi_\mathcal{E}^{(d)} \phi\right).$$

Here, $\partial_\mathcal{T}^{(i)}$ is defined for any $u_{i,h} \in W_{i,h}$, $i \in \{1,\ldots,d\}$ as

$$\partial_\mathcal{T}^{(i)} u_{i,h}(x) = \sum_{K\in\mathcal{T}} 1_K (\partial_\mathcal{T}^{(i)} u_{i,h})_K, \quad \partial_\mathcal{T}^{(i)} u_{i,h}\Big|_K = \frac{u_{i,h}|_{\sigma_{K,i+}} - u_{i,h}|_{\sigma_{K,i-}}}{h}, \ K \in \mathcal{T}.$$

Note that for any $r_h \in Q_h$ and $u_{i,h} \in W_{i,h}$, there hold

$$\overline{\eth_{\mathcal{D}_i} r_h} = \partial_\mathcal{T}^{(i)} \{\!\{r_h\}\!\}^{(i)} \quad \text{and} \quad \eth_{\mathcal{B}_{i,i}} u_{i,h} = \partial_\mathcal{T}^{(i)} u_{i,h}.$$

Divergence operator. For $\boldsymbol{u}_h \in \mathbf{W}_h$ and $\boldsymbol{v}_h \in \mathbf{Q}_h$ we define the following discrete divergence operators adjoint to the above discrete gradient operators

$$\mathrm{div}_{\mathcal{T}}^{\mathbf{W}} \boldsymbol{u}_h(x) = \sum_{i=1}^{d} \partial_{\mathcal{T}}^{(i)} u_{i,h}(x) \quad \text{and} \quad \mathrm{div}_{\mathcal{T}}^{Q} \boldsymbol{v}_h(x) = \sum_{i=1}^{d} \partial_{\mathcal{T}}^{(i)} \left\{\!\!\left\{ v_{i,h} \right\}\!\!\right\}^{(i)}(x) = \sum_{i=1}^{d} \overline{\eth_{\mathcal{D}_i} v_{i,h}}(x).$$

It is easy to observe for any $\boldsymbol{v}_h \in Q_h$ that

$$\mathrm{div}_{\mathcal{T}}^{\mathbf{W}} \left\{\!\!\left\{ \boldsymbol{v}_h \right\}\!\!\right\} = \mathrm{div}_{\mathcal{T}}^{Q} \boldsymbol{v}_h. \tag{2.2}$$

Upwind flux. Given a velocity field $\boldsymbol{u}_h \in \mathbf{Q}_h \cap \mathbf{W}_h$, the upwind flux function for $r_h \in Q_h$ is given by

$$\mathrm{Up}[r_h, \boldsymbol{u}_h]_\sigma = r_h^{\mathrm{in}} (u_\sigma)^+ + r_h^{\mathrm{out}} (u_\sigma)^-,$$

where

$$r^{\pm} = \frac{1}{2}(r \pm |r|), \quad u_\sigma = \begin{cases} \left\{\!\!\left\{ \boldsymbol{u}_h \right\}\!\!\right\} \cdot \boldsymbol{n}, & \text{if } \boldsymbol{u}_h \in \mathbf{Q}_h, \\ \boldsymbol{u}_h \cdot \boldsymbol{n}, & \text{if } \boldsymbol{u}_h \in \mathbf{W}_h. \end{cases}$$

To approximate nonlinear convective terms we apply the following diffusive upwind flux

$$F_h^{\varepsilon}[r_h, \boldsymbol{u}_h]_\sigma = \mathrm{Up}[r_h, \boldsymbol{u}_h]_\sigma - h^{\varepsilon} \left[\!\left[r_h \right]\!\right]_\sigma, \quad \varepsilon > -1.$$

For $\boldsymbol{\phi}_h \in \mathbf{Q}_h$ we define a vector-valued upwind flux componentwise

$$\mathbf{Up}[\boldsymbol{\phi}_h, \boldsymbol{u}_h] = \left(\mathrm{Up}[\phi_{1,h}, \boldsymbol{u}_h], \cdots, \mathrm{Up}[\phi_{d,h}, \boldsymbol{u}_h]\right), \quad \mathbf{F}_h^{\varepsilon}[\boldsymbol{\phi}_h, \boldsymbol{u}_h] = \left(F_h^{\varepsilon}[\phi_{1,h}, \boldsymbol{u}_h], \cdots, F_h^{\varepsilon}[\phi_{d,h}, \boldsymbol{u}_h]\right).$$

2.4 Preliminary estimates and inequalities

In this section we present a preliminary material. First, it is easy to check that the following integration by parts formulae hold, see e.g. [16, Lemma 2.1].

Lemma 2.1. *Let $r_h, \phi_h \in Q_h$, and $\boldsymbol{u}_h, \boldsymbol{\phi}_h \in \mathbf{W}_h$. Then*

$$\int_{\mathbb{T}^d} r_h \mathrm{div}_{\mathcal{T}}^{\mathbf{W}} \boldsymbol{u}_h \,\mathrm{d}x = -\int_{\mathbb{T}^d} \boldsymbol{u}_h \cdot \nabla_{\mathcal{D}} r_h \,\mathrm{d}x, \quad \int_{\mathbb{T}^d} r_h \partial_{\mathcal{T}}^{(i)} u_{i,h} \,\mathrm{d}x = -\int_{\mathbb{T}^d} u_{i,h} \eth_{\mathcal{D}_i} r_h \,\mathrm{d}x. \tag{2.3a}$$

Next, we report the following useful lemmas whose proofs are presented in Appendix A.

Lemma 2.2. *For any $r_h \in Q_h$, $\boldsymbol{v}_h \in \mathbf{Q}_h$, $\boldsymbol{u}_h \in \mathbf{W}_h$, $\psi \in W^{1,2}(\mathbb{T}^d)$ and $\boldsymbol{U} \in W^{1,2}(\mathbb{T}^d; \mathbb{R}^d)$, there hold*

$$\int_{\mathbb{T}^d} r_h \mathrm{div}_x \boldsymbol{U} \,\mathrm{d}x = \int_{\mathbb{T}^d} r_h \mathrm{div}_{\mathcal{T}}^{\mathbf{W}} \Pi_{\mathcal{E}} \boldsymbol{U} \,\mathrm{d}x, \tag{2.4}$$

$$\int_{\mathbb{T}^d} \boldsymbol{v}_h \cdot \nabla_x \psi \,\mathrm{d}x = \int_{\mathbb{T}^d} \boldsymbol{v}_h \cdot \nabla_{\mathcal{T}}^{\Pi_{\mathcal{E}}} \psi \,\mathrm{d}x. \tag{2.5}$$

Lemma 2.3. *For any $\boldsymbol{u}_h \in \mathbf{W}_h$, $\boldsymbol{v}_h \in \mathbf{Q}_h$ and $\psi \in W^{1,2}(\mathbb{T}^d)$ there hold*

$$\int_{\mathbb{T}^d} \boldsymbol{u}_h \cdot \nabla_x \psi \,\mathrm{d}x = -\int_{\mathbb{T}^d} \Pi_{\epsilon} \psi \; \mathrm{div}_{\mathcal{T}}^{\mathbf{W}} \boldsymbol{u}_h \,\mathrm{d}x, \tag{2.6}$$

$$\int_{\mathbb{T}^d} \boldsymbol{v}_h \cdot \nabla_x \psi \,\mathrm{d}x = -\sum_{i=1}^{d} \int_{\mathbb{T}^d} \Pi_{\mathcal{E}}^{(i)} \psi \; \eth_{\mathcal{D}_i} v_{i,h} \,\mathrm{d}x. \tag{2.7}$$

Lemma 2.4. *For any $\boldsymbol{u}_h \in \mathbf{W}_h$, $\boldsymbol{v}_h \in \mathbf{Q}_h$ and $\boldsymbol{U} \in W^{2,2}(\mathbb{T}^d;\mathbb{R}^d)$, we have*

$$\int_{\mathbb{T}^d} \Pi_Q \boldsymbol{u}_h \cdot \Delta_x \boldsymbol{U} \,\mathrm{d}x = -\sum_{i=1}^d \sum_{j=1}^d \sum_{\epsilon = D_\sigma | D_{\sigma'} \in \widetilde{\mathcal{E}}_{j,i}} \int_{D_\epsilon} \eth_{\mathcal{B}_{j,i}} u_{j,h} \left(\frac{(\Pi_{\mathcal{E}}^{(i)} \partial_i U_j)_{D_\sigma} + (\Pi_{\mathcal{E}}^{(i)} \partial_i U_j)_{D_{\sigma'}}}{2} \right) \mathrm{d}x, \tag{2.8a}$$

$$\int_{\mathbb{T}^d} \boldsymbol{u}_h \cdot \nabla_x \mathrm{div}_x \boldsymbol{U} \,\mathrm{d}x = -\int_{\mathbb{T}^d} \mathrm{div}_{\mathcal{T}}^{\mathbf{W}} \boldsymbol{u}_h \Pi_\epsilon (\mathrm{div}_x \boldsymbol{U}) \,\mathrm{d}x, \tag{2.8b}$$

$$\int_{\mathbb{T}^d} \boldsymbol{v}_h \cdot \Delta_x \boldsymbol{U} \,\mathrm{d}x = -\int_{\mathbb{T}^d} \nabla_{\mathcal{D}} \boldsymbol{v}_h : \Pi_{\mathcal{E}} \nabla_x \boldsymbol{U} \,\mathrm{d}x, \tag{2.8c}$$

$$\int_{\mathbb{T}^d} \{\!\{ \boldsymbol{v}_h \}\!\} \cdot \nabla_x \mathrm{div}_x \boldsymbol{U} \,\mathrm{d}x = -\int_{\mathbb{T}^d} \Pi_\epsilon \mathrm{div}_x \boldsymbol{U} \; \mathrm{div}_{\mathcal{T}}^{Q} \boldsymbol{v}_h \,\mathrm{d}x. \tag{2.8d}$$

Lemma 2.5. *Let $\boldsymbol{v}_h \in \mathbf{Q}_h$, $\boldsymbol{u}_h \in \mathbf{W}_h$, $\boldsymbol{U} \in W^{2,2}(\mathbb{T}^d;\mathbb{R}^d)$, and $\boldsymbol{\Phi} \in W^{3,2}(\mathbb{T}^d;\mathbb{R}^d)$. Then for any $i, j \in \{1, \ldots, d\}$, we have*

$$\|\Pi_Q \boldsymbol{u}_h - \boldsymbol{u}_h\|_{L^2} \le \frac{h}{2} \|\nabla_{\mathcal{B}} \boldsymbol{u}_h\|_{L^2}, \quad \|\{\!\{ \boldsymbol{v}_h \}\!\} - \boldsymbol{v}_h\|_{L^2} \le \frac{h}{2} \|\nabla_{\mathcal{D}} \boldsymbol{v}_h\|_{L^2}, \tag{2.9a}$$

$$\|\Pi_{\mathcal{E}}^{(i)} \partial_i U_j - \partial_i U_j\|_{L^2} \le h \|\boldsymbol{U}\|_{W^{2,2}} \tag{2.9b}$$

$$\|\mathrm{div}_x \boldsymbol{U} - \Pi_\epsilon \mathrm{div}_x \boldsymbol{U}\|_{L^2} \le h \|\boldsymbol{U}\|_{W^{2,2}}, \quad \|\Pi_\epsilon \mathrm{div}_x \boldsymbol{U} - \Pi_{\mathcal{E}}^{(i)} \mathrm{div}_x \boldsymbol{U}\|_{L^2} \le h \|\boldsymbol{U}\|_{W^{2,2}}. \tag{2.9c}$$

$$\|\nabla_x \mathrm{div}_x \boldsymbol{\Phi} - \nabla_Q \mathrm{div}_h \Pi_Q \boldsymbol{\Phi}\|_{L^2} \le h \|\boldsymbol{\Phi}\|_{W^{3,2}}, \quad \|\Delta_x \boldsymbol{\Phi} - \mathrm{div}_{\mathcal{T}}^{\mathbf{W}} \nabla_{\mathcal{D}} \Pi_Q \boldsymbol{\Phi}\|_{L^2} \le h \|\boldsymbol{\Phi}\|_{W^{3,2}}. \tag{2.9d}$$

2.5 Finite volume and finite difference methods

We proceed by presenting a finite volume and a finite difference numerical method that will be used to approximate the Navier–Stokes system (1.1)–(1.3). Both methods have been already successfully applied in numerical simulations, see, e.g., [12]. In our recent work [11, 12, 23], the convergence was shown for $\gamma > 1$ via the concept of dissipative measure-valued solutions. However, the error analysis was missing for the finite volume method and suboptimal for the finite difference method.

2.5.1 Finite volume method

We introduce the finite volume (FV) method approximating the Navier–Stokes system (1.1)–(1.3).

Definition 2.6 (FV scheme)**.** *Given the initial data* (1.3)*, we set $(\varrho_h^0, \varrho_h^0 \boldsymbol{u}_h^0) = (\Pi_Q \varrho_0, \Pi_Q[\varrho_0 \boldsymbol{u}_0])$. The* FV *approximation $(\varrho_h^n, \boldsymbol{u}_h^n) \in Q_h \times \mathbf{Q}_h$, $n = 1, \ldots, N$, of the Navier–Stokes system* (1.1)–(1.3) *is a solution of the following system of algebraic equations:*

$$\int_{\mathbb{T}^d} D_t \varrho_h^n \phi_h \,\mathrm{d}x - \int_{\mathcal{E}} F_h^\varepsilon[\varrho_h^n, \boldsymbol{u}_h^n] \, [\![\phi_h]\!] \,\mathrm{dS}(x) = 0 \quad \textit{for all} \;\; \phi_h \in Q_h, \tag{2.10a}$$

$$\begin{aligned} &\int_{\mathbb{T}^d} D_t(\varrho_h^n \boldsymbol{u}_h^n) \cdot \boldsymbol{\phi}_h \,\mathrm{d}x - \int_{\mathcal{E}} \mathbf{F}_h^\varepsilon[\varrho_h^n \boldsymbol{u}_h^n, \boldsymbol{u}_h^n] \cdot [\![\boldsymbol{\phi}_h]\!] \,\mathrm{dS}(x) - \int_{\mathbb{T}^d} p_h^n \mathrm{div}_h \boldsymbol{\phi}_h \,\mathrm{d}x \\ &= -\mu \int_{\mathbb{T}^d} \nabla_{\mathcal{D}} \boldsymbol{u}_h^n : \nabla_{\mathcal{D}} \boldsymbol{\phi}_h \,\mathrm{d}x - \nu \int_{\mathbb{T}^d} \mathrm{div}_{\mathcal{T}}^Q \boldsymbol{u}_h^n \, \mathrm{div}_{\mathcal{T}}^Q \boldsymbol{\phi}_h \,\mathrm{d}x \quad \textit{for all} \;\; \boldsymbol{\phi}_h \in \mathbf{Q}_h, \end{aligned} \tag{2.10b}$$

where $\nu = \frac{d-2}{d}\mu + \lambda$.

2.5.2 Finite difference MAC method

We proceed by presenting the finite difference MAC scheme that is based on a staggered grid approach. On the one hand, the discrete density ϱ_h and pressure $p_h = p(\varrho_h)$ are approximated on the primary grid $\mathcal{T}$. On the other hand, the i^{th} component of the velocity field $u_{i,h}$ is approximated on the i^{th} dual grid $\mathcal{D}_i$. The MAC scheme reads as follows.

Definition 2.7 (MAC scheme)**.** *Given the initial data* (1.3)*, we consider* $(\varrho_h^0, \varrho_h^0 \Pi_Q \boldsymbol{u}_h^0) = (\Pi_Q \varrho_0, \Pi_Q[\varrho_0 \boldsymbol{u}_0])$*. The* MAC approximation *of the Navier–Stokes system* (1.1)–(1.3) *is a sequence* $(\varrho_h^n, \boldsymbol{u}_h^n) \in Q_h \times \mathbf{W}_h$*,* $n = 1, 2, \ldots, N$*, which solves the following system of algebraic equations:*

$$\int_{\mathbb{T}^d} D_t \varrho_h^n \phi_h \,\mathrm{d}x - \int_{\mathcal{E}} F_h^{\varepsilon}[\varrho_h^n, \boldsymbol{u}_h^n] \, [\![\phi_h]\!] \,\mathrm{dS}(x) = 0 \quad \text{for all} \quad \phi_h \in Q_h, \tag{2.11a}$$

$$\begin{aligned}
&\int_{\mathbb{T}^d} D_t(\varrho_h^n \Pi_Q \boldsymbol{u}_h^n) \cdot \overline{\boldsymbol{\phi}_h} \,\mathrm{d}x - \int_{\mathcal{E}} \boldsymbol{Up}[\varrho_h^n \Pi_Q \boldsymbol{u}_h^n, \boldsymbol{u}_h^n] \cdot [\![\overline{\boldsymbol{\phi}_h}]\!] \,\mathrm{dS}(x) \\
&+ \mu \int_{\mathbb{T}^d} \nabla_{\mathcal{B}} \boldsymbol{u}_h^n : \nabla_{\mathcal{B}} \boldsymbol{\phi}_h \,\mathrm{d}x + \nu \int_{\mathbb{T}^d} \mathrm{div}_{\mathcal{T}}^{\mathbf{W}} \boldsymbol{u}_h^n \,\mathrm{div}_{\mathcal{T}}^{\mathbf{W}} \boldsymbol{\phi}_h \,\mathrm{d}x - \int_{\mathbb{T}^d} p_h^n \mathrm{div}_{\mathcal{T}}^{\mathbf{W}} \boldsymbol{\phi}_h \,\mathrm{d}x \\
&= -h^{\varepsilon+1} \sum_{i=1}^d \sum_{j=1}^d \int_{\mathbb{T}^d} \{\!\!\{ \overline{u_{i,h}}^n \}\!\!\}^{(j)} (\eth_{\mathcal{D}_j} \varrho_h) \eth_{\mathcal{D}_j} \overline{\phi_{i,h}} \,\mathrm{d}x, \ \text{for all } \boldsymbol{\phi}_h = (\phi_{1,h}, \ldots, \phi_{d,h}) \in \mathbf{W}_h,
\end{aligned} \tag{2.11b}$$

where $\nu = \frac{d-2}{d}\mu + \lambda$.

In what follows, we will denote by $\varrho_h(t), \boldsymbol{u}_h(t)$ the piecewise constant approximations of $\varrho_h^n, \boldsymbol{u}_h^n$, $n = 0, 1, \ldots, N$ on the time interval $[0, T]$, see Section 2.1. We note that both methods, the FV method (2.10) as well as the MAC method (2.11), preserve the positivity of density and conserve the mass

$$\varrho_h(t) > 0 \ \text{ and } \int_{\mathbb{T}^d} \varrho_h(t) \,\mathrm{d}x = M \qquad \text{for all} \ \ t \in (0, T), \tag{2.12}$$

where $M := \int_{\mathbb{T}^d} \varrho_0 \,\mathrm{d}x$ denotes the fluid mass, see e.g. [12, Lemma 11.2].

2.6 Energy stability

The essential feature of any numerical scheme is its stability. We now recall the energy stability of both numerical methods introduced above, see [12, Theorem 11.1 and 14.1]

Lemma 2.8 (Energy estimates)**.** *Let* $(\varrho_h, \boldsymbol{u}_h)$ *be a numerical solution obtained either by the FV scheme* (2.10) *or by the MAC scheme* (2.11) *with* $\gamma > 1$*. Then for all* $\tau \in (0, T)$*, it holds*

$$\int_{\mathbb{T}^d} \left(\frac{1}{2} \varrho_h \left| \Pi_Q \boldsymbol{u}_h \right|^2 + P(\varrho_h) \right)(\tau) \,\mathrm{d}x + \mu \int_0^\tau \int_{\mathbb{T}^d} \left| \nabla_h \boldsymbol{u}_h \right|^2 \mathrm{d}x\mathrm{d}t + \nu \int_0^\tau \int_{\mathbb{T}^d} \left| \mathrm{div}_h \boldsymbol{u}_h \right|^2 \,\mathrm{d}x\mathrm{d}t \le E_0, \tag{2.13}$$

where $E_0 = \int_{\mathbb{T}^d} \left(\frac{1}{2} \varrho_0 |\boldsymbol{u}_0|^2 + P(\varrho_0) \right) \mathrm{d}x$ *is the initial energy and*

$$(\nabla_h \boldsymbol{u}_h, \mathrm{div}_h \boldsymbol{u}_h) = \begin{cases} (\nabla_{\mathcal{D}} \boldsymbol{u}_h, \mathrm{div}_{\mathcal{T}}^{Q} \boldsymbol{u}_h) & \text{for } \boldsymbol{u}_h \in \mathbf{Q}_h \text{ in the FV scheme;} \\ (\nabla_{\mathcal{B}} \boldsymbol{u}_h, \mathrm{div}_{\mathcal{T}}^{\mathbf{W}} \boldsymbol{u}_h) & \text{for } \boldsymbol{u}_h \in \mathbf{W}_h \text{ in the MAC scheme.} \end{cases}$$

Moreover, there exists $c > 0$ *which may depend on the fluid mass* M *and the initial energy* E_0 *but is independent of the parameters* h *and* Δt *such that*

$$\| \varrho_h \left| \Pi_Q \boldsymbol{u}_h \right|^2 \|_{L^\infty L^1} \le c, \quad \| \varrho_h \|_{L^\infty L^\gamma} \le c, \quad \| \varrho_h \Pi_Q \boldsymbol{u}_h \|_{L^\infty L^{\frac{2\gamma}{\gamma+1}}} \le c, \tag{2.14a}$$

$$\| \mathrm{div}_h \boldsymbol{u}_h \|_{L^2 L^2} \le c, \quad \| \nabla_h \boldsymbol{u}_h \|_{L^2 L^2} \le c, \quad \| \boldsymbol{u}_h \|_{L^2 L^6} \le c. \tag{2.14b}$$

2.7 Consistency formulation

The next important ingredient of our approach is the consistency formulation of the numerical scheme.

Lemma 2.9 (Consistency formulation)**.** *Let* $(\varrho_h, \boldsymbol{u}_h)$ *be either a solution of the FV scheme* (2.10) *or the MAC scheme* (2.11) *with* $\Delta t \approx h \in (0,1)$, $\gamma > 1$ *and* $\varepsilon > -1$.

Then for all $\tau \in (0,T)$, $\phi \in L^\infty(0,T; W^{2,\infty}(\mathbb{T}^d))$, $\partial_t^2 \phi \in L^\infty((0,T)\times\mathbb{T}^d)$ *and* $\boldsymbol{\phi} \in L^\infty(0,T; W^{2,\infty}(\mathbb{T}^d;\mathbb{R}^d))$, $\partial_t^2 \boldsymbol{\phi} \in L^\infty((0,T)\times\mathbb{T}^d;\mathbb{R}^d)$ *there holds*

$$\left[\int_{\mathbb{T}^d} \varrho_h \phi \,\mathrm{d}x\right]_{t=0}^{\tau} = \int_0^\tau \int_{\mathbb{T}^d} \left(\varrho_h \partial_t \phi + \varrho_h \Pi_Q \boldsymbol{u}_h \cdot \nabla_x \phi\right) \mathrm{d}x\mathrm{d}t \; + e_\varrho(\tau, \Delta t, h, \phi), \tag{2.15a}$$

$$\begin{aligned}\left[\int_{\mathbb{T}^d} \varrho_h \Pi_Q \boldsymbol{u}_h \cdot \boldsymbol{\phi} \,\mathrm{d}x\right]_{t=0}^{\tau} &= \int_0^\tau \int_{\mathbb{T}^d} \left(\varrho_h \Pi_Q \boldsymbol{u}_h \cdot \partial_t \boldsymbol{\phi} + \varrho_h \Pi_Q \boldsymbol{u}_h \otimes \Pi_Q \boldsymbol{u}_h : \nabla_x \boldsymbol{\phi} + p_h \mathrm{div}_x \boldsymbol{\phi}\right) \mathrm{d}x\mathrm{d}t \\ &- \mu \int_0^\tau \int_{\mathbb{T}^d} \nabla_h \boldsymbol{u}_h : \nabla_x \boldsymbol{\phi} \,\mathrm{d}x\mathrm{d}t - \nu \int_0^\tau \int_{\mathbb{T}^d} \mathrm{div}_h \boldsymbol{u}_h \, \mathrm{div}_x \boldsymbol{\phi} \,\mathrm{d}x\mathrm{d}t \; + e_{\boldsymbol{m}}(\tau, \Delta t, h, \boldsymbol{\phi}),\end{aligned} \tag{2.15b}$$

where the consistency errors are bounded as follows:

$$\begin{aligned} |e_\varrho(\tau, \Delta t, h, \phi)| &\leq \begin{cases} C_\varrho\big(\Delta t + h + h^{1+\varepsilon} + h^{1+\beta_D}\big) & \text{for the FV method} \\ C_\varrho\big(\Delta t + h^{1+\varepsilon} + h^{1+\beta_D}\big) & \text{for the MAC method} \end{cases} \\ |e_{\boldsymbol{m}}(\tau, \Delta t, h, \boldsymbol{\phi})| &\leq \begin{cases} C_{\boldsymbol{m}}\big(\sqrt{\Delta t} + h + h^{1+\varepsilon} + h^{1+\beta_M}\big) & \text{for the FV method} \\ C_{\boldsymbol{m}}\big(\sqrt{\Delta t} + h + h^{1+\varepsilon} + h^{1+\beta_M} + h^{1+\varepsilon+\beta_D}\big) & \text{for the MAC method.} \end{cases} \end{aligned} \tag{2.15c}$$

Here, the constant C_ϱ *depends on*

$$\text{the initial energy } E_0, T, \text{ and } \|\phi\|_{L^\infty(0,T;W^{2,\infty}(\mathbb{T}^d))}, \; \|\partial_t^2 \phi\|_{L^\infty((0,T)\times\mathbb{T}^d)},$$

and $C_{\boldsymbol{m}}$ *depends on*

$$E_0, T, \|\boldsymbol{\phi}\|_{L^\infty(0,T;W^{2,\infty}(\mathbb{T}^d;\mathbb{R}^d))}, \; \|\partial_t^2 \boldsymbol{\phi}\|_{L^\infty((0,T)\times\mathbb{T}^d;\mathbb{R}^d)}.$$

Further, the exponents β_D *and* β_M *are given by*

$$\beta_D = \begin{cases} \max\left\{-\frac{3\varepsilon+3+d}{6\gamma}, \frac{\gamma-2}{2\gamma} d\right\}, & \text{if } \gamma \in (1,2), \\ 0, & \text{if } \gamma \geq 2, \end{cases} \qquad \beta_M = \begin{cases} -\frac{3\varepsilon+3+d}{6\gamma}, & \text{if } \gamma \in (1,2), \\ \frac{\gamma-3}{3\gamma} d, & \text{if } \gamma \in [2,3), \\ 0, & \text{if } \gamma \geq 3 \text{ for } d = 3, \\ 0, & \text{if } \gamma > 2 \text{ for } d = 2. \end{cases} \tag{2.15d}$$

Remark 1. *Consistency formulation for the FV and MAC method was introduced in [12, Theorem 11.2] and [12, Theorem 14.2], respectively. Instead of an abstract consistency error identified in [12], Lemma 2.9 provides an explicit bound in terms of the numerical step and regularity of the associated test function. Moreover, we improve the result of [12] by requiring less regularity of the test functions.*

Proof of Lemma 2.9. The consistency errors arising from the time derivative term can be evaluated in the following way. First, by a direct calculation, we obtain

$$
\begin{aligned}
&\int_0^{t^{n+1}}\int_{\mathbb{T}^d} D_t r_h(t)\Pi_Q\varphi(t)\,\mathrm{d}x\mathrm{dt} = \int_0^{t^{n+1}}\int_{\mathbb{T}^d} \frac{r_h(t)-r_h(t-\Delta t)}{\Delta t}\varphi(t)\,\mathrm{d}x\mathrm{dt}\\
&= \frac{1}{\Delta t}\int_0^{t^{n+1}}\int_{\mathbb{T}^d} r_h(t)\varphi(t)\,\mathrm{d}x\mathrm{dt} - \frac{1}{\Delta t}\int_{-\Delta t}^{t^n}\int_{\mathbb{T}^d} r_h(t)\varphi(t+\Delta t)\,\mathrm{d}x\mathrm{dt}\\
&= -\int_0^{t^{n+1}}\int_{\mathbb{T}^d} r_h(t)D_t\varphi(t+\Delta t)\,\mathrm{d}x\mathrm{dt} + \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\int_{\mathbb{T}^d} r_h(t)\varphi(t+\Delta t)\,\mathrm{d}x\mathrm{dt}\\
&\quad - \frac{1}{\Delta t}\int_{-\Delta t}^{0}\int_{\mathbb{T}^d} r_h(t)\varphi(t+\Delta t)\,\mathrm{d}x\mathrm{dt}\\
&= -\int_0^{t^{n+1}}\int_{\mathbb{T}^d} r_h(t)D_t\varphi(t+\Delta t)\,\mathrm{d}x\mathrm{dt} + \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\int_{\mathbb{T}^d} r_h(t)\varphi(t)\,\mathrm{d}x\mathrm{dt} - \frac{1}{\Delta t}\int_0^{\Delta t}\int_{\mathbb{T}^d} r_h^0\varphi(t)\,\mathrm{d}x\mathrm{dt}\\
&= -\int_0^{t^{n+1}}\int_{\mathbb{T}^d} r_h(t)\partial_t\varphi(t)\,\mathrm{d}x\mathrm{dt} + \int_{\mathbb{T}^d} \underbrace{r_h(\tau)}_{=r_h^n\ \forall\tau\in[t^n,t^{n+1})}\varphi(\tau)\,\mathrm{d}x - \int_{\mathbb{T}^d} r_h^0\varphi(0)\,\mathrm{d}x + I_1 + I_2 + I_3,
\end{aligned}
\tag{2.16}
$$

for any $\tau\in[t_n,t_{n+1})$, $n=1,\ldots,N_T$, where

$$
\begin{aligned}
I_1 &= \int_{\mathbb{T}^d} r_h^0\frac{1}{\Delta t}\int_0^{\Delta t}(\varphi(0)-\varphi(t))\,\mathrm{dt}\,\mathrm{d}x \lesssim \Delta t\|\partial_t\varphi\|_{L^\infty L^\infty}\|r_h^0\|_{L^1},\\
I_2 &= \int_{\mathbb{T}^d}\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\big(r_h(t)\varphi(t+\Delta t)-r_h(\tau)\varphi(\tau)\big)\mathrm{dt}\,\mathrm{d}x\\
&= \int_{\mathbb{T}^d}\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} r_h^n\big(\varphi(t+\Delta t)-\varphi(\tau)\big)\mathrm{dt}\,\mathrm{d}x \lesssim \Delta t\|r_h^n\|_{L^1}\|\partial_t\varphi\|_{L^\infty L^\infty},\\
I_3 &= \int_0^{t^{n+1}}\int_{\mathbb{T}^d} r_h(t)\left(\partial_t\varphi(t)-D_t\varphi(t+\Delta t)\right)\,\mathrm{d}x\mathrm{dt} = \int_{\mathbb{T}^d}\sum_{k=0}^{n}\int_{t^k}^{t^{k+1}} r_h(t)\big(\partial_t\varphi(t)-D_t\varphi(t+\Delta t)\big)\mathrm{dt}\,\mathrm{d}x\\
&\le \Delta t\|\partial_t^2\varphi\|_{L^\infty L^\infty}\|r_h\|_{L^\infty L^1}.
\end{aligned}
$$

Collecting the above estimates we obtain from (2.16) that

$$
\left[\int_{\mathbb{T}^d} r_h\varphi\,\mathrm{d}x\right]_0^\tau - \int_0^{t^{n+1}}\int_{\mathbb{T}^d}\left(D_t r_h(t)\Pi_Q\varphi(t) + r_h(t)\partial_t\varphi(t)\right)\,\mathrm{d}x\mathrm{dt} \le \Delta t\|\partial_t^2\varphi\|_{L^\infty L^\infty}\|r_h\|_{L^\infty L^1},
\tag{2.17}
$$

whenever $\tau\in[t_n,t_{n+1})$, where r_h stands for ϱ_h or $\varrho_h\boldsymbol{u}_h$.

Analogously as in the proofs of [12, Theorem 11.2] and [12, Theorem 14.2], we obtain

$$
\left[\int_{\mathbb{T}^d}\varrho_h\phi\,\mathrm{d}x\right]_{t=0}^\tau = \int_0^{t^{n+1}}\int_{\mathbb{T}^d}\left(\varrho_h\partial_t\phi + \varrho_h\Pi_Q\boldsymbol{u}_h\cdot\nabla_x\phi\right)\,\mathrm{d}x\mathrm{dt} \ + e_\varrho(\tau,\Delta t,h,\phi),
\tag{2.18a}
$$

$$
\begin{aligned}
\left[\int_{\mathbb{T}^d}\varrho_h\Pi_Q\boldsymbol{u}_h\cdot\boldsymbol{\phi}\,\mathrm{d}x\right]_{t=0}^\tau &= \int_0^{t^{n+1}}\int_{\mathbb{T}^d}\left(\varrho_h\Pi_Q\boldsymbol{u}_h\cdot\partial_t\boldsymbol{\phi} + \varrho_h\Pi_Q\boldsymbol{u}_h\otimes\Pi_Q\boldsymbol{u}_h:\nabla_x\boldsymbol{\phi}\right)\,\mathrm{d}x\mathrm{dt}\\
&\quad + \int_0^{t^{n+1}}\int_{\mathbb{T}^d}\big(p_h\mathbb{I} - \mu\nabla_h\boldsymbol{u}_h - \nu\mathrm{div}_h\boldsymbol{u}_h\big):\nabla_x\boldsymbol{\phi}\,\mathrm{d}x\mathrm{dt} \ + e^*_{\boldsymbol{m}}(\tau,\Delta t,h,\phi),
\end{aligned}
\tag{2.18b}
$$

where $e^*_{\boldsymbol{m}}$ is controlled by

$$|e^*_{\boldsymbol{m}}(\tau,\Delta t,h,\boldsymbol{\phi})| \leq \begin{cases} C_{\boldsymbol{m}}\big(\Delta t + h + h^{1+\varepsilon} + h^{1+\beta_M}\big) & \text{for the FV method ,}\\ C_{\boldsymbol{m}}\big(\Delta t + h + h^{1+\varepsilon} + h^{1+\beta_M} + h^{1+\varepsilon+\beta_D}\big) & \text{for the MAC method.}\end{cases}$$

In order to derive (2.15a) it suffices to realize that the time integral from τ to t^{n+1} at the right hand side of (2.18a) is of order $\mathcal{O}(\Delta t)$. Indeed

$$\begin{aligned} &\int_\tau^{t^{n+1}}\int_{\mathbb{T}^d}\Big(\varrho_h\partial_t\phi + \varrho_h\Pi_Q\boldsymbol{u}_h\cdot\nabla_x\phi\Big)\,\mathrm{d}x\mathrm{dt}\\ &\leq \big(\|\partial_t\phi\|_{L^\infty L^\infty}\|\varrho_h^n\|_{L^\infty L^1} + \|\nabla_x\phi\|_{L^\infty L^\infty}\|\varrho_h^n\Pi_Q\boldsymbol{u}_h^n\|_{L^\infty L^1}\big)\int_\tau^{t^{n+1}} 1\mathrm{dt} \lesssim \Delta t. \end{aligned} \tag{2.19}$$

Combining (2.18a) and (2.19) yields (2.15a).

Similarly, to get (2.15b) we need the following estimate

$$\begin{aligned} &\int_\tau^{t^{n+1}}\int_{\mathbb{T}^d}\Big(\varrho_h\Pi_Q\boldsymbol{u}_h\cdot\partial_t\boldsymbol{\phi} + \big(\varrho_h\Pi_Q\boldsymbol{u}_h\otimes\Pi_Q\boldsymbol{u}_h + p_h\mathbb{I} - \mu\nabla_h\boldsymbol{u}_h - \nu\mathrm{div}_h\boldsymbol{u}_h\big):\nabla_x\boldsymbol{\phi}\Big)\,\mathrm{d}x\mathrm{dt}\\ &\leq \int_\tau^{t^{n+1}}\|\varrho_h^n\Pi_Q\boldsymbol{u}_h^n\|_{L^1(\mathbb{T}^d)}\|\partial_t\boldsymbol{\phi}\|_{L^\infty(\mathbb{T}^d)}\mathrm{dt} + \int_\tau^{t^{n+1}}\|\varrho_h^n\Pi_Q\boldsymbol{u}_h^n\otimes\Pi_Q\boldsymbol{u}_h^n + p_h^n\mathbb{I}\|_{L^1(\mathbb{T}^d)}\|\nabla_x\boldsymbol{\phi}\|_{L^\infty(\mathbb{T}^d)}\mathrm{dt}\\ &+\int_\tau^{t^{n+1}}\|\mu\nabla_h\boldsymbol{u}_h^n + \nu\mathrm{div}_h\boldsymbol{u}_h^n\|_{L^1(\mathbb{T}^d)}\|\nabla_x\boldsymbol{\phi}\|_{L^\infty(\mathbb{T}^d)}\mathrm{dt}\\ &\leq \Delta t\|\varrho_h^n\Pi_Q\boldsymbol{u}_h^n\|_{L^\infty L^1}\|\partial_t\boldsymbol{\phi}\|_{L^\infty L^\infty} + \Delta t\|\varrho_h^n\Pi_Q\boldsymbol{u}_h^n\otimes\Pi_Q\boldsymbol{u}_h^n + p_h^n\ \|_{L^\infty L^1}\|\nabla_x\boldsymbol{\phi}\|_{L^\infty L^\infty}\\ &+\|\nabla_x\boldsymbol{\phi}\|_{L^\infty L^\infty}\|\mu\nabla_h\boldsymbol{u}_h^n + \nu\mathrm{div}_h\boldsymbol{u}_h^n\|_{L^2L^1}\left(\int_\tau^{t^{n+1}}1^2\mathrm{dt}\right)^{1/2} \lesssim \sqrt{\Delta t}. \end{aligned}$$

Substituting the above estimate into (2.18b) we obtain (2.15b), which completes the proof. □

Lemma 2.10 (Consistency formulation for a bounded numerical solution)**.** *Let the assumptions of Lemma 2.9 hold. Moreover, let ϱ_h and $\boldsymbol{u}_h$ be uniformly bounded, i.e., there exist positive constants $\overline{\varrho}$ and $\overline{u}$ such that*

$$\varrho_h \leq \overline{\varrho} \text{ and } |\boldsymbol{u}_h| \leq \overline{u}. \tag{2.20}$$

Then for all $\tau\in(0,T)$, $\phi\in L^\infty(0,T;W^{2,\infty}(\mathbb{T}^d))$, $\partial_t^2\phi\in L^\infty((0,T)\times\mathbb{T}^d)$ and $\boldsymbol{\phi}\in L^\infty(0,T;W^{2,\infty}(\mathbb{T}^d;\mathbb{R}^d))\cap L^2(0,T;W^{3,2}(\mathbb{T}^d;\mathbb{R}^d))$, $\partial_t^2\boldsymbol{\phi}\in L^\infty((0,T)\times\mathbb{T}^d;\mathbb{R}^d)$, there holds

$$\left[\int_{\mathbb{T}^d}\varrho_h\phi\,\mathrm{d}x\right]_{t=0}^\tau = \int_0^\tau\int_{\mathbb{T}^d}\left(\varrho_h\partial_t\phi + \varrho_h\Pi_Q\boldsymbol{u}_h\cdot\nabla_x\phi\right)\mathrm{d}x\mathrm{dt} + e_\varrho(\tau,\Delta t,h,\phi), \tag{2.21a}$$

$$\begin{aligned}\left[\int_{\mathbb{T}^d}\varrho_h\Pi_Q\boldsymbol{u}_h\cdot\boldsymbol{\phi}\,\mathrm{d}x\right]_{t=0}^\tau &= \int_0^\tau\int_{\mathbb{T}^d}\left(\varrho_h\Pi_Q\boldsymbol{u}_h\cdot\partial_t\boldsymbol{\phi} + \varrho_h\Pi_Q\boldsymbol{u}_h\otimes\Pi_Q\boldsymbol{u}_h:\nabla_x\boldsymbol{\phi} + p_h\mathrm{div}_x\boldsymbol{\phi}\right)\mathrm{d}x\mathrm{dt}\\ &+\int_0^\tau\int_{\mathbb{T}^d}\boldsymbol{u}_h\cdot\left(\mu\Delta_x\boldsymbol{\phi} + \nu\nabla_x\mathrm{div}_x\boldsymbol{\phi}\right)\mathrm{d}x\mathrm{dt} + e_{\boldsymbol{m}}(\tau,\Delta t,h,\boldsymbol{\phi}),\end{aligned} \tag{2.21b}$$

where the consistency errors can be bounded as follows

$$|e_\varrho(\tau,\Delta t,h,\phi)| \leq C_\varrho(\Delta t + h), \quad |e_{\boldsymbol{m}}(\tau,\Delta t,h,\boldsymbol{\phi})| \leq C_{\boldsymbol{m}}(\Delta t + h) \tag{2.21c}$$

Here, the constant C_ϱ depends on

$$\overline{\varrho}, \overline{u}, E_0, T, \|\phi\|_{L^\infty(0,T;W^{2,\infty}(\mathbb{T}^d))}, \ \|\partial_t^2 \phi\|_{L^\infty((0,T)\times\mathbb{T}^d)},$$

and $C_{\boldsymbol{m}}$ depends on

$$\overline{\varrho}, \overline{u}, E_0, T, \|\boldsymbol{\phi}\|_{L^\infty(0,T;W^{2,\infty}(\mathbb{T}^d;\mathbb{R}^d))}, \ \|\boldsymbol{\phi}\|_{L^2(0,T;W^{3,2}(\mathbb{T}^d;\mathbb{R}^d))}, \|\partial_t^2 \boldsymbol{\phi}\|_{L^\infty((0,T)\times\mathbb{T}^d;\mathbb{R}^d)}.$$

Proof. We will present the proof for the FV method, the proof for the MAC method is analogous. First, we denote the errors of the inviscid fluxes as

$$e_1 = \int_0^{t^{n+1}} \int_{\mathbb{T}^d} \varrho_h \Pi_Q \boldsymbol{u}_h \cdot \nabla_x \phi \,\mathrm{d}x\mathrm{dt} - \int_0^{t^{n+1}} \int_{\mathcal{E}} F_h^\varepsilon[\varrho_h, \boldsymbol{u}_h] \, [\![\Pi_Q \phi]\!] \ \mathrm{dS}(x)\mathrm{dt}, \tag{2.22}$$

$$\begin{aligned} e_2 = &\int_0^{t^{n+1}} \int_{\mathbb{T}^d} \varrho_h \Pi_Q \boldsymbol{u}_h \otimes \Pi_Q \boldsymbol{u}_h : \nabla_x \boldsymbol{\phi} \,\mathrm{d}x\mathrm{dt} - \int_0^{t^{n+1}} \int_{\mathcal{E}} \mathbf{F}_h^\varepsilon[\varrho_h \boldsymbol{u}_h, \boldsymbol{u}_h] \cdot [\![\Pi_Q \boldsymbol{\phi}]\!] \ \mathrm{dS}(x)\mathrm{dt} \\ &+ \int_0^{t^{n+1}} \int_{\mathbb{T}^d} p_h \mathrm{div}_x \boldsymbol{\phi} - p_h \mathrm{div}_h \Pi_Q \boldsymbol{\phi} \,\mathrm{d}x\mathrm{dt}. \end{aligned} \tag{2.23}$$

Analogously as in the proof of [12, Theorem 11.3] we get

$$|e_1| \leq c(\|\phi\|_{L^\infty W^{2,\infty}}) h \|\varrho_h\|_{L^2L^2} \quad \text{and} \quad |e_2| \leq c(\|\boldsymbol{\phi}\|_{L^\infty W^{2,\infty}}) h \|\varrho_h \boldsymbol{u}_h\|_{L^2L^2}.$$

In view of assumption (2.20) the errors e_1 and e_2 are controlled by

$$\begin{aligned} |e_1| &\leq c(\|\phi\|_{L^\infty W^{2,\infty}}) h \|\varrho_h\|_{L^2L^2} \leq c(\|\phi\|_{L^\infty W^{2,\infty}}, \overline{\varrho}) h, \\ |e_2| &\leq c(\|\boldsymbol{\phi}\|_{L^\infty W^{2,\infty}}) h \|\varrho_h \boldsymbol{u}_h\|_{L^2L^2} \leq c(\|\boldsymbol{\phi}\|_{L^\infty W^{2,\infty}}, \overline{\varrho}, \overline{u}) h. \end{aligned} \tag{2.24}$$

Now, summing up (2.22) and (2.17) with $r_h = \varrho_h$, and recalling the estimates (2.19) and (2.24) implies (2.21a). Moreover, summing up (2.23) and (2.17) with $r_h = \varrho_h \boldsymbol{u}_h$ we get

$$\begin{aligned} \left[\int_{\mathbb{T}^d} \varrho_h \boldsymbol{u}_h \cdot \boldsymbol{\phi} \,\mathrm{d}x\right]_0^\tau = &\int_0^\tau \int_{\mathbb{T}^d} \varrho_h \boldsymbol{u}_h \cdot \partial_t \boldsymbol{\phi} + \big(\varrho_h \boldsymbol{u}_h \otimes \boldsymbol{u}_h + p_h \mathbb{I}\big) : \nabla_x \boldsymbol{\phi} + \boldsymbol{u}_h \cdot (\mu \Delta_x \boldsymbol{\phi} + \nu \nabla_x \mathrm{div}_x \boldsymbol{\phi}) \,\mathrm{d}x\mathrm{dt} \\ &+ e_2 + e_3 + e_4, \end{aligned} \tag{2.25}$$

where e_2 is given in (2.23). The error terms e_3 and e_4 can be estimated in the following way

$$\begin{aligned} |e_3| &= \left| - \int_0^{t^{n+1}} \int_{\mathbb{T}^d} \boldsymbol{u}_h \cdot (\mu \Delta_x \boldsymbol{\phi} + \nu \nabla_x \mathrm{div}_x \boldsymbol{\phi}) + \big(\mu \nabla_{\mathcal{D}} \boldsymbol{u}_h : \nabla_{\mathcal{D}} \Pi_Q \boldsymbol{\phi} + \nu \mathrm{div}_h \boldsymbol{u}_h \mathrm{div}_h \Pi_Q \boldsymbol{\phi}\big) \,\mathrm{d}x \right| \\ &= \left| \int_0^{t^{n+1}} \int_{\mathbb{T}^d} \mu \boldsymbol{u}_h \cdot (\mathrm{div}_{\mathcal{T}}^{\mathbf{W}} \nabla_{\mathcal{D}} \Pi_Q \boldsymbol{\phi} - \Delta_x \boldsymbol{\phi}) + \nu \boldsymbol{u}_h \cdot (\nabla_Q \mathrm{div}_h \Pi_Q \boldsymbol{\phi} - \nabla_x \mathrm{div}_x \boldsymbol{\phi}) \,\mathrm{d}x \right| \\ &\leq c(\|\boldsymbol{\phi}\|_{L^2 W^{3,2}}, \overline{u}) h, \\ |e_4| &= \left| \int_\tau^{t^{n+1}} \int_{\mathbb{T}^d} \varrho_h \Pi_Q \boldsymbol{u}_h \cdot \partial_t \boldsymbol{\phi} + \big(\varrho_h \Pi_Q \boldsymbol{u}_h \otimes \Pi_Q \boldsymbol{u}_h + p_h \mathbb{I}\big) : \nabla_x \boldsymbol{\phi} + \boldsymbol{u}_h \cdot (\mu \Delta_x \boldsymbol{\phi} + \nu \nabla_x \mathrm{div}_x \boldsymbol{\phi}) \,\mathrm{d}x\mathrm{dt} \right| \\ &\leq \Delta t \|\boldsymbol{\phi}\|_{C^1} \left(\|\varrho_h \Pi_Q \boldsymbol{u}_h\|_{L^\infty L^1} + \|\varrho_h \,|\Pi_Q \boldsymbol{u}_h|^2\|_{L^\infty L^1} + \|p_h\|_{L^\infty L^1}\right) + \overline{u} \|\boldsymbol{\phi}\|_{L^\infty W^{2,\infty}} \int_\tau^{t^{n+1}} \mathrm{dt} \\ &\leq c(\|\boldsymbol{\phi}\|_{L^\infty W^{2,\infty}}, \|\boldsymbol{\phi}\|_{C^1}, \overline{\varrho}, \overline{u}) \Delta t. \end{aligned}$$

Consequently, collecting the estimates of e_2, e_3 and e_4 we observe that (2.21b) follows from (2.25), which completes the proof. □

3 Error estimates

This section is the heart of the paper. We prove the main result – the convergence rates for the FV (2.10) and MAC (2.11) schemes. If, in addition, the numerical solutions are uniformly bounded, the convergence rates can be improved to the first order.

Theorem 3.1 (Convergence rates)**.** *Let $\gamma > 1$ and the initial data $(\varrho_0, \boldsymbol{u}_0)$ satisfy*

$$\varrho_0 \in W^{k,2}(\mathbb{T}^d),\ \varrho_0 > 0 \text{ in } \mathbb{T}^d, \qquad \boldsymbol{u}_0 \in W^{k,2}(\mathbb{T}^d;\mathbb{R}^d), \quad k \geq 6.$$

Suppose that the Navier–Stokes system (1.1) *admits a classical solution $(\varrho, \boldsymbol{u})$ defined on $[0,T] \times \mathbb{T}^d$, with the initial data $(\varrho_0, \boldsymbol{u}_0)$. Further, let $(\varrho_h, \boldsymbol{u}_h)$ be a numerical solution obtained either by the FV scheme* (2.10) *or by the MAC scheme* (2.11) *emanating from the projected initial data $(\varrho_h^0, \boldsymbol{u}_h^0)$.*

Then there exists a positive number

$$c = c(T, \|(\varrho_0, \boldsymbol{u}_0)\|_{W^{k,2}(\mathbb{T}^d;\mathbb{R}^{d+1})}, \inf \varrho_0, \|(\varrho, \boldsymbol{u})\|_{C([0,T]\times\mathbb{T}^d;\mathbb{R}^{d+1})})$$

such that

$$\begin{aligned}&\sup_{0\leq t\leq T} \mathfrak{E}(\varrho_h, \boldsymbol{u}_h|\varrho, \boldsymbol{u}) + \mu \int_0^T \int_{\mathbb{T}^d} |\nabla_h \boldsymbol{u}_h - \nabla_x \boldsymbol{u}|^2 \,\mathrm{d}x\mathrm{d}t + \nu \int_0^T \int_{\mathbb{T}^d} |\mathrm{div}_h \boldsymbol{u}_h - \mathrm{div}_x \boldsymbol{u}|^2 \,\mathrm{d}x\mathrm{d}t \\ &\leq c(h^A + \sqrt{\Delta t}),\end{aligned} \tag{3.1}$$

$$\begin{aligned}&\|\varrho_h - \varrho\|_{L^\infty L^\gamma} + \|\varrho_h \boldsymbol{u}_h - \varrho \boldsymbol{u}\|_{L^\infty L^{\frac{2\gamma}{\gamma+1}}} \lesssim c(\sqrt{\Delta t} + h)^{1/2} + c(\sqrt{\Delta t} + h^A)^{1/\gamma} && \text{for } \gamma \leq 2, \\ &\|\varrho_h - \varrho\|_{L^\infty L^2} + \|\varrho_h \boldsymbol{u}_h - \varrho \boldsymbol{u}\|_{L^\infty L^{\frac{2\gamma}{\gamma+1}}} \lesssim c(\sqrt{\Delta t} + h^A)^{1/2} && \text{for } \gamma > 2,\end{aligned} \tag{3.2}$$

and

$$\|\boldsymbol{u}_h - \boldsymbol{u}\|_{L^2L^2} \lesssim c(\sqrt{\Delta t} + h^A)^{1/2}. \tag{3.3}$$

The convergence rate A reads

$$A = \begin{cases} A_{FV} := \min\{1, 1+\varepsilon, 1+\beta_D, 1+\beta_M\} & \text{for the FV method,} \\ A_{MAC} := \min\{1, 1+\varepsilon, 1+\beta_D, 1+\beta_M, 1+\varepsilon+\beta_D\} & \text{for the MAC method.} \end{cases} \tag{3.4}$$

Here the constants β_D and β_M are given in (2.15d).

Remark 2. *Let us discuss the obtained convergence rate $\mathcal{O}(h^A)$ for the choice $\Delta t = h$ and different values of $\gamma > 1$, $d = 2, 3$.*

- *For the case $d = 2$, we obtain the following convergence rate A:*
  - *Let $\gamma \geq 2$. Then for any $\varepsilon \geq 0$ both numerical methods have the first order convergence rate, i.e. $A = 1$.*
  - *Let $\gamma \in (1, 2)$. The convergence rates are different for the FV and MAC schemes.*
    - *$A_{FV} = \min\left\{1 - \frac{5+3\varepsilon}{6\gamma}, 1, 1+\varepsilon\right\}$. Choosing the optimal value of ε, $\varepsilon = -\frac{5}{3+6\gamma} \in (-\frac{5}{9}, -\frac{1}{3})$, the convergence rate $A_{FV} = 1+\varepsilon$ varies between $\frac{4}{9}$ for $\gamma \searrow 1$ and $\frac{2}{3}$ for $\gamma \nearrow 2$.*
    - *$A_{MAC} = \min\left\{1 - \frac{5+3\varepsilon}{6\gamma}, 1, 1+\varepsilon, 1+\varepsilon - \frac{5+3\varepsilon}{6\gamma}\right\}$ reaches its maximum value $\frac{6\gamma-5}{6\gamma} > 0$ at $\varepsilon = 0$. Thus, the convergence rate varies between $\frac{1}{6}$ for $\gamma \searrow 1$ and $\frac{7}{12}$ for $\gamma \nearrow 2$.*

- *For the case $d = 3$, we obtain the following convergence rate A:*
  - *Let $\gamma \geq 3$. Then for any $\varepsilon \geq 0$ both methods have first order convergence rates, i.e. $A = 1$.*
  - *Let $\gamma \in [2, 3)$. Then for any $\varepsilon \geq \frac{2\gamma-3}{\gamma}$ we have $A = \frac{2\gamma-3}{\gamma}$ and the convergence rate varies between $\frac{1}{2}$ for $\gamma = 2$ and 1 for $\gamma \nearrow 3$.*
  - *Let $\gamma \in (1, 2)$.*
    - * $A_{FV} = \min\left\{1 - \frac{2+\varepsilon}{2\gamma}, 1, 1+\varepsilon\right\}$. Choosing an optimal value of ε, $\varepsilon = -\frac{2}{1+2\gamma} \in (-\frac{2}{3}, -\frac{2}{5})$, $A_{FV} = 1 + \varepsilon$ and varies between $\frac{1}{3}$ for $\gamma \searrow 1$ and $\frac{3}{5}$ for $\gamma \nearrow 2$.*
    - * $A_{MAC} = \min\left\{1 - \frac{2+\varepsilon}{2\gamma}, 1, 1+\varepsilon, 1+\varepsilon - \frac{2+\varepsilon}{2\gamma}\right\}$ reaches its maximum value $\frac{\gamma-1}{\gamma} > 0$ at $\varepsilon = 0$. Note that A_{MAC} varies between 0 when $\gamma \searrow 1$ and $\frac{1}{2}$ when $\gamma \nearrow 2$.*

Remark 3. *In view of the above results, the convergence rates available in the literature, see e.g. [14, 15, 23], are not optimal. Indeed, for $d = 3$ and $\gamma = \frac{3}{2}$, they degenerate to 0. Moreover, no error analysis is available for $\gamma < \frac{3}{2}$. Our approach yields error estimates also for $\gamma \in (1, \frac{3}{2}]$. In addition, we have better convergence rates, e.g., for $d = 3$ and $\gamma = \frac{3}{2}$, where the convergence errors are $\mathcal{O}(h^{\frac{3}{4}})$ and $\mathcal{O}(h^{\frac{1}{3}})$ for the FV and MAC schemes, respectively.*

Proof of Theorem 3.1. First, by a straightforward but lengthy calculation, see Appendix D, we observe the following relative energy inequality

$$
\begin{aligned}
&[\mathfrak{E}(\varrho_h, \boldsymbol{u}_h | \varrho, \boldsymbol{u})]_0^\tau + \int_0^\tau \int_{\mathbb{T}^d} \left(\mu \left|\nabla_h \boldsymbol{u}_h\right|^2 + \nu \left|\operatorname{div}_h \boldsymbol{u}_h\right|^2\right) \mathrm{d}x\mathrm{d}t \\
&\leq \int_0^\tau \int_{\mathbb{T}^d} \left(\varrho_h \partial_t \frac{\left|\boldsymbol{u}\right|^2}{2} + \varrho_h \Pi_Q \boldsymbol{u}_h \cdot \nabla_x \frac{\left|\boldsymbol{u}\right|^2}{2}\right) \mathrm{d}x\mathrm{d}t \; + e_\varrho\left(\tau, \Delta t, h, \left|\boldsymbol{u}\right|^2/2\right) \\
&- \int_0^\tau \int_{\mathbb{T}^d} \left(\varrho_h \partial_t P'(\varrho) + \varrho_h \Pi_Q \boldsymbol{u}_h \cdot \nabla_x P'(\varrho)\right) \mathrm{d}x \; - e_\varrho(\tau, \Delta t, h, P'(\varrho)) \\
&- \int_0^\tau \int_{\mathbb{T}^d} \left(\varrho_h \Pi_Q \boldsymbol{u}_h \cdot \partial_t \boldsymbol{u} + \varrho_h \Pi_Q \boldsymbol{u}_h \otimes \Pi_Q \boldsymbol{u}_h : \nabla_x \boldsymbol{u} + p_h \operatorname{div}_x \boldsymbol{u}\right) \mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \int_{\mathbb{T}^d} \left(\mu \nabla_h \boldsymbol{u}_h : \nabla_x \boldsymbol{u} + \nu \operatorname{div}_h \boldsymbol{u}_h \operatorname{div}_x \boldsymbol{u}\right) \mathrm{d}x\mathrm{d}t \; + e_{\boldsymbol{m}}(\tau, \Delta t, h, -\boldsymbol{u}) \\
&+ \int_0^\tau \int_{\mathbb{T}^d} \partial_t \big(\varrho P'(\varrho) - P(\varrho)\big) \mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{3.5}
$$

Next, we observe the following identities

$$
\begin{aligned}
&\varrho_h \Pi_Q \boldsymbol{u}_h \cdot \nabla_x \frac{\left|\boldsymbol{u}\right|^2}{2} - \varrho_h \Pi_Q \boldsymbol{u}_h \otimes \Pi_Q \boldsymbol{u}_h : \nabla_x \boldsymbol{u} \\
&\quad = -\varrho_h (\Pi_Q \boldsymbol{u}_h - \boldsymbol{u}) \otimes (\Pi_Q \boldsymbol{u}_h - \boldsymbol{u}) : \nabla_x \boldsymbol{u} - \varrho_h (\Pi_Q \boldsymbol{u}_h - \boldsymbol{u}) \cdot (\boldsymbol{u} \cdot \nabla_x \boldsymbol{u}),
\end{aligned}
$$

$$
P''(\varrho) = \frac{1}{\varrho} p'(\varrho), \quad \varrho P'(\varrho) - P(\varrho) = p(\varrho), \quad \partial_t(\varrho P'(\varrho) - P(\varrho)) = \partial_t p(\varrho).
$$

Then by substituting the above equalities into (3.5) and denoting

$$
e_S = e_\varrho\left(\tau, \Delta t, h, \left|\boldsymbol{u}\right|^2/2\right) - e_\varrho(\tau, \Delta t, h, P'(\varrho)) + e_{\boldsymbol{m}}(\tau, \Delta t, h, -\boldsymbol{u}),
$$

we obtain

$$
\begin{aligned}
&[\mathfrak{E}(\varrho_h,\boldsymbol{u}_h|\varrho,\boldsymbol{u})]_0^\tau + \int_0^\tau \int_{\mathbb{T}^d} \left(\mu\,|\nabla_h \boldsymbol{u}_h - \nabla_x \boldsymbol{u}|^2 + \nu\,|\mathrm{div}_h \boldsymbol{u}_h - \mathrm{div}_x \boldsymbol{u}|^2\right)\,\mathrm{d}x\mathrm{d}t \\
&\le e_S + \int_0^\tau \int_{\mathbb{T}^d} \varrho_h(\boldsymbol{u} - \Pi_Q \boldsymbol{u}_h)\cdot(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla_x \boldsymbol{u})\,\mathrm{d}x\mathrm{d}t \\
&- \int_0^\tau \int_{\mathbb{T}^d} \varrho_h(\Pi_Q \boldsymbol{u}_h - \boldsymbol{u})\otimes(\Pi_Q \boldsymbol{u}_h - \boldsymbol{u}) : \nabla_x \boldsymbol{u}\,\mathrm{d}x\mathrm{d}t \\
&+ \mu \int_0^\tau \int_{\mathbb{T}^d} \left(|\nabla_x \boldsymbol{u}|^2 - \nabla_h \boldsymbol{u}_h : \nabla_x \boldsymbol{u}\right)\,\mathrm{d}x\mathrm{d}t \\
&+ \nu \int_0^\tau \int_{\mathbb{T}^d} \left(|\mathrm{div}_x \boldsymbol{u}|^2 - \mathrm{div}_h \boldsymbol{u}_h \mathrm{div}_x \boldsymbol{u}\right)\,\mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \int_{\mathbb{T}^d} \left(\partial_t p(\varrho) - \varrho_h \frac{\partial_t p(\varrho)}{\varrho} - \varrho_h \Pi_Q \boldsymbol{u}_h \cdot \frac{\nabla_x p(\varrho)}{\varrho} - p_h \mathrm{div}_x \boldsymbol{U}\right)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{3.6}
$$

As $(\varrho,\boldsymbol{u})$ satisfies the Navier–Stokes system (1.1), we know that

$$
\varrho(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla_x \boldsymbol{u}) = \mu\Delta_x \boldsymbol{u} + \nu\nabla_x \mathrm{div}_x \boldsymbol{u} - \nabla_x p(\varrho).
$$

Substituting this equality into (3.6) we get

$$
\begin{aligned}
&[\mathfrak{E}(\varrho_h,\boldsymbol{u}_h|\varrho,\boldsymbol{u})]_0^\tau + \int_0^\tau \int_{\mathbb{T}^d} \left(\mu\,|\nabla_h \boldsymbol{u}_h - \nabla_x \boldsymbol{u}|^2 + \nu\,|\mathrm{div}_h \boldsymbol{u}_h - \mathrm{div}_x \boldsymbol{u}|^2\right)\,\mathrm{d}x\mathrm{d}t \\
&\le e_S + \int_0^\tau \int_{\mathbb{T}^d} (\varrho_h - \varrho)(\boldsymbol{u} - \Pi_Q \boldsymbol{u}_h)\cdot(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla_x \boldsymbol{u})\,\mathrm{d}x\mathrm{d}t, \\
&- \int_0^\tau \int_{\mathbb{T}^d} \varrho_h(\Pi_Q \boldsymbol{u}_h - \boldsymbol{u})\otimes(\Pi_Q \boldsymbol{u}_h - \boldsymbol{u}) : \nabla_x \boldsymbol{u}\,\mathrm{d}x\mathrm{d}t, \\
&+ \mu \int_0^\tau \int_{\mathbb{T}^d} \left(|\nabla_x \boldsymbol{u}|^2 - \nabla_h \boldsymbol{u}_h : \nabla_x \boldsymbol{u} + (\boldsymbol{u} - \Pi_Q \boldsymbol{u}_h)\cdot\Delta_x \boldsymbol{u}\right)\,\mathrm{d}x\mathrm{d}t \\
&+ \nu \int_0^\tau \int_{\mathbb{T}^d} \left(|\mathrm{div}_x \boldsymbol{u}|^2 - \mathrm{div}_h \boldsymbol{u}_h \mathrm{div}_x \boldsymbol{u} + (\boldsymbol{u} - \Pi_Q \boldsymbol{u}_h)\cdot\nabla_x \mathrm{div}_x \boldsymbol{u}\right)\,\mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \int_{\mathbb{T}^d} \left(\frac{\varrho - \varrho_h}{\varrho}\partial_t p(\varrho) - \frac{\varrho_h}{\varrho}\Pi_Q \boldsymbol{u}_h \cdot \nabla_x p(\varrho) - p_h \mathrm{div}_x \boldsymbol{u}\right)\,\mathrm{d}x\mathrm{d}t - \int_0^\tau \int_{\mathbb{T}^d} (\boldsymbol{u} - \Pi_Q \boldsymbol{u}_h)\cdot\nabla_x p(\varrho)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Rearranging the terms on the right hand side, we arrive at

$$
[\mathfrak{E}(\varrho_h,\boldsymbol{u}_h|\varrho,\boldsymbol{u})]_0^\tau + \int_0^\tau \int_{\mathbb{T}^d} \left(\mu\,|\nabla_h \boldsymbol{u}_h - \nabla_x \boldsymbol{u}|^2 + \nu\,|\mathrm{div}_h \boldsymbol{u}_h - \mathrm{div}_x \boldsymbol{u}|^2\right)\,\mathrm{d}x\mathrm{d}t \le e_S + \sum_{i=1}^5 R_i^E,
$$

where the integrals $R_i^E, i = 1,\cdots,5$, read

$$
\begin{aligned}
R_1^E &= \int_0^\tau \int_{\mathbb{T}^d} (\varrho_h - \varrho)(\boldsymbol{u} - \Pi_Q \boldsymbol{u}_h)\cdot(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla_x \boldsymbol{u} + \frac{\nabla_x p(\varrho)}{\varrho})\,\mathrm{d}x\mathrm{d}t \\
&= \int_0^\tau \int_{\mathbb{T}^d} (\varrho_h - \varrho)(\boldsymbol{u} - \Pi_Q \boldsymbol{u}_h)\cdot\frac{\mathrm{div}_x \mathbb{S}(\nabla_x \boldsymbol{u})}{\varrho}\,\mathrm{d}x\mathrm{d}t \\
R_2^E &= -\int_0^\tau \int_{\mathbb{T}^d} \varrho_h(\Pi_Q \boldsymbol{u}_h - \boldsymbol{u})\otimes(\Pi_Q \boldsymbol{u}_h - \boldsymbol{u}) : \nabla_x \boldsymbol{u}\,\mathrm{d}x\mathrm{d}t, \\
R_3^E &= -\mu \int_0^\tau \int_{\mathbb{T}^d} (\nabla_h \boldsymbol{u}_h : \nabla_x \boldsymbol{u} + \Pi_Q \boldsymbol{u}_h \cdot \Delta_x \boldsymbol{u})\,\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

$$R_4^E = -\nu \int_0^\tau \int_{\mathbb{T}^d} (\mathrm{div}_h \boldsymbol{u}_h \mathrm{div}_x \boldsymbol{u} + \Pi_Q \boldsymbol{u}_h \cdot \nabla_x \mathrm{div}_x \boldsymbol{u}) \,\mathrm{d}x\mathrm{d}t,$$
$$R_5^E = -\int_0^\tau \int_{\mathbb{T}^d} \big(p_h - p'(\varrho)(\varrho_h - \varrho) - p(\varrho)\big) \mathrm{div}_x \boldsymbol{u} \,\mathrm{d}x\mathrm{d}t.$$

Next, for $i = 1, \cdots, 5$ we analyze R_i^E such that it can be controlled either by the relative energy or the mesh parameter h.

Term R_1^E. Applying Hölder's inequality and Lemma B.4 we obtain

$$\left|R_1^E\right| \leq \frac{1}{\underline{r}} \|\mathrm{div}_x \mathbb{S}(\nabla_x \boldsymbol{u})\|_{L^\infty((0,T)\times\mathbb{T}^d)} \left(C_0 \int_0^\tau \mathfrak{E}(\varrho_h, \boldsymbol{u}_h | \varrho, \boldsymbol{u}) \mathrm{d}t + C_1 \delta \|\nabla_h \boldsymbol{u}_h - \nabla_x \boldsymbol{u}\|_{L^2}^2 + C_2 \delta h^2 \right)$$
$$= C_0^* \int_0^\tau \mathfrak{E}(\varrho_h, \boldsymbol{u}_h | \varrho, \boldsymbol{u}) \mathrm{d}t + C_1^* \delta \|\nabla_h \boldsymbol{u}_h - \nabla_x \boldsymbol{u}\|_{L^2}^2 + C_2^* \delta h^2,$$

where $C_0^* > 0$ depends on $\|\boldsymbol{u}\|_{L^\infty W^{2,\infty}}$, $\|\varrho\|_{C([0,T]\times\mathbb{T}^d)}, M, E_0, \gamma$, δ, and $\underline{r} = \min_{[0,T]\times\mathbb{T}^d} \varrho$;
$C_1^* > 0$ depends on $\|\boldsymbol{u}\|_{L^\infty W^{2,\infty}}$, M, E_0, and γ;
$C_2^* > 0$ depends on $\|\boldsymbol{u}\|_{L^\infty W^{2,\infty}}$, M, E_0, γ, and $\|\nabla_x \boldsymbol{u}\|_{L^\infty((0,T)\times\mathbb{T}^d)}$.

Term R_2^E. Thanks to Hölder's inequality we observe the following estimate.

$$\left|R_2^E\right| \leq C \int_0^\tau \mathfrak{E}(\varrho_h, \boldsymbol{u}_h | \varrho, \boldsymbol{u}) \mathrm{d}t,$$

where C depends on $\|\nabla_x \boldsymbol{u}\|_{L^\infty((0,T)\times\mathbb{T}^d)}$.

Term R_3^E. We analyze the third term R_3^E in two cases.

First, we consider the case of the FV scheme. In this case $\boldsymbol{u}_h \in \mathbf{Q}_h$, $\nabla_h \boldsymbol{u}_h = \nabla_{\mathcal{D}} \boldsymbol{u}_h$ and $\Pi_Q \boldsymbol{u}_h = \boldsymbol{u}_h$. Thus,

$$\left|R_3^E\right| = \mu \left| \int_0^\tau \int_{\mathbb{T}^d} (\nabla_h \boldsymbol{u}_h : \nabla_x \boldsymbol{u} + \Pi_Q \boldsymbol{u}_h \cdot \Delta_x \boldsymbol{u}) \,\mathrm{d}x\mathrm{d}t \right| = \mu \left| \int_0^\tau \int_{\mathbb{T}^d} \left(\nabla_{\mathcal{D}} \boldsymbol{u}_h : \nabla_x \boldsymbol{u} + \boldsymbol{u}_h \cdot \mathrm{div}_{\mathcal{T}}^{\mathbf{W}} \Pi_{\mathcal{E}} \nabla_x \boldsymbol{u} \right) \mathrm{d}x\mathrm{d}t \right|$$
$$= \mu \left| \int_0^\tau \int_{\mathbb{T}^d} \nabla_{\mathcal{D}} \boldsymbol{u}_h : (\nabla_x \boldsymbol{u} - \Pi_{\mathcal{E}} \nabla_x \boldsymbol{u}) \,\mathrm{d}x\mathrm{d}t \right| \leq \mu h \|\nabla_h \boldsymbol{u}_h\|_{L^2L^2} \|\boldsymbol{u}\|_{L^2W^{2,2}},$$

where we have used the equality (2.4), the integration by parts formula (2.3a), and the estimate (2.9b).

Second, we consider the case of $\boldsymbol{u}_h \in \mathbf{W}_h$ obtained by the MAC scheme. In this case, $\nabla_h \boldsymbol{u}_h = \nabla_{\mathcal{B}} \boldsymbol{u}_h$ and the term R_3^E can be estimated in the following way

$$\left|R_3^E\right| = \mu \left| \int_0^\tau \int_{\mathbb{T}^d} (\nabla_h \boldsymbol{u}_h : \nabla_x \boldsymbol{u} + \Pi_Q \boldsymbol{u}_h \cdot \Delta_x \boldsymbol{u}) \,\mathrm{d}x\mathrm{d}t \right|$$
$$= \mu \left| \int_0^\tau \sum_{i=1}^d \sum_{j=1}^d \sum_{\epsilon = D_\sigma | D_{\sigma'} \in \widetilde{\mathcal{E}}_{j,i}} \int_{D_\epsilon} \eth_{\mathcal{B}_{j,i}} u_{j,h} \left(\partial_i U_j - \frac{(\Pi_{\mathcal{E}}^{(i)} \partial_i u_j)_{D_\sigma} + (\Pi_{\mathcal{E}}^{(i)} \partial_i u_j)_{D_{\sigma'}}}{2} \right) \mathrm{d}x\mathrm{d}t \right|$$
$$\leq \mu h \|\nabla_h \boldsymbol{u}_h\|_{L^2L^2} \|\boldsymbol{u}\|_{L^2W^{2,2}},$$

where we have applied (2.8a), Hölder's inequality and the estimate (2.9b).

Consequently, we have for both cases

$$\left|R_3^E\right| \leq Ch,$$

where the constant C depends on μ, the initial energy E_0 and $\|\boldsymbol{U}\|_{L^2W^{2,2}}$.

Term R_4^E. We analyze the term R_4^E also in two cases.

First, for $\boldsymbol{u}_h \in \mathbf{Q}_h$ obtained by the FV method, we have $\mathrm{div}_h \boldsymbol{u}_h = \mathrm{div}_{\mathcal{T}}^{Q} \boldsymbol{u}_h$, $\Pi_Q \boldsymbol{u}_h = \boldsymbol{u}_h$ and thus

$$
\begin{aligned}
\left|R_4^E\right| &= \nu \left| \int_0^\tau \int_{\mathbb{T}^d} \left(\mathrm{div}_h \boldsymbol{u}_h \mathrm{div}_x \boldsymbol{u} + \Pi_Q \boldsymbol{u}_h \cdot \nabla_x \mathrm{div}_x \boldsymbol{u}\right) \,\mathrm{d}x\mathrm{d}t \right| \\
&= \nu \left| \int_0^\tau \int_{\mathbb{T}^d} \left(\mathrm{div}_{\mathcal{T}}^{Q} \boldsymbol{u}_h \mathrm{div}_x \boldsymbol{u} + \boldsymbol{u}_h \cdot \nabla_x \mathrm{div}_x \boldsymbol{u} - \left(\Pi_\epsilon \mathrm{div}_x \boldsymbol{u}\ \mathrm{div}_{\mathcal{T}}^{Q} \boldsymbol{v}_h + \{\!\{\boldsymbol{v}_h\}\!\} \cdot \nabla_x \mathrm{div}_x \boldsymbol{u}\right)\right) \,\mathrm{d}x\mathrm{d}t \right| \\
&= \nu \left| \int_0^\tau \int_{\mathbb{T}^d} \left(\mathrm{div}_{\mathcal{T}}^{Q} \boldsymbol{u}_h (\mathrm{div}_x \boldsymbol{u} - \Pi_\epsilon \mathrm{div}_x \boldsymbol{u}) + (\boldsymbol{u}_h - \{\!\{\boldsymbol{u}_h\}\!\}) \cdot \nabla_x \mathrm{div}_x \boldsymbol{u}\right) \mathrm{d}x\mathrm{d}t \right| \\
&\le h \left(\|\mathrm{div}_h \boldsymbol{u}_h\|_{L^2L^2} + \|\nabla_h \boldsymbol{u}_h\|_{L^2L^2}\right) \|\boldsymbol{u}\|_{L^2W^{2,2}},
\end{aligned}
$$

where we have used the identity (2.8d), Hölder's inequality, the estimates (2.9a) and (2.9c).

Second, for the case of $\boldsymbol{u}_h \in \mathbf{W}_h$ we have

$$
\begin{aligned}
\left|R_4^E\right| &= \nu \left| \int_0^\tau \int_{\mathbb{T}^d} \left(\mathrm{div}_h \boldsymbol{u}_h \mathrm{div}_x \boldsymbol{u} + \Pi_Q \boldsymbol{u}_h \cdot \nabla_x \mathrm{div}_x \boldsymbol{u}\right) \,\mathrm{d}x\mathrm{d}t \right| \\
&= \nu \left| \int_0^\tau \int_{\mathbb{T}^d} \left(\mathrm{div}_{\mathcal{T}}^{\mathbf{W}} \boldsymbol{u}_h \mathrm{div}_x \boldsymbol{u} - \left(\mathrm{div}_{\mathcal{T}}^{\mathbf{W}} \boldsymbol{u}_h \Pi_\epsilon \mathrm{div}_x \boldsymbol{u} + \boldsymbol{u}_h \cdot \nabla_x \mathrm{div}_x \boldsymbol{u}\right) + \Pi_Q \boldsymbol{u}_h \cdot \nabla_x \mathrm{div}_x \boldsymbol{u}\right) \,\mathrm{d}x\mathrm{d}t \right| \\
&= \nu \left| \int_0^\tau \int_{\mathbb{T}^d} \left(\mathrm{div}_{\mathcal{T}}^{\mathbf{W}} \boldsymbol{u}_h (\mathrm{div}_x \boldsymbol{u} - \Pi_\epsilon \mathrm{div}_x \boldsymbol{u}) + (\Pi_Q \boldsymbol{u}_h - \boldsymbol{u}_h) \cdot \nabla_x \mathrm{div}_x \boldsymbol{u}\right) \mathrm{d}x\mathrm{d}t \right| \\
&\le \nu h \left(\|\mathrm{div}_h \boldsymbol{u}_h\|_{L^2L^2} + \|\nabla_h \boldsymbol{u}_h\|_{L^2L^2}\right) \|\boldsymbol{u}\|_{L^2W^{2,2}},
\end{aligned}
$$

where (2.8b), Hölder's inequality, the estimates (2.9a) and (2.9c) were applied.

Consequently, we have for both cases

$$
\left|R_4^E\right| \le Ch,
$$

where the constant C depends on ν, initial energy E_0, and $\|\boldsymbol{u}\|_{L^2W^{2,2}}$.

Term R_5^E. The estimate of R_5^E is straightforward by applying Hölder's inequality, i.e.,

$$
\left|R_5^E\right| \le C \int_0^\tau \mathfrak{E}(\varrho_h, \boldsymbol{u}_h | \varrho, \boldsymbol{u}) \mathrm{d}t,
$$

where C depends on $\|\mathrm{div}_x \boldsymbol{u}\|_{L^\infty((0,T)\times\mathbb{T}^d)}$.

Consequently, collecting the above estimates of R_i^E for $i = 1, \cdots, 5$, we find

$$
\begin{aligned}
&\mathfrak{E}(\varrho_h, \boldsymbol{u}_h | \varrho, \boldsymbol{u})(\tau) + \int_0^\tau \int_{\mathbb{T}^d} \left((\mu - C_1^*\delta) \left|\nabla_h \boldsymbol{u}_h - \nabla_x \boldsymbol{u}\right|^2 + \nu \left|\mathrm{div}_h \boldsymbol{u}_h - \mathrm{div}_x \boldsymbol{u}\right|^2\right) \,\mathrm{d}x\mathrm{d}t \\
&\le e_S + \mathfrak{E}(\varrho_h, \boldsymbol{u}_h | r, \boldsymbol{u})(0) + C_0^* \int_0^\tau \mathfrak{E}(\varrho_h, \boldsymbol{u}_h | r, \boldsymbol{u}) \mathrm{d}t + C_2^* \delta h^2.
\end{aligned}
\tag{3.7}
$$

Applying the standard projection error estimates we get

$$
\mathfrak{E}(\varrho_h, \boldsymbol{u}_h | \varrho, \boldsymbol{u})(0) \le Ch^2, \tag{3.8}
$$

where C depends on $\|\varrho_0\|_C$ and $\|\boldsymbol{u}_0\|_{L^2W^{1,2}}$.

Consequently, by choosing $\delta < \frac{\mu}{C_1^*}$, substituting (3.8) into (3.7), using Gronwall's lemma and recalling the consistency error (2.15c), we may infer that

$$
\mathfrak{E}(\varrho_h, \boldsymbol{u}_h | \varrho, \boldsymbol{u})(\tau) + \int_0^\tau \int_{\mathbb{T}^d} \left(\left|\nabla_h \boldsymbol{u}_h - \nabla_x \boldsymbol{u}\right|^2 + \left|\mathrm{div}_h \boldsymbol{u}_h - \mathrm{div}_x \boldsymbol{u}\right|^2\right) \,\mathrm{d}x\mathrm{d}t \le C e^{\frac{\tau C_0^*}{1 - \Delta t C_0^*}} (\sqrt{\Delta t} + h^A)
$$

for $\Delta t < \frac{1}{C_0^*}$. Here, the constant C depends on $\|\varrho\|_{L^\infty W^{2,\infty}}, \|\boldsymbol{u}\|_{L^\infty W^{2,\infty}}$ and the exponent A is given by (3.4).

Finally, we combine the above estimate with Lemma C.1 and Lemma B.2 in order to obtain (3.2) and (3.3), respectively. Note that E_0 and M are bounded by the norm $\|(\varrho_0, \boldsymbol{u}_0)\|_{W^{k,2}(\mathbb{T}^d;R^{d+1})}$. Due to Proposition 1.1 all terms depending on the norms of the exact solution $(\varrho, \boldsymbol{u})$ as well as $\underline{r}$ are bounded by a constant $c = c(T, \|(\varrho_0, \boldsymbol{u}_0)\|_{W^{k,2}(\mathbb{T}^d;R^{d+1})}, \|(\varrho, \boldsymbol{u})\|_{C([0,T]\times\mathbb{T}^d;R^{d+1})})$ which finishes the proof. □

Finally, we observe that under the assumption that the numerical solutions $(\varrho_h, \boldsymbol{u}_h)$ are uniformly bounded, the above error estimates can be improved. Indeed, applying Lemma 2.10, Lemma C.1 and Lemma B.2 we derive the first order error rate.

Theorem 3.2 (Error rates for bounded numerical solutions)**.** *In addition to the hypotheses of Theorem 3.1, let the numerical solution $(\varrho_h, \boldsymbol{u}_h)$ be uniformly bounded,*

$$\|\varrho_h\|_{L^\infty((0,T)\times\mathbb{T}^d)} \leq \overline{\varrho} \quad \text{and} \quad \|\boldsymbol{u}_h\|_{L^\infty((0,T)\times\mathbb{T}^d;\mathbb{R}^d)} \leq \overline{u}. \tag{3.9}$$

Then there exists a positive number

$$c = c\left(T, \|(\varrho_0, \boldsymbol{u}_0)\|_{W^{k,2}(\mathbb{T}^d;\mathbb{R}^{d+1})}, \inf \varrho_0, \overline{\varrho}, \overline{u}, \right)$$

such that

$$\sup_{0\leq t\leq\tau} \mathfrak{E}(\varrho_h, \boldsymbol{u}_h|\varrho, \boldsymbol{u}) + \mu \int_0^\tau \int_{\mathbb{T}^d} |\nabla_h \boldsymbol{u}_h - \nabla_x \boldsymbol{u}|^2 \,\mathrm{d}x\mathrm{d}t + \nu \int_0^\tau \int_{\mathbb{T}^d} |\mathrm{div}_h \boldsymbol{u}_h - \mathrm{div}_x \boldsymbol{u}|^2 \,\mathrm{d}x\mathrm{d}t$$
$$\leq c(h + \Delta t)$$

for all $\tau \in [0, T]$, and

$$\|\varrho_h - \varrho\|_{L^\infty L^2} + \|\varrho_h \boldsymbol{u}_h - \varrho \boldsymbol{u}\|_{L^\infty L^2} + \|\boldsymbol{u}_h - \boldsymbol{u}\|_{L^2 L^2} \lesssim c(\Delta t^{\frac{1}{2}} + h^{\frac{1}{2}}).$$

4 Conclusion

In this paper we have presented improved error estimates for two well-known numerical methods applied to compressible Navier–Stokes equations. Specifically, we consider the upwind finite volume method and the Marker-and-Cell (MAC) method with implicit time discretization and piecewise constant approximation in space. However, the approach presented in the paper can be applied also to other well-known numerical methods for compressible Navier–Stokes equations.

The novelty of our approach lies in the use of continuous form of the relative energy inequality combined with a refined consistency analysis. Thus, following the framework of the Lax equivalence theorem it suffices to show the (energy) stability, cf. Lemma 2.8, and the consistency of a numerical scheme, cf. Lemma 2.9, in order to obtain the convergence rates for the scheme. Indeed, the consistency errors directly yield global errors in the relative energy. To obtain the corresponding error estimates we only assume that the initial data are sufficiently regular and a strong solution exists globaly in time. The error estimates presented in Theorem 3.1 improves the results already presented in the literature [15, 14, 23], see Remarks 2,3 for a detailed discussion. In particular, our error estimates hold for the full range of the adiabatic coefficient $\gamma > 1$.

Moreover, we have considered a natural hypothesis on uniformly bounded numerical solutions and proved that the error estimates can be further improved, cf. Theorem 3.2. Indeed, we prove that both numerical methods converge with the first order in time and mesh parameter in terms of the relative energy and with the half order in the $L^\infty(0, T; L^2(\mathbb{T}^d))$-norm for the density and momentum, as well as in the $L^2((0, T) \times \mathbb{T}^d)$-norm for the velocity.

Appendix

A Proof of the preliminary lemmas

In this section we present the proofs of Lemmas 2.2 – 2.5.

Proof of Lemma 2.2. First, we calculate

$$\int_{\mathbb{T}^d} r_h \mathrm{div}_x \boldsymbol{U} \,\mathrm{d}x = \sum_{K\in\mathcal{T}} r_K \int_K \mathrm{div}_x \boldsymbol{U} \,\mathrm{d}x = \sum_{K\in\mathcal{T}} r_K \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma \boldsymbol{U}\cdot\boldsymbol{n} \,\mathrm{dS}(x)$$

$$= \sum_{K\in\mathcal{T}} r_K \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \Pi_{\mathcal{E}} \boldsymbol{U}\cdot\boldsymbol{n} = \sum_{K\in\mathcal{T}} r_K |K| \mathrm{div}_{\mathcal{T}}^{\mathbf{W}} \Pi_{\mathcal{E}} \boldsymbol{U} = \int_{\mathbb{T}^d} r_h \mathrm{div}_{\mathcal{T}}^{\mathbf{W}} \Pi_{\mathcal{E}} \boldsymbol{U} \,\mathrm{d}x.$$

Analogously, we find

$$\int_{\mathbb{T}^d} \boldsymbol{v}_h \cdot \nabla_x \psi \,\mathrm{d}x = \sum_{K\in\mathcal{T}} \boldsymbol{v}_K \cdot \int_K \nabla_x \psi \,\mathrm{d}x = \sum_{K\in\mathcal{T}} \boldsymbol{v}_K \cdot \Big(\sum_{\sigma\in\mathcal{E}(K)} \int_\sigma \psi \boldsymbol{n} \,\mathrm{dS}(x) \Big)$$

$$= \sum_{K\in\mathcal{T}} \boldsymbol{v}_K \cdot \left(\sum_{i=1}^d \sum_{\sigma\in\mathcal{E}_i(K)} |\sigma| \Pi_{\mathcal{E}}^{(i)} \psi \boldsymbol{n} \right) = \sum_{i=1}^d \sum_{K\in\mathcal{T}} v_{i,h}|_K \ \left(|K| \partial_{\mathcal{T}}^{(i)} \Pi_{\mathcal{E}}^{(i)} \psi \right)$$

$$= \sum_{i=1}^d \int_{\mathbb{T}^d} v_{i,h} \, \partial_{\mathcal{T}}^{(i)} \Pi_{\mathcal{E}}^{(i)} \psi \,\mathrm{d}x = \int_{\mathbb{T}^d} \boldsymbol{v}_h \cdot \nabla_{\mathcal{T}}^{\Pi\mathcal{E}} \psi \,\mathrm{d}x,$$

which completes the proof. □

Proof of Lemma 2.3. First, we calculate

$$\int_{\mathbb{T}^d} \boldsymbol{u}_h \cdot \nabla_x \psi \,\mathrm{d}x = \sum_{i=1}^d \sum_{\sigma\in\mathcal{E}_i} \int_{D_\sigma} u_{i,h} \partial_i \psi \,\mathrm{d}x = \sum_{i=1}^d \sum_{\sigma\in\mathcal{E}_i} u_{i,h} \left(\int_{\epsilon^+} \psi \,\mathrm{dS}(x) - \int_{\epsilon^-} \psi \,\mathrm{dS}(x) \right)$$

$$= \sum_{i=1}^d \sum_{\sigma\in\mathcal{E}_i} u_{i,h} \left(\int_{D_{\epsilon^+}} \Pi_\epsilon \psi \,\mathrm{d}x - \int_{D_{\epsilon^-}} \Pi_\epsilon \psi \,\mathrm{d}x \right) / h,$$

where ϵ^- and ϵ^+ are the left and right edges of D_σ in the i^{th}-direction of the canonical system for $\sigma \in \mathcal{E}_i$. Note that $D_{\epsilon^\pm} \subset \mathcal{T}$ are elements of the primary grid $\mathcal{T}$. Then we can rewrite the above relation as

$$\int_{\mathbb{T}^d} \boldsymbol{u}_h \cdot \nabla_x \psi \,\mathrm{d}x = \sum_{i=1}^d \sum_{\sigma\in\mathcal{E}_i} |D_\sigma| u_{i,h} \eth_{\mathcal{D}_i} \Pi_\epsilon \psi \,\mathrm{d}x = -\sum_{i=1}^d \sum_{K\in\mathcal{T}} |K| \partial_{\mathcal{T}}^{(i)} u_{i,h} \Pi_\epsilon \psi \,\mathrm{d}x = -\int_{\mathbb{T}^d} \Pi_\epsilon \psi \ \mathrm{div}_{\mathcal{T}}^{\mathbf{W}} \boldsymbol{u}_h \,\mathrm{d}x,$$

where we have used (2.3a). This proves (2.6).

The proof of (2.7) follows from (2.5) and (2.3a), specifically,

$$\int_{\mathbb{T}^d} \boldsymbol{v}_h \cdot \nabla_x \psi \,\mathrm{d}x = \int_{\mathbb{T}^d} \boldsymbol{v}_h \cdot \nabla_{\mathcal{T}}^{\Pi\mathcal{E}} \psi \,\mathrm{d}x = \sum_{i=1}^d \int_{\mathbb{T}^d} v_{i,h} \partial_{\mathcal{T}}^{(i)} \Pi_{\mathcal{E}}^{(i)} \psi \,\mathrm{d}x = -\sum_{i=1}^d \sum_{K\in\mathcal{T}} \int_K \eth_{\mathcal{D}_i} v_{i,h} \ \Pi_{\mathcal{E}}^{(i)} \psi \,\mathrm{d}x.$$

□

Proof of Lemma 2.4. First, we recall (2.4) and (2.3a) to derive the first equality

$$\begin{aligned}
&\int_{\mathbb{T}^d} \Pi_Q \boldsymbol{u}_h \cdot \Delta_x \boldsymbol{U} \,\mathrm{d}x = \int_{\mathbb{T}^d} \Pi_Q \boldsymbol{u}_h \cdot (\mathrm{div}_x \nabla_x \boldsymbol{U}) \,\mathrm{d}x = \int_{\mathbb{T}^d} \Pi_Q \boldsymbol{u}_h \cdot (\mathrm{div}_{\mathcal{T}}^{\mathbf{W}} \Pi_{\mathcal{E}} \nabla_x \boldsymbol{U}) \,\mathrm{d}x \\
&= -\int_{\mathbb{T}^d} \nabla_{\mathcal{D}} \Pi_Q \boldsymbol{u}_h : \Pi_{\mathcal{E}} \nabla_x \boldsymbol{U} \,\mathrm{d}x = -\sum_{i=1}^d \sum_{j=1}^d \sum_{\sigma \in \mathcal{E}_i} \int_{D_\sigma} \eth_{\mathcal{D}_i} \overline{u_{j,h}} \Pi_{\mathcal{E}}^{(i)} \partial_i U_j \,\mathrm{d}x \\
&= -\sum_{i=1}^d \sum_{j=1}^d \sum_{\sigma \in \mathcal{E}_i} \int_{D_\sigma} \left(\frac{1}{2} \sum_{\epsilon \in \widetilde{\mathcal{E}}_{j,i}(D_\sigma)} (\eth_{\mathcal{B}_{j,i}} u_{j,h})_{D_\epsilon} \right) \Pi_{\mathcal{E}}^{(i)} \partial_i U_j \,\mathrm{d}x \\
&= -\sum_{i=1}^d \sum_{j=1}^d \sum_{\epsilon = D_\sigma | D_{\sigma'} \in \widetilde{\mathcal{E}}_{j,i}} \int_{D_\epsilon} \eth_{\mathcal{B}_{j,i}} u_{j,h} \left(\frac{(\Pi_{\mathcal{E}}^{(i)} \partial_i U_j)_{D_\sigma} + (\Pi_{\mathcal{E}}^{(i)} \partial_i U_j)_{D_{\sigma'}}}{2} \right) \mathrm{d}x
\end{aligned}$$

Next, it is easy to check (2.8b) by setting $\psi = \mathrm{div}_x \boldsymbol{U}$ in (2.6), i.e.,

$$\int_{\mathbb{T}^d} \boldsymbol{u}_h \cdot \nabla_x \mathrm{div}_x \boldsymbol{U} \,\mathrm{d}x = -\int_{\mathbb{T}^d} \mathrm{div}_{\mathcal{T}}^{\mathbf{W}} \boldsymbol{u}_h \Pi_\epsilon (\mathrm{div}_x \boldsymbol{U}) \,\mathrm{d}x.$$

Further, thanks to (2.4) and (2.3a), we observe (2.8c), i.e.,

$$\int_{\mathbb{T}^d} \boldsymbol{v}_h \cdot \Delta_x \boldsymbol{U} \,\mathrm{d}x = \int_{\mathbb{T}^d} \boldsymbol{v}_h \cdot (\mathrm{div}_x \nabla_x \boldsymbol{U}) \,\mathrm{d}x = \int_{\mathbb{T}^d} \boldsymbol{v}_h \cdot (\mathrm{div}_{\mathcal{T}}^{\mathbf{W}} \Pi_{\mathcal{E}} \nabla_x \boldsymbol{U}) \,\mathrm{d}x = -\int_{\mathbb{T}^d} \nabla_{\mathcal{D}} \boldsymbol{v}_h : \Pi_{\mathcal{E}} \nabla_x \boldsymbol{U} \,\mathrm{d}x.$$

Finally, by setting $(\{\!\{\boldsymbol{v}_h\}\!\}, \mathrm{div}_x \boldsymbol{U})$ as $(\boldsymbol{u}_h, \psi)$ into (2.6) we get (2.8d), i.e.

$$\int_{\mathbb{T}^d} \{\!\{\boldsymbol{v}_h\}\!\} \cdot \nabla_x \mathrm{div}_x \boldsymbol{U} \,\mathrm{d}x = -\int_{\mathbb{T}^d} \Pi_\epsilon \mathrm{div}_x \boldsymbol{U} \ \mathrm{div}_{\mathcal{T}}^{\mathbf{W}} (\{\!\{\boldsymbol{v}_h\}\!\}) \,\mathrm{d}x = -\int_{\mathbb{T}^d} \Pi_\epsilon \mathrm{div}_x \boldsymbol{U} \ \mathrm{div}_{\mathcal{T}}^{Q} \boldsymbol{v}_h \,\mathrm{d}x,$$

where we have used the identity (2.2). □

Proof of Lemma 2.5. Note that the estimates stated in (2.9b) – (2.9d) hold due to the standard interpolation error; whence we omit the proof. Now we prove (2.9a). First, by a direct calculation, we have

$$\begin{aligned}
&\|\Pi_Q \boldsymbol{u}_h - \boldsymbol{u}_h\|_{L^2}^2 = \sum_{K \in \mathcal{T}} \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_i(K)} |D_{\sigma,K}| \left(\frac{u_{i,\sigma_{K,i+}} + u_{i,\sigma_{K,i-}}}{2} - u_{i,\sigma} \right)^2 \\
&= \frac{1}{4} \sum_{K \in \mathcal{T}} \sum_{i=1}^d \left(\frac{u_{i,\sigma_{K,i+}} - u_{i,\sigma_{K,i-}}}{2} \right)^2 \sum_{\sigma \in \mathcal{E}_i(K)} |D_{\sigma,K}| = \frac{h^2}{4} \sum_{K \in \mathcal{T}} |K| \sum_{i=1}^d \left(\partial_{\mathcal{T}}^{(i)} u_{i,h} \right)^2 \leq \frac{h^2}{4} \|\nabla_{\mathcal{B}} \boldsymbol{u}_h\|_{L^2}^2,
\end{aligned}$$

where we have used the fact that $\eth_{\mathcal{B}_{i,i}} = \partial_{\mathcal{T}}^{(i)}$ in the last inequality, which proves the first estimate of (2.9a). Analogously, we compute

$$\begin{aligned}
&\|\{\!\{\boldsymbol{v}_h\}\!\} - \boldsymbol{v}_h\|_{L^2}^2 = \sum_{K \in \mathcal{T}} \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_i(K)} |D_{\sigma,K}| \left(\frac{v_{i,h}^{\mathrm{in}} + v_{i,h}^{\mathrm{out}}}{2} - v_{i,h}^{\mathrm{in}} \right)^2 \\
&= \frac{h^2}{4} \sum_{K \in \mathcal{T}} \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_i(K)} |D_{\sigma,K}| (\eth_{\mathcal{D}_i} v_{i,h})^2 = \frac{h^2}{4} \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_i} |D_\sigma| (\eth_{\mathcal{D}_i} v_{i,h})^2 \leq \frac{h^2}{4} \|\nabla_{\mathcal{D}} \boldsymbol{v}_h\|_{L^2}^2,
\end{aligned}$$

which proves the second estimate of (2.9a). This concludes the proof of Lemma 2.5. □

B Sobolev-Poincaré type inequality

First, we recall [12, Theorem 17] for a generalized Sobolev-Poincaré inequality.

Lemma B.1 ([12])**.** *For a structure mesh let $\gamma > 1$ and $\varrho_h \geq 0$ satisfy*

$$0 < c_M \leq \int_{\mathbb{T}^d} \varrho_h \,\mathrm{d}x \text{ and } \int_{\mathbb{T}^d} \varrho_h^\gamma \,\mathrm{d}x \leq c_E,$$

where $\gamma > 1$, c_M and c_E are positive constants. Then there exists $c = c(c_M, c_E, \gamma)$ independent of h such that

$$\|f_h\|^2_{L^q(\mathbb{T}^d)} \leq c \left(\|\nabla_h f_h\|^2_{L^2(\mathbb{T}^d)} + \int_{\mathbb{T}^d} \varrho_h |f_h|^2 \,\mathrm{d}x \right).$$

Now we are ready to show the following lemma.

Lemma B.2. *Under the assumption of Lemma B.1 let $(\varrho_h, \boldsymbol{u}_h)$ be a solution obtained either by the FV method (2.10) or the MAC method (2.11). Let $\boldsymbol{U} \in W^{2,\infty}(\mathbb{T}^d; \mathbb{R}^d)$, then there exists $C_1 = C_1(M, E_0, \gamma) > 0$ and $C_2 = C_2(M, E_0, \gamma, \|\nabla_x \boldsymbol{U}\|_{L^\infty}, \|\boldsymbol{U}\|_{W^{2,\infty}}) > 0$ such that*

$$\|\boldsymbol{u}_h - \boldsymbol{U}\|^2_{L^2} \leq C_1 \left(\|\nabla_h \boldsymbol{u}_h - \nabla_x \boldsymbol{U})\|^2_{L^2(\mathbb{T}^d)} + \int_{\mathbb{T}^d} \varrho_h |\boldsymbol{u}_h - \boldsymbol{U}|^2 \,\mathrm{d}x \right) + C_2 h^2, \tag{B.1}$$

$$\|\Pi_Q \boldsymbol{u}_h - \boldsymbol{U}\|^2_{L^2} \leq C_1 \left(\|\nabla_h \boldsymbol{u}_h - \nabla_x \boldsymbol{U}_h\|^2_{L^2(\mathbb{T}^d)} + \int_{\mathbb{T}^d} \varrho_h |\boldsymbol{u}_h - \boldsymbol{U}|^2 \,\mathrm{d}x \right) + C_2 h^2, \tag{B.2}$$

where M and E_0 are the fluid mass and initial energy.

Proof. Firstly, by setting $f_h = \boldsymbol{u}_h - \boldsymbol{U}_h$ for some $\boldsymbol{U}_h$ belonging to the same discrete space as $\boldsymbol{u}_h$ in Lemma B.1 we know that

$$\|\boldsymbol{u}_h - \boldsymbol{U}_h\|^2_{L^2(\mathbb{T}^d)} \leq C_1 \left(\|\nabla_h (\boldsymbol{u}_h - \boldsymbol{U}_h)\|^2_{L^2(\mathbb{T}^d)} + \int_{\mathbb{T}^d} \varrho_h |\boldsymbol{u}_h - \boldsymbol{U}_h|^2 \,\mathrm{d}x \right),$$

where the constant C_1 depends on $c_M \equiv M$, $c_E \equiv E_0$ and γ. Note that the choices of c_M and c_E are owing to the mass conservation (2.12) and energy stability (2.13).

Next, for $\boldsymbol{u}_h \in Q_h$ and $\boldsymbol{u}_h \in \mathbf{W}_h$ we set $\boldsymbol{U}_h = \Pi_Q \boldsymbol{U} \in Q_h$ and $\boldsymbol{U}_h = \Pi_{\mathcal{E}} \boldsymbol{U} \in \mathbf{W}_h$, respectively. Then by the triangular inequality and projection error we derive

$$\begin{aligned}
&\|\boldsymbol{u}_h - \boldsymbol{U}\|^2_{L^2} \leq \|\boldsymbol{u}_h - \boldsymbol{U}_h\|^2_{L^2} + \|\boldsymbol{U}_h - \boldsymbol{U}\|^2_{L^2} \\
&\leq C_1 \left(\|\nabla_h (\boldsymbol{u}_h - \boldsymbol{U}_h)\|^2_{L^2(\mathbb{T}^d)} + \int_{\mathbb{T}^d} \varrho_h |\boldsymbol{u}_h - \boldsymbol{U}_h|^2 \,\mathrm{d}x \right) + (h \|\nabla_x \boldsymbol{U}\|_{L^2})^2 \\
&\leq C_1 \left(\|\nabla_h \boldsymbol{u}_h - \nabla_x \boldsymbol{U}\|^2_{L^2(\mathbb{T}^d)} + \int_{\mathbb{T}^d} \varrho_h |\boldsymbol{u}_h - \boldsymbol{U}|^2 \,\mathrm{d}x \right) \\
&\quad + C_1 \left(\|\nabla_x \boldsymbol{U} - \nabla_h \boldsymbol{U}_h\|^2_{L^2(\mathbb{T}^d)} + \int_{\mathbb{T}^d} \varrho_h |\boldsymbol{U}_h - \boldsymbol{U}|^2 \,\mathrm{d}x \right) + h^2 \|\nabla_x \boldsymbol{U}\|^2_{L^2} \\
&\leq C_1 \left(\|\nabla_h \boldsymbol{u}_h - \nabla_x \boldsymbol{U}\|^2_{L^2(\mathbb{T}^d)} + \int_{\mathbb{T}^d} \varrho_h |\boldsymbol{u}_h - \boldsymbol{U}|^2 \,\mathrm{d}x \right) \\
&\quad + C_1 \left(h^2 \|\boldsymbol{U}\|^2_{W^{2,\infty}} + h^2 \|\nabla_x \boldsymbol{U}\|^2_{L^\infty} \int_{\mathbb{T}^d} \varrho_h \,\mathrm{d}x \right) + h^2 \|\nabla_x \boldsymbol{U}\|^2_{L^2}
\end{aligned}$$

$$= C_1 \left(\|\nabla_h \boldsymbol{u}_h - \nabla_x \boldsymbol{U}\|^2_{L^2(\mathbb{T}^d)} + \int_{\mathbb{T}^d} \varrho_h |\boldsymbol{u}_h - \boldsymbol{U}|^2 \,\mathrm{d}x \right) + C_2 h^2,$$

where C_2 depends on $C_1, \|\boldsymbol{U}\|_{W^{2,\infty}}, \|\nabla_x \boldsymbol{U}\|_{L^\infty}, M$, and $\|\nabla_x \boldsymbol{U}\|_{L^2}$, which proves (B.1).

Finally, we proceed with the proof of (B.2). On the one hand, for the case of $\boldsymbol{u}_h \in Q_h$ we have $\Pi_Q \boldsymbol{u}_h = \boldsymbol{u}_h$, meaning (B.2) automatically holds as it is the same as (B.1). On the other hand, for the case of $\boldsymbol{u}_h \in \mathbf{W}_h$ we employ (B.1) and the triangular inequality to derive

$$\begin{aligned}
&\|\Pi_Q \boldsymbol{u}_h - \boldsymbol{U}\|^2_{L^2} \le \|\Pi_Q \boldsymbol{u}_h - \boldsymbol{u}_h\|^2_{L^2} + \|\boldsymbol{u}_h - \boldsymbol{U}\|^2_{L^2} \\
&\le h^2 \|\mathrm{div}_h \boldsymbol{u}_h\|^2_{L^2} + C_1 \left(\|\nabla_h \boldsymbol{u}_h - \nabla_x \boldsymbol{U}\|^2_{L^2(\mathbb{T}^d)} + \int_{\mathbb{T}^d} \varrho_h |\boldsymbol{u}_h - \boldsymbol{U}|^2 \,\mathrm{d}x \right) + C_2 h^2 \\
&\lesssim C_1 \left(\|\nabla_h \boldsymbol{u}_h - \nabla_x \boldsymbol{U}\|^2_{L^2(\mathbb{T}^d)} + \int_{\mathbb{T}^d} \varrho_h |\boldsymbol{u}_h - \boldsymbol{U}|^2 \,\mathrm{d}x \right) + C_2 h^2,
\end{aligned}$$

where we have used the fact that $\|\mathrm{div}_h \boldsymbol{u}_h\|^2_{L^2} \lesssim E_0$ in view of (2.14b), which completes the proof. □

Next, we recall [12, Lemma 14.3] in order to show the following statement formulated in Lemma B.4.

Lemma B.3 ([12])**.** *Let $\gamma > 1$, $\underline{r} = \frac{1}{2} \min_{(t,x)\in Q_T} r > 0$ and $\overline{r} = 2 \max_{(t,x)\in Q_T} r$. Then there exists $C = C(\underline{r}, \overline{r}) > 0$ such that*

$$(\varrho - r)^2 1_{\mathrm{ess}}(\varrho) + (1 + \varrho^\gamma) 1_{\mathrm{res}}(\varrho) \le C \mathbb{E}(\varrho | r),$$

where $\mathbb{E}(\varrho|r) = P(\varrho) - P'(r)(\varrho - r) - P(r)$ and

$$(1_{\mathrm{ess}}(\varrho), 1_{\mathrm{res}}(\varrho)) = \begin{cases} (1,0) & \text{if } \varrho \in [\underline{r}, \overline{r}], \\ (0,1) & \text{if } \varrho \in \mathbb{R}^+ \backslash [\underline{r}, \overline{r}]. \end{cases} \tag{B.3}$$

Now we are ready to show the following lemma.

Lemma B.4. *Let $(\varrho_h, \boldsymbol{u}_h)$ be a solution obtained either by the FV method* (2.10) *or the MAC method* (2.11)*, and let $\boldsymbol{U} \in L^\infty(0,T; W^{2,\infty}(\mathbb{T}^d; \mathbb{R}^d))$. Then there holds*

$$\int_0^\tau \int_{\mathbb{T}^d} |(\varrho_h - r)(\Pi_Q \boldsymbol{u}_h - \boldsymbol{U})| \,\mathrm{d}x\mathrm{d}t \le C_0 \int_0^\tau \mathfrak{E}(\varrho_h, \boldsymbol{u}_h | r, \boldsymbol{U}) \mathrm{d}t + C_1 \delta \|\nabla_h \boldsymbol{u}_h - \nabla_x \boldsymbol{U}\|^2_{L^2} + C_2 \delta h^2,$$

where C_1, C_2 are the same as in Lemma B.2, and C_0 depends on $\underline{r}, \overline{r}, \delta$, M, E_0, γ.

Proof. First, thanks to Lemma B.3 we observe

$$\begin{aligned}
&\int_0^\tau \int_{\mathbb{T}^d} 1_{\mathrm{res}}(\varrho_h)\, \varrho_h \,\mathrm{d}x\mathrm{d}t = \int_0^\tau \int_{\mathbb{T}^d} 1_{\varrho_h < \underline{r}}\, \varrho_h \,\mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\mathbb{T}^d} 1_{\varrho_h > \overline{r}}\, \varrho_h \,\mathrm{d}x\mathrm{d}t \\
&\le \underline{r} \int_0^\tau \int_{\mathbb{T}^d} 1_{\varrho_h < \underline{r}}\, 1 \,\mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\mathbb{T}^d} 1_{\varrho_h > \overline{r}}\, \varrho_h^\gamma \,\mathrm{d}x\mathrm{d}t \le C \int_0^\tau \mathfrak{E}(\varrho_h, \boldsymbol{u}_h | r, \boldsymbol{U}) \mathrm{d}t,
\end{aligned}$$

where $C = C(\underline{r}, \overline{r})$ is given in Lemma B.3.

Next, using the triangular inequality, Young's inequality, the above estimate, Lemma B.2 and Lemma B.3 we find

$$\int_0^\tau \int_{\mathbb{T}^d} |(\varrho_h - r)(\Pi_Q \boldsymbol{u}_h - \boldsymbol{U})| \,\mathrm{d}x\mathrm{d}t$$

$$
\begin{aligned}
&\le \int_0^\tau \int_{\mathbb{T}^d} 1_{\rm ess}(\varrho_h)|(\varrho_h - r)(\Pi_Q \boldsymbol{u}_h - \boldsymbol{U})| \,{\rm d}x{\rm dt} + \int_0^\tau \int_{\mathbb{T}^d} 1_{\varrho_h<\underline{r}}\overline{r}|\Pi_Q \boldsymbol{u}_h - \boldsymbol{U}| \,{\rm d}x{\rm dt} \\
&\quad + \int_0^\tau \int_{\mathbb{T}^d} 1_{\varrho_h>\overline{r}}\varrho_h|\Pi_Q \boldsymbol{u}_h - \boldsymbol{U}| \,{\rm d}x{\rm dt} \\
&\le \int_0^\tau \int_{\mathbb{T}^d} 1_{\rm ess}(\varrho_h)\frac{1}{2}\big((\varrho_h - r)^2 + \varrho_h|\Pi_Q \boldsymbol{u}_h - \boldsymbol{U}|^2/\underline{r}\big) \,{\rm d}x{\rm dt} \\
&\quad + \int_0^\tau \int_{\mathbb{T}^d} 1_{\varrho_h<\underline{r}}\frac{1}{2}\big(\frac{1}{\delta}\overline{r}^2 + \delta|\Pi_Q \boldsymbol{u}_h - \boldsymbol{U}|^2\big) \,{\rm d}x{\rm dt} \\
&\quad + \int_0^\tau \int_{\mathbb{T}^d} 1_{\varrho_h>\overline{r}}\frac{1}{2}\big(\varrho_h + \varrho_h|\Pi_Q \boldsymbol{u}_h - \boldsymbol{U}|^2\big) \,{\rm d}x{\rm dt} \\
&\lesssim C_0 \int_0^\tau \mathfrak{E}(\varrho_h, \boldsymbol{u}_h|r, \boldsymbol{U}){\rm dt} + C_1\delta\|\nabla_h \boldsymbol{u}_h - \nabla_x \boldsymbol{U}\|_{L^2}^2 + C_2\delta h^2,
\end{aligned}
$$

where C_0 depends on $\underline{r}$, $C(\underline{r}, \overline{r})$, δ, and C_1. We have completed the proof. □

C Relative energy norm

In this section we show how to control the errors in the conservative variables by the relative energy.

Lemma C.1. *Let $\gamma > 1$ and $(r, \boldsymbol{U})$ satisfy*

$$
\underline{r} = \frac{1}{2} \min_{(t,x)\in Q_T} r, \quad \overline{r} = 2 \max_{(t,x)\in Q_T} r, \quad \overline{U} = \max_{(t,x)\in Q_T} |\boldsymbol{U}|
$$

for some positive constants $\overline{u}, \underline{r}, \overline{r}$.

- *If $\varrho > 0$ and $\int_{\mathbb{T}^d} \varrho^\gamma \,{\rm d}x \le E_0$ hold, then*

$$
\|\varrho - r\|_{L^\gamma} + \|\boldsymbol{m} - \boldsymbol{M}\|_{L^{\frac{2\gamma}{\gamma+1}}} \lesssim (\mathfrak{E}(\varrho, \boldsymbol{u}|r, \boldsymbol{U}))^{\frac{1}{2}} + (\mathfrak{E}(\varrho, \boldsymbol{u}|r, \boldsymbol{U}))^{\frac{1}{\gamma}} \text{ for } \gamma \le 2; \tag{C.1a}
$$

$$
\|\varrho - r\|_{L^2} + \|\boldsymbol{m} - \boldsymbol{M}\|_{L^{\frac{2\gamma}{\gamma+1}}} \lesssim (\mathfrak{E}(\varrho, \boldsymbol{u}|r, \boldsymbol{U}))^{\frac{1}{2}} \text{ for } \gamma \ge 2, \tag{C.1b}
$$

  where $m = \varrho\boldsymbol{u}$ and $\boldsymbol{M} = r\boldsymbol{U}$.

- *In addition, let $\varrho < \overline{\varrho}$. Then*

$$
\|\varrho - r\|_{L^2} + \|\boldsymbol{m} - \boldsymbol{M}\|_{L^2} \lesssim (\mathfrak{E}(\varrho, \boldsymbol{u}|r, \boldsymbol{U}))^{\frac{1}{2}}. \tag{C.2}
$$

Proof. First, by the triangular inequality and Lemma B.3 we obtain for $\gamma \le 2$ that

$$
\begin{aligned}
&\|\varrho - r\|_{L^\gamma} \le \|(\varrho - r)1_{\rm ess}(\varrho)\|_{L^\gamma} + \|(\varrho - r)1_{\rm res}(\varrho)\|_{L^\gamma} \lesssim \|(\varrho - r)1_{\rm ess}(\varrho)\|_{L^2} + \|(\varrho - r)1_{\rm res}(\varrho)\|_{L^\gamma} \\
&\lesssim \big(\mathbb{E}(\varrho|r)\big)^{1/2} + (\|\varrho\|_{L^\gamma} + \|r\|_{L^\gamma})\, 1_{\rm res}(\varrho) \lesssim \big(\mathbb{E}(\varrho|r)\big)^{1/2} + \left(\int_{\mathbb{T}^d} \varrho^\gamma\, 1_{\rm res}(\varrho) \,{\rm d}x\right)^{1/\gamma} + \left(\int_{\mathbb{T}^d} 1_{\rm res}(\varrho) \,{\rm d}x\right)^{1/\gamma} \\
&\lesssim \big(\mathbb{E}(\varrho|r)\big)^{1/2} + \big(\mathbb{E}(\varrho|r)\big)^{1/\gamma} \le \big(\mathfrak{E}(\varrho, \boldsymbol{u}|r, \boldsymbol{U})\big)^{1/2} + \big(\mathfrak{E}(\varrho, \boldsymbol{u}|r, \boldsymbol{U})\big)^{1/\gamma},
\end{aligned}
$$

where $1_{\rm ess}(\varrho)$ and $1_{\rm res}(\varrho)$ are given in Lemma B.3. Further, utilizing the above estimate with the triangular inequality, Hölder's inequality, and the L^γ bound on ϱ, we find

$$
\|\boldsymbol{m} - \boldsymbol{M}\|_{L^{\frac{2\gamma}{\gamma+1}}} \le \|\varrho(\boldsymbol{u} - \boldsymbol{U})\|_{L^{\frac{2\gamma}{\gamma+1}}} + \|(\varrho - r)\boldsymbol{U}\|_{L^{\frac{2\gamma}{\gamma+1}}} \lesssim \|\sqrt{\varrho}\|_{L^{2\gamma}}\|\sqrt{\varrho}(\boldsymbol{u} - \boldsymbol{U})\|_{L^2} + \|\varrho - r\|_{L^\gamma}\|\boldsymbol{U}\|_{L^{\frac{2\gamma}{\gamma-1}}}
$$

$$\lesssim \|\varrho\|_{L^\gamma}^{1/2}\|\varrho|\boldsymbol{u}-\boldsymbol{U}|^2\|_{L^1}^{1/2} + \|\varrho - r\|_{L^\gamma}\|\boldsymbol{U}\|_{L^\infty} \lesssim \big(\mathfrak{E}(\varrho,\boldsymbol{u}|r,\boldsymbol{U})\big)^{1/2} + \big(\mathfrak{E}(\varrho,\boldsymbol{u}|r,\boldsymbol{U})\big)^{1/\gamma}$$

which proves (C.1a).

Next, again by the triangular inequality and Lemma B.3 we observe for $\gamma \geq 2$ that

$$\begin{aligned}
\|\varrho - r\|_{L^2} &\leq \|(\varrho - r)1_{\rm ess}(\varrho)\|_{L^2} + \|(\varrho - r)1_{\rm res}(\varrho)\|_{L^2} \\
&\lesssim \big(\mathbb{E}(\varrho|r)\big)^{1/2} + \left(\int_{\mathbb{T}^d} \varrho^2\, 1_{\varrho>\bar{r}}\,\mathrm{d}x\right)^{1/2} + \left(\int_{\mathbb{T}^d} 1_{\rm res}(\varrho)\,\mathrm{d}x\right)^{1/2} \\
&\lesssim \big(\mathbb{E}(\varrho|r)\big)^{1/2} + \left(\int_{\mathbb{T}^d} \varrho^\gamma\, 1_{\varrho>\bar{r}}\,\mathrm{d}x\right)^{1/2} \lesssim \big(\mathbb{E}(\varrho|r)\big)^{1/2},
\end{aligned}$$

where we have used the fact that $\varrho^2 \leq \varrho^\gamma$ for large ϱ with $\gamma \geq 2$. Further, it is easy to check that

$$\begin{aligned}
&\|\boldsymbol{m} - \boldsymbol{M}\|_{L^{\frac{2\gamma}{\gamma+1}}} \leq \|\varrho(\boldsymbol{u}-\boldsymbol{U})\|_{L^{\frac{2\gamma}{\gamma+1}}} + \|(\varrho - r)\boldsymbol{U}\|_{L^{\frac{2\gamma}{\gamma+1}}} \\
&\lesssim \|\sqrt{\varrho}\|_{L^{2\gamma}}\|\sqrt{\varrho}(\boldsymbol{u}-\boldsymbol{U})\|_{L^2} + \|\varrho - r\|_{L^2}\|\boldsymbol{U}\|_{L^{2\gamma}} \\
&\lesssim \|\varrho\|_{L^\gamma}^{1/2}\|\varrho|\boldsymbol{u}-\boldsymbol{U}|^2\|_{L^1}^{1/2} + \|\varrho - r\|_{L^2}\|\boldsymbol{U}\|_{L^\infty} \lesssim \big(\mathfrak{E}(\varrho,\boldsymbol{u}|r,\boldsymbol{U})\big)^{1/2}
\end{aligned}$$

which proves (C.1b).

When assuming an upper bound on ϱ, we derive via Lemma B.3 that

$$\|\varrho - r\|_{L^2} \leq \|(\varrho - r)1_{\rm ess}(\varrho)\|_{L^2} + \|(\varrho - r)1_{\rm res}(\varrho)\|_{L^2} \lesssim \big(\mathbb{E}(\varrho|r)\big)^{1/2} + \|1_{\rm res}(\varrho)\|_{L^2} \lesssim \big(\mathbb{E}(\varrho|r)\big)^{1/2}$$

which implies

$$\begin{aligned}
&\|\boldsymbol{m} - \boldsymbol{M}\|_{L^2} \leq \|\varrho(\boldsymbol{u}-\boldsymbol{U})\|_{L^2} + \|(\varrho - r)\boldsymbol{U}\|_{L^2}\|\sqrt{\varrho}\|_{L^\infty}\|\sqrt{\varrho}(\boldsymbol{u}-\boldsymbol{U})\|_{L^2} + \|\varrho - r\|_{L^2}\|\boldsymbol{U}\|_{L^\infty} \\
&\lesssim \big(\mathfrak{E}(\varrho,\boldsymbol{u}|r,\boldsymbol{U})\big)^{1/2}.
\end{aligned}$$

Combining the above two estimates we get (C.2) and complete the proof. □

D Derivation of the relative energy

In this section we show the relative energy inequality (3.5). We start with the reformulation of the relative energy.

$$\mathfrak{E}(\varrho_h,\boldsymbol{u}_h|r,\boldsymbol{U}) = \int_{\mathbb{T}^d} \left(\frac{1}{2}\varrho_h\left|\Pi_Q\boldsymbol{u}_h - \boldsymbol{U}\right|^2 + P(\varrho_h) - P'(r)(\varrho_h - r) - P(r)\right)\mathrm{d}x = \sum_{i=1}^4 T_i,$$

where

$$\begin{aligned}
T_1 &= \int_{\mathbb{T}^d}\left(\frac{1}{2}\varrho_h\left|\Pi_Q\boldsymbol{u}_h\right|^2 + P(\varrho_h)\right)\mathrm{d}x, &\quad T_2 &= \int_{\mathbb{T}^d}\varrho_h\left(\frac{1}{2}\left|\boldsymbol{U}\right|^2 - P'(r)\right)\mathrm{d}x, \\
T_3 &= -\int_{\mathbb{T}^d}\varrho_h\Pi_Q\boldsymbol{u}_h\cdot\boldsymbol{U}\,\mathrm{d}x, &\quad T_4 &= \int_{\mathbb{T}^d}(rP'(r) - P(r))\,\mathrm{d}x.
\end{aligned}$$

Next, we collect the energy estimate (2.13), and set the test function $\phi = \left(\frac{1}{2}\left|\boldsymbol{U}\right|^2 - P'(r)\right)$ in the consistency formulation (2.15a), as well as $\boldsymbol{\phi} = -\boldsymbol{U}$ in the consistency formulation (2.15b) to get respectively the following

$$[T_1]_{t=0}^\tau = \left[\int_{\mathbb{T}^d}\left(\frac{1}{2}\varrho_h\left|\Pi_Q\boldsymbol{u}_h\right|^2 + P(\varrho_h)\right)\mathrm{d}x\right]_{t=0}^\tau \leq -\mu\int_0^\tau\int_{\mathbb{T}^d}\left|\nabla_h\boldsymbol{u}_h\right|^2\mathrm{d}x\mathrm{d}t - \nu\int_0^\tau\int_{\mathbb{T}^d}\left|\mathrm{div}_h\boldsymbol{u}_h\right|^2\mathrm{d}x\mathrm{d}t,$$

$$
[T_2]_{t=0}^{\tau} = \left[\int_{\mathbb{T}^d} \varrho_h \underbrace{\left(\frac{1}{2} |\boldsymbol{U}|^2 - P'(r) \right)}_{\text{test function in (2.15a)}} \mathrm{d}x \right]_{t=0}^{\tau}
$$

$$
= \int_0^\tau \int_{\mathbb{T}^d} \left(\varrho_h \partial_t \frac{|\boldsymbol{U}|^2}{2} + \varrho_h \Pi_Q \boldsymbol{u}_h \cdot \nabla_x \frac{|\boldsymbol{U}|^2}{2} \right) \mathrm{d}x\mathrm{d}t + e_\varrho \left(\tau, \Delta t, h, |\boldsymbol{U}|^2 /2 \right)
$$

$$
- \int_0^\tau \int_{\mathbb{T}^d} \left(\varrho_h \partial_t P'(r) + \varrho_h \Pi_Q \boldsymbol{u}_h \cdot \nabla_x P'(r) \right) \mathrm{d}x\mathrm{d}t - e_\varrho(\tau, \Delta t, h, P'(r)),
$$

$$
[T_3]_{t=0}^{\tau} = \left[\int_{\mathbb{T}^d} \varrho_h \Pi_Q \boldsymbol{u}_h \cdot \underbrace{(-\boldsymbol{U})}_{\text{test function in (2.15b)}} \mathrm{d}x \right]_{t=0}^{\tau}
$$

$$
= - \int_0^\tau \int_{\mathbb{T}^d} \left(\varrho_h \Pi_Q \boldsymbol{u}_h \cdot \partial_t \boldsymbol{U} + \varrho_h \Pi_Q \boldsymbol{u}_h \otimes \Pi_Q \boldsymbol{u}_h : \nabla_x \boldsymbol{U} + p_h \mathrm{div}_x \boldsymbol{U} \right) \mathrm{d}x\mathrm{d}t
$$

$$
+ \int_0^\tau \int_{\mathbb{T}^d} \left(\mu \nabla_h \boldsymbol{u}_h : \nabla_x \boldsymbol{U} + \nu \mathrm{div}_h \boldsymbol{u}_h \mathrm{div}_x \boldsymbol{U} \right) \mathrm{d}x\mathrm{d}t + e_{\boldsymbol{m}}(\tau, \Delta t, h, -\boldsymbol{U}).
$$

Moreover, the term T_4 reads

$$
[T_4]_{t=0}^{\tau} = \left[\int_{\mathbb{T}^d} \left(rP'(r) - P(r) \right) \mathrm{d}x \right]_{t=0}^{\tau} = \int_0^\tau \int_{\mathbb{T}^d} \partial_t (rP'(r) - P(r))\, \mathrm{d}x\mathrm{d}t.
$$

Summing up the above terms we get (3.5)

$$
[\mathfrak{E}(\varrho_h, \boldsymbol{u}_h | r, \boldsymbol{U})]_0^T + \int_0^\tau \int_{\mathbb{T}^d} \left(\mu \left| \nabla_h \boldsymbol{u}_h \right|^2 + \nu \left| \mathrm{div}_h \boldsymbol{u}_h \right|^2 \right) \mathrm{d}x\mathrm{d}t
$$

$$
\le \int_0^\tau \int_{\mathbb{T}^d} \left(\varrho_h \partial_t \frac{|\boldsymbol{U}|^2}{2} + \varrho_h \Pi_Q \boldsymbol{u}_h \cdot \nabla_x \frac{|\boldsymbol{U}|^2}{2} \right) \mathrm{d}x\mathrm{d}t + e_\varrho \left(\tau, \Delta t, h, |\boldsymbol{U}|^2 /2 \right)
$$

$$
- \int_0^\tau \int_{\mathbb{T}^d} \left(\varrho_h \partial_t P'(r) + \varrho_h \Pi_Q \boldsymbol{u}_h \cdot \nabla_x P'(r) \right) \mathrm{d}x\mathrm{d}t - e_\varrho(\tau, \Delta t, h, P'(r))
$$

$$
- \int_0^\tau \int_{\mathbb{T}^d} \left(\varrho_h \Pi_Q \boldsymbol{u}_h \cdot \partial_t \boldsymbol{U} + \varrho_h \Pi_Q \boldsymbol{u}_h \otimes \Pi_Q \boldsymbol{u}_h : \nabla_x \boldsymbol{U} + p_h \mathrm{div}_x \boldsymbol{U} \right) \mathrm{d}x\mathrm{d}t
$$

$$
+ \int_0^\tau \int_{\mathbb{T}^d} \left(\mu \nabla_h \boldsymbol{u}_h : \nabla_x \boldsymbol{U} + \nu \mathrm{div}_h \boldsymbol{u}_h \mathrm{div}_x \boldsymbol{U} \right) \mathrm{d}x\mathrm{d}t + e_{\boldsymbol{m}}(\tau, \Delta t, h, -\boldsymbol{U})
$$

$$
+ \int_0^\tau \int_{\mathbb{T}^d} \partial_t (rP'(r) - P(r))\, \mathrm{d}x\mathrm{d}t.
$$

References

[1] D. Breit, E. Feireisl, and M. Hofmanová. Local strong solutions to the stochastic compressible Navier–Stokes system. *Comm. Partial Differential Equations*, **43**(2):313–345, 2018.

[2] V. Dolejší and M. Feistauer. *Discontinuous Galerkin Method.* Volume 48 of Springer Series in Computational Mathematics, Springer, 2015.

[3] M. Feistauer. *Mathematical Methods in Fluid Dynamics.* Volume 67 of Pitman Monographs and Surveys in Pure and Applied Mathematics, Longman Scientific & Technical, Harlow, 1993.

[4] M. Feistauer, J. Felcman, I. Straškraba. *Mathematical and Computational Methods for Compressible Flow.* The Clarendon Press, Oxford University Press, 2003.

[5] R. Eymard, T. Gallouët, and R. Herbin. Finite volume methods. *Handbook of numerical analysis* **7**: 713–1018, 2000.

[6] E. Feireisl, R. Hošek, D. Maltese, and A. Novotný. Error estimates for a numerical method for the compressible Navier–Stokes system on sufficiently smooth domains. *ESAIM Math. Model. Numer. Anal.* **51**(1): 279–319, 2017.

[7] E. Feireisl. *Dynamics of Viscous Compressible Fluids.* Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, 2004.

[8] E. Feireisl, B.J. Jin, and A. Novotný. Relative entropies, suitable weak solutions, and weak strong uniqueness for the compressible Navier–Stokes system. *J. Math. Fluid Mech.* **14**(4): 717–730, 2012.

[9] E. Feireisl, T. G. Karper, and M. Pokorný, *Mathematical Theory of Compressible Viscous Fluids: Analysis and Numerics.* Birkhäuser–Verlag, Basel 2017.

[10] E. Feireisl, and M. Lukáčová-Medviďová. Convergence of a mixed finite element–finite volume scheme for the isentropic Navier–Stokes system via the dissipative measure–valued solutions. *Found. Comput. Math.* **18**(3): 703–730, 2018.

[11] E. Feireisl, M. Lukáčová-Medviďová, H. Mizerová, and B. She. Convergence of a finite volume scheme for the compressible Navier–Stokes system. *ESAIM: M2AN* **53**(6): 1957–1979, 2019.

[12] E. Feireisl, M. Lukáčová–Medviďová, H. Mizerová, and B. She. *Numerical Analysis of Compressible Fluid Flows.* Springer–Verlag, 2021.

[13] I. Gallagher. A remark on smooth solutions of the weakly compressible periodic Navier–Stokes equations. *J. Math. Kyoto Univ.* **40**(3): 525–540, 2000.

[14] T. Gallouët, D. Maltese, and A. Novotný. Error estimates for the implicit MAC scheme for the compressible Navier–Stokes equations. *Numer. Math.* **141**: 495–567, 2019.

[15] T. Gallouët, R. Herbin, D. Maltese, and A. Novotný. Error estimates for a numerical approximation to the compressible barotropic Navier–Stokes equations. *IMA J. Numer. Anal.* **36**(2): 543–592, 2016.

[16] R. Hošek and B. She. Stability and consistency of a finite difference scheme for compressible viscous isentropic flow in multi-dimension. *J. Numer. Math.* **26**(3): 111–140, 2018.

[17] V. Jovanović. An error estimate for a numerical scheme for the compressible Navier–Stokes system. *Kragujevac J. Math.* **30**: 263–275, 2007.

[18] T. Karper. A convergent FEM-DG method for the compressible Navier–Stokes equations. *Numer. Math.* **125**(3): 441–510, 2013.

[19] Y. Kwon and A. Novotný. Consistency, convergence and error estimates for a mixed finite element–finite volume scheme to compressible Navier–Stokes equations with general inflow/outflow boundary data. *IMA J. Numer. Anal.* **42**(1): 107–164, 2022.

[20] P.L. Lions. Mathematical topics in fluid mechanics. Vol. 2: Compressible models. Oxford University Press, 1998.

[21] B. Liu. The analysis of a finite element method with streamline diffusion for the compressible Navier–Stokes equations. *SIAM J. Numer. Anal.* **38**:1–16, 2000.

[22] B. Liu. On a finite element method for three-dimensional unsteady compressible viscous flows. *Numer. Methods Partial Differ. Eq.* **20**: 432–449, 2004.

[23] H. Mizerová and B. She. Convergence and error estimates for a finite difference scheme for the multi-dimensional compressible Navier–Stokes system. *J. Sci. Comput.* **84**(1): No.25, 2020.

[24] Y. Sun, C. Wang, and Z. Zhang. A Beale–Kato–Majda blow–up criterion for the 3-D compressible Navier–Stokes equations. *J. Math. Pures. Appl.* **95**(1): 36–47, 2011.

[25] A. Valli and M. Zajaczkowski. Navier–Stokes equations for compressible fluids: Global existence and qualitative properties of the solutions in the general case. *Commun. Math. Phys.* **103**: 259–296,1986.

[26] P.I. Plotnikov and W. Weigant. Isothermal Navier–Stokes equations and Radon transform. *SIAM J. Math. Anal.* **47**(1): 626–653, 2015.

[27] E.F. Toro. *Riemann Solvers and Numerical Methods for Fluid Dynamics. A practical introduction.* Springer, 2009.