INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

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Preprint No. 64-2019

PRAHA 2019

JOINT NUMERICAL RANGE OF COMMUTING TUPLES IS NOT CONVEX

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Abstract. The result in the title is proved.

Let H be a Hilbert space and $T_1, \dots, T_n$ an n-tuple of bounded linear operators acting on H. Denote by $W(T_1, \dots, T_n)$ the joint numerical radius defined by

$$W(T_1, \dots, T_n) = \big\{ (\langle T_1 x, x \rangle, \dots, \langle T_n x, x \rangle) : x \in H, \|x\| = 1 \big\}.$$

By the classical result of Hausdorff and Toeplitz, the numerical range of a single Hilbert space operator is always a convex set. However, it is well known that for $n \geq 2$ the joint numerical range can be non-convex. Apparently, Hausdorff knew this already in 1918; for a simple example see e.g. [4], p.138.

The convexity of various types of (joint) numerical ranges is the basic question and has been studied intensely, see e.e. [3], [5], [6], [7], [8] and references therein. In many of these results, the commutativity of the operators played an important role. For example, it is well known that the joint numerical range of each commuting tuple of normal operators is convex, while there are non-commuting tuples of selfadjoint operators with non-convex joint numerical range even in a two-dimensional space. In [3], the convexity of the joint numerical range of doubly commuting matrices was proved. It is also well known that spectral properties of commuting tuples are much better than those of non-commuting tuples. So there was a hope that the joint numerical range of a commuting n-tuple of operators might be convex in general. Surprisingly, this question remained open for a long time.

The purpose of this note is to fill this gap and show that the joint numerical range of commuting tuples is not convex in general.

Example 1. There exists a triple of mutually commuting Hilbert space operators such that their joint numerical range is not convex.

1991 *Mathematics Subject Classification.* Primary 47A05; Secondary 47A16.
Key words and phrases. joint numerical range, commuting tuple, non-convexity.
This work was supported by the grant No. 17-27844S of GA CR and RVO:67985840.

Proof. Let $H=\mathbb{C}^4$ with the standard basis e_1,e_2,e_3,e_4. Let

$$T_1=\begin{pmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad T_2=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\quad\text{and}\quad T_3=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix}.$$

Let H_0 be the two-dimensional subspace spanned by e_1 and e_2. Clearly $R(T_j)\subset H_0$ and $N(T_j)\supset H_0$ for all $j=1,2,3$ (where $R(T_j)$ denotes the range and $N(T_j)$ the kernel of T_j, respectively). So we have $T_jT_k=0$ for all $j,k=1,2,3$. In particular, the operators T_1,T_2,T_3 are mutually commuting.

We show that the numerical range of the triple (T_1,T_2,T_3) is not convex. Clearly

$$W(T_1,T_2,T_3)=\bigl\{(c\bar a,d\bar a,c\bar b):a,b,c,d\in\mathbb{C},|a|^2+|b|^2+|c|^2+|d|^2=1\bigr\}.$$

In particular, for $a=0,b=\sqrt2/2,c=\sqrt2/2,d=0$ we have $(0,0,1/2)\in W(T_1,T_2,T_3)$. Similarly, for $a=\sqrt2/2,b=0=c,d=\sqrt2/2$ we have $(0,1/2,0)\in W(T_1,T_2,T_3)$.

We show that the midpoint $(0,1/4,1/4)$ does not belong to $W(T_1,T_2,T_3)$. Suppose on the contrary that there exist $a,b,c,d\in\mathbb{C}$ with $|a|^2+|b|^2+|c|^2+|d|^2=1$ and $c\bar a=0$, $d\bar a=1/4$ and $c\bar b=1/4$. So either $c=0$ or $a=0$. If $c=0$ then $c\bar b=0$, a contradiction. If $a=0$ then $d\bar a=0$, again a contradiction. So $(0,1/4,1/4)\notin W(T_1,T_2,T_3)$ and $W(T_1,T_2,T_3)$ is not convex.

Problem 2. Is the joint numerical range of a pair of commuting Hilbert space operators always convex?

It was proved in [2] that the joint numerical range of a commuting tuple of operators on a two-dimensional Hilbert space is convex. The three-dimensional case remains open.

Problem 3. Let $(T_1,\dots,T_n)$ be a commuting n-tuple of 3×3 matrices. Is their joint numerical range $W(T_1,\dots,T_n)$ convex?

Remark 4. In [1], E. Asplund and V. Ptak considered another numerical range. If $T_1,\dots,T_n$ are operators acting on a Hilbert space H, then define

$$\widetilde W(T_1,\dots,T_n)=\bigl\{(\langle T_1x,y\rangle,\dots,\langle T_nx,y\rangle):x,y\in H,\|x\|\le1,\|y\|\le1\bigr\}.$$

They proved that $\widetilde W(T_1,T_2)$ is convex for each pair of Hilbert space operators T_1,T_2.

In fact the matrices T_1,T_2,T_3 constructed in Example 1 show that in general $\widetilde W(T_1,T_2,T_3)$ is not convex, even for mutually commuting operators T_1,T_2,T_3.

Theorem 5. Let T_1,T_2,T_3 be the matrices considered in Example 1. Then $\widetilde W(T_1,T_2,T_3)$ is not convex.

Proof. The proof is analogous. We have

$$\widetilde{W}(T_1,T_2,T_3)=\big\{(c\bar{a}',d\bar{a}',c\bar{b}'):a,b,c,d,a',b',c',d'\in\mathbb{C},|a|^2+|b|^2+|c|^2+|d|^2\le 1,\\ |a'|^2+|b'|^2+|c'|^2+|d'|^2\le 1\big\}.$$

As in Example 1, we have $(0,0,1/2)\in\widetilde{W}(T_1,T_2,T_3)$, $(0,1/2,0)\in\widetilde{W}(T_1,T_2,T_3)$ and $(0,1/4,1/4)\notin\widetilde{W}(T_1,T_2,T_3)$. So $\widetilde{W}(T_1,T_2,T_3)$ is not convex.

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