INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

Inverse problems for symmetric doubly stochastic matrices whose Suleĭmanova spectra are bounded below by 1/2

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Preprint No. 60-2019

PRAHA 2019

INVERSE PROBLEMS FOR SYMMETRIC DOUBLY STOCHASTIC MATRICES WHOSE SULEĬMANOVA SPECTRA ARE BOUNDED BELOW BY 1/2

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Abstract. A new sufficient condition for a list of real numbers to be the spectrum of a symmetric doubly stochastic matrix is presented; this is a contribution to the classical spectral inverse problem for symmetric doubly stochastic matrices that is still open in its full generality. It is proved that whenever $\lambda_2, \ldots, \lambda_n$ are non-positive real numbers with $1 + \lambda_2 + \ldots + \lambda_n \geqslant 1/2$, then there exists a symmetric, doubly stochastic matrix whose spectrum is precisely $(1, \lambda_2, \ldots, \lambda_n)$. We point out that this criterion is incomparable to the classical sufficient conditions due to Perfect–Mirsky, Soules, and their modern refinements due to Nadar *et al.* We also provide some examples and applications of our results.

1. Introduction

A square matrix with real entries is termed *stochastic*, when it has all entries non-negative and each row adds up to 1. Stochastic matrices are conveniently interpreted as transition matrices of finite-state Markov chains (hence the terminology). The aim of this note is to consider the inverse eigenvalue problem for doubly stochastic matrices (also called *bistochastic* in the literature; a stochastic matrix is *doubly stochastic* if its transpose is stochastic too). Doubly stochastic matrices may be interpreted as transition matrices of finite-state symmetric Markov chains. Permutation matrices are a paradigm example of a class of doubly stochastic matrices; according to the Birkhoff—von Neumann theorem, the set of $n \times n$ doubly stochastic matrices is the convex hull of the set of permutation matrices of an n-element set.

Inverse eigenvalue problems for classes of matrices such as (symmetric or not) matrices with non-negative entries, stochastic, or doubly stochastic are well-rooted in the literature, having their origin in the works of Suleĭmanova [19, 20] and, independently, Perfect ([14, 15]) with an important subsequent continuation by Perfect and Mirsky [16]. Recently, the problem has gained new impetus as reflected by a plethora of new sufficient conditions ([2, 5, 6, 7, 10, 11]). We refer to Mourad's paper [9] for a good overview concerning the said problems.

Date: October 18, 2019.

2010 *Mathematics Subject Classification.* 65F18 (primary), and 15A18, 15A12 (secondary).

Key words and phrases. doubly stochastic matrix, bistochastic matrix, inverse problem, SDIEP, Suleĭmanova spectrum.

More explicitly, the symmetric doubly stochastic eigenvalue inverse problem (SDIEP) asks the following:

Let $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ be a list of real numbers. In what circumstances does there exist a symmetric doubly stochastic matrix whose spectrum consists of these numbers?

Quite naturally, the inverse problems for other classes of matrices are formulated totally analogously. In this short note, we focus only on providing a new sufficient condition to address SDIEP.

Since any matrix A solving the above problem has non-negative entries, by the classical Frobenius–Perron theorem, A must have a non-negative eigenvalue λ_1 (that is called *the Perron eigenvalue* of A) such that $\lambda_1 \geqslant |\lambda|$ for any other eigenvalue λ of A (and the eigenvector associated to λ_1 has positive entries, it is also known as *the Perron eigenvector*). It is to be noted that already for stochastic matrices, $\lambda_1 = 1$ is the Perron eigenvalue to which corresponds the eigenvector comprising only 1s. Consequently, without loss of generality we shall restrict ourselves to λs from the interval $[-1, 1]$.

Since the trace of (any power of) a square matrix with non-negative entries is non-negative, for a list $(1, \lambda_2, \ldots, \lambda_n)$ of real numbers to form a spectrum of a solution to SDIEP, it is necessary that $1 + \lambda_2^k + \ldots + \lambda_n^k \geqslant 0$ for any $k \in \mathbb{N}$. Perfect and Mirsky [16] provided a fairly general sufficient condition for the possibility of solving SDIEP for a list of real numbers $1 = \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant -1$. Namely, they proved that as long as

$$\frac{1}{n} + \frac{1}{n(n-1)}\lambda_2 + \ldots + \frac{1}{2\cdot 1}\lambda_n \geqslant 0, \tag{1.1}$$

there exists a symmetric doubly stochastic $n \times n$ matrix whose spectrum is equal to $\{1, \lambda_2, \ldots, \lambda_n\}$. This condition was subsequently refined by Soules [18], who gave a finer condition depending on the remainder of n when divided by 2.

More specifically, let m be such that $n = 2m + 1$ in the case n is odd and $n = 2m + 2$ in the case where n is even. If

$$\frac{1}{n} + \frac{n-m-1}{n(m+1)}\lambda_2 + \sum_{k=1}^{n} \frac{1}{(k+1)k}\lambda_{n-2k+2} \geqslant 0, \tag{1.2}$$

then there exists a symmetric doubly stochastic $n \times n$ matrix whose spectrum coincides with $\{1, \lambda_2, \ldots, \lambda_n\}$.

Soules' condition was refined further by Nader *et al.* who arrived at a more complicated condition that depends on the remainder $n \bmod 4$ ([11, Theorem 5]) that covers a large class of cases.

Following the terminology introduced by Paparella [13], we call a list of real numbers $\sigma = (1, \lambda_2, \ldots, \lambda_n)$ a *normalised Suleĭmanova spectrum*, whenever $\lambda_j \leqslant 0$ for $j = 2, \ldots, n$ and $1 + \lambda_2 + \ldots + \lambda_n \geqslant 0$.

Fiedler ([4, Theorem 2.4]) showed that normalised Suleĭmanova spectra are realisable by symmetric non-negative matrices. Soto and Ccapa ([17, Theorem 3.3]) proved that as long as $\lambda_j < 0$ for $j = 2, \ldots, n$ every normalised Suleĭmanova spectrum is realisable as a spectrum of a stochastic matrix (not necessarily symmetric).

Johnson and Paparella ([6, Problem 6.2. and Theorem 6.3.]) proved, among other things, that every normalised Suleĭmanova spectrum is realisable as a spectrum of a symmetric doiubly stochastic matrix for all Hadamard orders (recall that the order of a Hadamard matrix must be 1, 2 or a multiple of 4). In particular, when $n = 2^k$ for some k, the resulting matrix A is trisymmetric ([6, Corollary 6.5]), that is, it satisfies any two of the following three conditions: *symmetric*, *persymmetric* ($AK = KA^T$), or *centrosymmetric* ($AK = KA$), where K is the exchange matrix (backward identity); satisfying any two of the above three conditions always implies that the remaining third condition holds.

However it is to be noted that a normalised Suleĭmanova spectrum with so that $1 + \lambda_2 + \ldots \lambda_n = 0$ may fail to be realisable within the class of symmetric doubly stochastic matrices. Indeed, if n is an odd number, *e.g.*, the list $(1, 0, 0, \ldots, 0, -1)$ cannot be a spectrum of any symmetric doubly stochastic matrix ([11, Corollary 1]).

For the sake of brevity, we call a Suleĭmanova spectrum $\sigma = (1, \lambda_2, \ldots, \lambda_n)$ *δ-normalised* ($\delta > 0$), whenever

$$1 + \lambda_2 + \ldots + \lambda_n \geqslant \delta \tag{1.3}$$

and $\lambda_j \leqslant 0$ for $j = 2, \ldots, n$.

The main result of this note is to prove that 1/2-normalised Suleĭmanova spectra may be indeed realised as spectra of symmetric, doubly stochastic matrices.

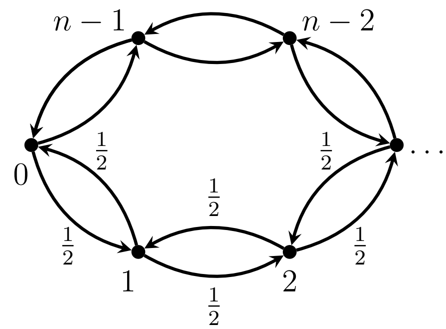
Theorem. *Let $\sigma = (1, \lambda_2, \ldots, \lambda_n)$ be a 1/2-normalised Suleĭmanova spectrum, that is, $\lambda_2, \ldots, \lambda_n \leqslant 0$ and $1 + \lambda_2 + \ldots + \lambda_n \geqslant 1/2$. Then there exists a symmetric, doubly stochastic matrix whose spectrum is precisely σ.*

Remark 1.1. It may be worth mentioning that the above result has a natural counterpart when all the eigenvalues are non-negative. That is, whenever $\lambda_j \geqslant 0$ for $j = 2, \ldots, n$ one may want to consider the condition

$$1 + \lambda_2 + \ldots + \lambda_n \leqslant \gamma, \tag{1.4}$$

where $\gamma \geqslant 1$. In this case, we show that if $\sigma = (1, \lambda_2, \ldots, \lambda_n)$ satisfies (1.4) with $\gamma = \frac{3}{2}$, then there is a symmetric, doubly stochastic matrix with σ as its spectrum.

We discern that the idea for the proof has its origins in the theory of Markov chains. Roughly speaking, we consider a simple (symmetric) random walk on the discrete torus $\mathbb{Z}/n\mathbb{Z}$ ($n \in \mathbb{N}$) represented by the following graph that we denote by S_n:



The crux of the proof is to use the eigenvectors of the transition probability matrix P_n of the above Markov chain as a *generator* for a class of new symmetric, doubly stochastic matrices. More specifically, we construct the desired matrices as Schur forms of the type $Q\Lambda Q^T$, where the column vectors of Q are suitably chosen eigenvectors of P_n and Λ is a diagonal matrix containing the Suleĭmanova spectrum. This very idea is not specific to Q, however it exploits the properties of Q, that is, symmetry and orthogonality. Such approach offers a room for improvement—we address this further in Remark 2.8.

By appealing to this heuristics and backed with some numerical evidence we raise the following problem.

Problem 1.2. Let $\delta > 0$. Can every δ-normalised Suleĭmanova spectrum be realised as the spectrum of a symmetric doubly stochastic matrix?

The main result of the note asserts that the answer is positive for $\delta \geqslant 1/2$.

2. Proof of the main result

We consider the symmetric random walk on the graph S_n described in the Introduction. The corresponding transition probability matrix P_n is then given by

$$P_n = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \cdots & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 & 0 \\ & \ddots & \ddots & \ddots & & & & \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & 0 \end{bmatrix}.$$

Notation. In this section, we index the eigenvalues to start from 0 rather than 1 so that $\sigma = (\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$ rather than $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ as done in the previous section. This assures that the formulae in this section are clearer and more compact.

A rote calculation shows that the eigenvalues λ_k and the corresponding (real) eigenvectors u_k ($k \in \{0, \ldots, n-1\}$) of P_n are given by

$$\lambda_k = \cos\Big(\frac{2\pi k}{n}\Big), \qquad u_k^{(j)} = \cos\Big(\frac{2\pi kj}{n}\Big) \qquad (k, j \in \{0, \ldots, n-1\}) \tag{2.1}$$

(for details see, *e.g.*, [8, 12.3.1]). Due to the symmetry, $\lambda_k = \lambda_{n-k}$ and $u_k = u_{n-k}$, so the eigenvectors of P_n fail to span $\mathbb{R}^n$; in the next lemma we find another set of eigenvectors corresponding to P_n that do indeed form an orthonormal basis of $\mathbb{R}^n$.

Lemma 2.1. *Let $\lambda_0, \ldots, \lambda_{n-1}$ be the eigenvalues of P_n as found in* (2.1). *Set*

$$w_k^{(j)} = \sqrt{\frac{2}{n}} \sin\left(\frac{2\pi kj}{n} + \frac{\pi}{4}\right) \qquad (k, j \in \{0, \ldots, n-1\}).$$

Then $w_k = [w_k^{(j)}]_{0\leqslant j\leqslant n-1}$ are eigenvectors of P_n corresponding to $\lambda_0, \ldots, \lambda_{n-1}$. Furthermore, these eigenvectors form an orthonormal basis of $\mathbb{R}^n$.

Proof. It follows from [8, 12.3.1] that

$$\varphi_k = (1, e^{\frac{2\pi ik}{n}}, \ldots, e^{\frac{2\pi ikj}{n}} \ldots, e^{\frac{2\pi ik(n-1)}{n}})^{\mathrm{T}} \qquad (k \in \{0, \ldots, n-1\})$$

are (complex) eigenvectors corresponding to the eigenvalues $\lambda_0, \ldots, \lambda_{n-1}$ accordingly. Since both the real and imaginary parts of φ_k are eigenvectors of P_n too, we conclude that $u_k^{(j)} = \cos\left(\frac{2\pi kj}{n}\right)$ and $v_k^{(j)} = \sin\left(\frac{2\pi kj}{n}\right)$ are the coordinates of the (real) eigenvectors u_k and v_k corresponding to λ_k for $k, j \in \{0, \ldots, n-1\}$. However, it is to be noted that neither the system $(u_k)_{k=0}^{n-1}$ nor $(v_k)_{k=0}^{n-1}$ spans $\mathbb{R}^n$. As sums of eigenvectors corresponding to the same eigenvalue, if non-zero, are still eigenvectors, let us consider the eigenvector $u_k + v_k$ corresponding to λ_k. Evidently,

$$u_k^{(j)} + v_k^{(j)} = \sqrt{2} \sin\left(\frac{2\pi kj}{n} + \frac{\pi}{4}\right).$$

We then define the vectors w_k by

$$w_k^{(j)} = \sqrt{\frac{2}{n}} \sin\left(\frac{2\pi kj}{n} + \frac{\pi}{4}\right) \qquad (k, j \in \{0, \ldots, n-1\}).$$

Conspicuously, w_k is a (real) eigenvector of P_n corresponding to λ_k.

In order to show that $(w_k)_{k=0}^{n-1}$ are pairwise orthogonal unit vectors, we invoke the identities $2\sin(x)\sin(y) = \cos(x-y) - \cos(x+y)$ and $\cos\left(x + \frac{\pi}{2}\right) = -\sin(x)$. Using them we arrive at

$$\begin{aligned}\langle w_k, w_l\rangle =& \frac{2}{n}\sum_{j=0}^{n-1} \sin\left(\frac{2\pi kj}{n} + \frac{\pi}{4}\right) \sin\left(\frac{2\pi lj}{n} + \frac{\pi}{4}\right) \\ =& \frac{1}{n}\left(-1 + \sum_{j=0}^{n} \cos\left(\frac{2\pi j(k-l)}{n}\right)\right) + \frac{1}{n}\sum_{j=0}^{n} \sin\left(\frac{2\pi j(k+l)}{n}\right),\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^n$.

Let m be a positive integer. Then $\sum_{j=0}^{n} \cos\left(\frac{2\pi mj}{n}\right)$ and $\sum_{j=0}^{n} \sin\left(\frac{2\pi mj}{n}\right)$ are the real and imaginary part of the expression

$$\sum_{j=0}^{n} e^{i\frac{2\pi mj}{n}} = \frac{e^{2i\pi\left(m+\frac{m}{n}\right)} - 1}{e^{2i\pi\frac{m}{n}} - 1} = 1$$

respectively. Consequently, $\langle w_k, w_l\rangle = \delta_{k,l}$, where $\delta_{k,l}$ denotes the Kronecker delta, and thus $(w_k)_{k=0}^{n-1}$ forms an orthonormal basis of $\mathbb{R}^n$. □

Before we proceed, a piece of notation is required. Let Q be the matrix whose columns are precisely the eigenvectors from the statement of Lemma 2.1, that is, $Q = [w_0\ w_1\ \ldots\ w_{n-1}]$. For a diagonal matrix $\Lambda = \operatorname{diag}(\lambda_0 = 1, \lambda_1, \ldots, \lambda_{n-1})$ with $\lambda_k \in [-1, 1]$ ($k \in \{1, \ldots, n-1\}$) we consider the product matrix

$$P(\Lambda) := Q\Lambda Q^{\mathrm{T}} = \sum_{j=0}^{n-1} \lambda_j w_j w_j^T. \tag{2.2}$$

Remark 2.2. We note that the matrix $P(\Lambda)$ is symmetric and its each row adds up to 1 since $P(\Lambda)w_0 = w_0$.

Now we present the general form of the entries of $Q\Lambda Q^T$, with an arbitrary orthogonal Q so that $Q = [q_0\ q_1\ \ldots\ q_{n-1}]$ with $q_0 = \frac{1}{\sqrt{n}}(1, \ldots, 1)^T$ and $q_k \in \mathbb{R}^n$ for each $k = 1, \ldots, n$.

Remark 2.3. Let us denote the entry of Q in the k^{th} row and l^{th} column by $q_l^{(k)}$ for $j, k \in \{0, \ldots, n-1\}$. Note that $Q = [q_l^{(k)}]_{k,l=0}^{n-1}$ and $Q\Lambda = [\lambda_l q_l^{(k)}]_{k,l}$, where $\lambda_0 = 1$. Hence,

$$(Q\Lambda Q^T)_{k,l} = \sum_{j=0}^{n-1} \lambda_j q_j^{(k)} q_j^{(l)} = \frac{1}{\sqrt{n}} q_0^{(k)} + \sum_{j=1}^{n-1} \lambda_j q_j^{(k)} q_j^{(l)}. \tag{2.3}$$

In our construction of Q, that is, where Q is a matrix whose columns are precisely the eigenvectors from the statement of Lemma 2.1, namely, $Q = [w_0\ w_1\ \ldots\ w_{n-1}]$ we have that

$$q_j^{(k)} = w_j^{(k)} = \sqrt{\frac{2}{n}} \sin\left(\frac{2\pi kj}{n} + \frac{\pi}{4}\right) \qquad (k, j \in \{0, \ldots, n-1\}), \tag{2.4}$$

in particular, $q_0^{(k)} = w_0^{(k)} = \frac{1}{\sqrt{n}}$.

To summarise let us record the form of the entries the matrix $P(\Lambda)$ in a lemma:

Lemma 2.4. *We have $P(\Lambda) = [p_{kl}]_{k,l=0}^{n-1}$, where*

$$p_{kl} = \frac{1}{n}\left(1 + 2\sum_{j=1}^{n-1} \lambda_j \sin\left(\frac{2\pi kj}{n} + \frac{\pi}{4}\right) \sin\left(\frac{2\pi lj}{n} + \frac{\pi}{4}\right)\right) \qquad (k, l \in \{0, \ldots n-1\}).$$

Proof. Follows from (2.3) and (2.4). □

Proposition 2.5. *Let $P(\Lambda)$ be as in* (2.2)*. The matrix $P(\Lambda)$ is doubly stochastic if and only if*

$$\sum_{j=1}^{n-1} \lambda_j \sin\left(\frac{2\pi kj}{n} + \frac{\pi}{4}\right) \sin\left(\frac{2\pi lj}{n} + \frac{\pi}{4}\right) \geqslant -\frac{1}{2} \tag{2.5}$$

for all $k \in \{0, \ldots n-1\}$ and $l \in \{k, \ldots, n-1\}$.

Proof. The matrix $P(\Lambda)$ is doubly-stochastic if and only if $p_{kl} \geqslant 0$ for all $k \in \{0, \dots n-1\}$ and $l \in \{k, \dots, n-1\}$ and this is indeed equivalent to (2.5) by Lemma 2.4. □

Finally, the main result and the statement from Remark 1.1 follows directly from the next corollary to Proposition 2.5.

Corollary 2.6. *Let $P(\Lambda)$ be as in* (2.2).

(1) *Suppose that $\lambda_i \leqslant 0$ for all $i \in \{1, \dots, n-1\}$. Then $P(\Lambda)$ is doubly stochastic as long as*

$$\sum_{i=1}^{n-1} \lambda_i \geqslant -\frac{1}{2}. \tag{2.6}$$

(2) *Suppose that $\lambda_i \geqslant 0$ for all $i \in \{1, \dots, n-1\}$. Then $P(\Lambda)$ is doubly stochastic as long as*

$$\sum_{i=1}^{n-1} \lambda_i \leqslant \frac{1}{2}. \tag{2.7}$$

Proof. First we show (1). Assume that (2.6) holds and set $S_j(k) = \sin\left(\frac{2\pi kj}{n} + \frac{\pi}{4}\right)$ for any $j \in \{1, \dots, n-1\}$ and $k \in \{0, \dots, n-1\}$. Clearly, $S_j(k)S_j(l) \leqslant 1$ for all $k, l \in \{0, \dots n-1\}$ and since all $\lambda_j \leqslant 0$, we have $\lambda_j S_j(k)S_j(l) \geqslant \lambda_j$. Hence, we arrive at the estimate

$$\begin{aligned} \sum_{j=1}^{n-1} \lambda_j S_j(k)S_j(l) \geqslant & \sum_{j=1}^{n-1} \lambda_j \\ \geqslant & -\frac{1}{2}. \end{aligned}$$

To show (2), assume (2.7) and use the fact that $S_j(k)S_j(l) \geqslant -1$. □

Remark 2.7. Note that

$$0 \leqslant \delta_{\min} := \min\left\{\sum_{j=0}^{n-1} \lambda_j \colon \sum_{j=1}^{n-1} \lambda_j S_j(k)S_j(l) \geqslant -\frac{1}{2} \text{ for all } k, l \in \{0, \dots n-1\}\right\} \leqslant \frac{1}{2}.$$

For the upper bound, let us note that $S_j(k) = 1$ if and only if n is divisible by 8 and $kj = n\left(\frac{1}{8} + m\right)$ for some $m \in \mathbb{N} \cup \{0\}$. Thus if n is not a multiple of 8 then $\delta_{\min} < \frac{1}{2}$.
For the lower bound we found that if $n = 5$ then for $\sigma = (1, -0.004, -0.002, -0.004, -0.51)$ so that $\sum_{j=0}^{4} \lambda_j = 0.48$ we have a negative element in the third row, third column, namely $P(\Lambda)_{22} \approx -0.0005 < 0$. These suggest that $\delta_{\min} \in (0.48, 0.5)$.

We leave the task of finding the exact value of $\delta_{\min}$ an *open problem.*

Remark 2.8. One may observe that our matrix Q is orthogonal and symmetric. Therefore, one may try to find different orthogonal symmetric matrix so that Problem 1.2 has a solution for $\delta \in (0, \frac{1}{2})$. This could be achieved by finding a constant $M \in (1, 2)$ that is an upper bound for the term $nq_j^{(k)} q_j^{(l)}$ in equation (2.3) for each j, k, $l \in \{0, 1, \dots, n-1\}$.

A natural example of a matrix that is both orthogonal and symmetric, is a *Householder matrix*, *i.e.*,

$$\begin{aligned} H(v) =& I - 2\frac{vv^T}{\|v\|^2} \\ =& \begin{bmatrix} \beta & \beta & \beta & \dots & \beta & \beta \\ \beta & 1-\alpha & -\alpha & \dots & -\alpha & -\alpha \\ \beta & -\alpha & 1-\alpha & \ddots & -\alpha & -\alpha \\ \dots & \dots & \ddots & \ddots & \ddots & \dots \\ \beta & -\alpha & -\alpha & \ddots & 1-\alpha & -\alpha \\ \beta & -\alpha & -\alpha & \dots & -\alpha & 1-\alpha \end{bmatrix} \end{aligned}$$

so that $v = (1-\sqrt{n}, 1, \dots, 1)^T$, $\alpha = \frac{1}{\sqrt{n}(\sqrt{n}-1)}$ and $\beta = \frac{1}{\sqrt{n}}$ (see [22, Section 4.2]). However, given that $\lambda_j \leqslant 0$ for each $j = 1, \dots, n-1$, the matrix $H(v)\Lambda H(v)$, as n becomes large fails to have non-negative entries as

$$(H(v)\Lambda H(v))_{k,k} = \frac{1}{n} + \alpha^2 \sum_{j=1 \ \& \ j\neq k}^{n-1} \lambda_j + (1-\alpha)^2 \lambda_k.$$

3. Examples and applications

Examples. Let $n \in \mathbb{N}$. In this section we provide some examples of Suleĭmanova spectra, $\sigma_n = (1, \lambda_2, \dots, \lambda_n)$, for which $\lambda_2, \dots, \lambda_n$ add up to $-\frac{1}{2}$ and, thus, yield symmetric doubly stochastic matrices obtained via our construction (2.2), but do not satisfy known sufficient conditions (*e.g.*, (1.1), (1.2), *etc.*) to obtain symmetric doubly stochastic matrices.

To wit, neither (1.1) nor (1.2) is satisfied for

- $\sigma_5 = (1, -0.02, -0.03, -0.05, -0.4)$ (odd dimension);
- $\sigma_6 = (1, -0.01, -0.02, -0.06, -0.08, -0.33)$ (even dimension),

respectively.

Let $\sigma_n = (1, \lambda_2, \dots, \lambda_n)$, where $\lambda_2 \geqslant \lambda_3 \dots \geqslant \lambda_n$. The *improved Soules' condition* when n is even, [11, Theorem 3, Notation 1, Observation 1], that is,

$$\frac{1}{n} + \frac{1}{n}\lambda_2 + \frac{\frac{n}{2} - \left[\frac{n+2}{4}\right]}{\frac{n}{2}\left[\frac{n+2}{4}\right]}\lambda_4 + \sum_{k=1}^{\left[\frac{n+2}{4}\right]-1} \frac{\lambda_{n-4k+4}}{k(k+1)} \geqslant 0$$

is not satisfied as witnessed by

$$\sigma_{10} = (1, -0.01, -0.01, -0.025, -0.03, -0.035, -0.04, -0.05, -0.08, -0.22)$$

(the square brackets in the above formula denote the integral part of a real number).

Let n be odd, *new condition 1* ([11, Theorem 4, Notation 1]; we adapt the naming conventions from the said paper), that is,

$$\frac{1}{n}+\frac{n-1}{n(n+1)}\lambda_2+\frac{\frac{n+1}{2}-\left[\frac{n+3}{4}\right]}{\frac{n+1}{2}\left[\frac{n+3}{4}\right]}\lambda_4+\sum_{k=1}^{\left[\frac{n+3}{4}\right]-1}\frac{\lambda_{n-4k+4}}{k(k+1)}\geqslant 0$$

is not satisfied as witnessed by the 1/2-normalised Suleĭmanova spectrum

$$\sigma_5=(1,-0.03,-0.03,-0.04,-0.4).$$

Next we give examples of spectra that do not satisfy *New condition 2* ([11, Theorem 5, Notation 2]) which is given by (3.1, 3.2, 3.3, 3.4) depending on the remainder $n \mod 4$. Let m be an integer greater than 1. If

(1) $n=4m$, then

$$\frac{1}{n}+\frac{1}{n}\lambda_2+\frac{2}{n}\lambda_4+\frac{\frac{n}{4}-\left[\frac{n+4}{8}\right]}{\frac{n}{4}\left[\frac{n+4}{8}\right]}\lambda_8+\sum_{k=1}^{\left[\frac{n+4}{8}\right]-1}\frac{\lambda_{n-8k+8}}{k(k+1)}\geqslant 0 \tag{3.1}$$

is not satisfied by the 1/2-normalised Suleĭmanova spectrum

$$\sigma_{16}=(1,-0.003,-0.003,-0.004,-0.007,-0.009,-0.02,-0.0209,-0.021,$$
$$-0.024,-0.026,-0.035,-0.042,-0.076,-0.0811,-0.128);$$

(2) $n=4m+2$, then

$$\frac{1}{n}+\frac{1}{n}\lambda_2+\frac{2(n-2)}{n(n+2)}\lambda_4+\frac{\frac{n+2}{4}-\left[\frac{n+6}{8}\right]}{\frac{n+2}{4}\left[\frac{n+6}{8}\right]}\lambda_8+\sum_{k=1}^{\left[\frac{n+6}{8}\right]-1}\frac{\lambda_{n-8k+8}}{k(k+1)}\geqslant 0 \tag{3.2}$$

is not satisfied by the 1/2-normalised Suleĭmanova spectrum

$$\sigma_{10}=(1,-0.01,-0.01,-0.01,-0.02,-0.02,-0.04,-0.07,-0.1,-0.22);$$

(3) $n=4m+3$, then

$$\frac{1}{n}+\frac{n-1}{n(n+1)}\lambda_2+\frac{2}{n+1}\lambda_4+\frac{\frac{n+1}{4}-\left[\frac{n+5}{8}\right]}{\frac{n+1}{4}\left[\frac{n+5}{8}\right]}\lambda_8+\sum_{k=1}^{\left[\frac{n+5}{8}\right]-1}\frac{\lambda_{n-8k+8}}{k(k+1)}\geqslant 0 \tag{3.3}$$

is not satisfied by the 1/2-normalised Suleĭmanova spectrum

$$\sigma_{11}=(1,-0.001,-0.004,-0.01,-0.01,-0.012,-0.013,-0.05,-0.09,-0.11,-0.2);$$

(4) $n=4m+1$, then

$$\frac{1}{n}+\frac{n-1}{n(n+1)}\lambda_2+\frac{2(n-1)}{(n+1)(n+3)}\lambda_4+\frac{\frac{n+3}{4}-\left[\frac{n+7}{8}\right]}{\frac{n+3}{4}\left[\frac{n+7}{8}\right]}\lambda_8+\sum_{k=1}^{\left[\frac{n+7}{8}\right]-1}\frac{\lambda_{n-8k+8}}{k(k+1)}\geqslant 0 \tag{3.4}$$

is not satisfied by the 1/2-normalised Suleĭmanova spectrum

$$\sigma_9=(1,-0.006,-0.018,-0.02,-0.028,-0.028,-0.053,-0.105,-0.242).$$

New condition 3 ([11, Conjecture 1, Example 1]) that for $n = 26$ takes the form

$$\frac{1}{26} + \frac{1}{26}\lambda_2 + \frac{6}{13 \cdot 7}\lambda_4 + \frac{3}{28}\lambda_8 + \frac{1}{4}\lambda_{16} + \frac{1}{2}\lambda_{26} \geqslant 0$$

is not satisfied by the 1/2-normalised Suleĭmanova spectrum

$$\sigma_{26} = (1, \underbrace{-0.004}_{\lambda_2}, -0.005, \underbrace{-0.006}_{\lambda_4}, -0.007, -0.01, -0.01, \underbrace{-0.011}_{\lambda_8}, -0.011, -0.011, -0.012,$$

$$-0.012, -0.015, -0.015, -0.016, \underbrace{-0.017}_{\lambda_{16}}, -0.019, -0.02, -0.022, -0.022, -0.025, -0.028,$$

$$-0.028, -0.032, -0.069, \underbrace{-0.073}_{\lambda_{26}}).$$

Applications to random generation of doubly stochastic matrices. Let $n \in \mathbb{N}$ and $\alpha \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Our construction provides a simple way to randomly generate symmetric doubly stochastic matrices via their spectrum. Namely, let $X_1, \ldots, X_{n-1}$ be independent random variables having probability distributions supported on $[0, 1]$ and let us consider $S_n := X_1 + \ldots + X_{n-1}$. Set

$$\lambda_i = \alpha \frac{X_i}{S_n} \quad (i \in \{1, \ldots, n-1\}).$$

Then $\sigma = (1, \lambda_1, \ldots, \lambda_{n-1})$ is a spectrum of a symmetric doubly stochastic matrix and the corresponding matrix may be obtained via (2.2). More algorithms to generate doubly stochastic matrices (not necessarily symmetric) can be found in [1]. For an elaborate discussion on spectral properties of random doubly stochastic matrices we refer the reader to [12].

Supplementary material. We supplemnt the material with a Python code organised in a `JupyterNotebook` available at

`https://github.com/Nty24/DoublyStochasticMatricesGenerator`

that generates further examples and counterexamples in the spirit of Section 3.

Acknowledgements. The first-named author would like to thank his colleagues, James Burridge, for the idea to investigate the eigenvectors of symmetric random walks, and Ittay Weiss, for great discussions that led to Remark 2.8. The authors would like to thank Pietro Paparella for clarifying the history of the results on realising normalised Suleĭmanova spactra. M.G.'s visit to Prague in August 2019, during which some part of the project was completed, was supported by funding received from GAČR project 17-27844S; RVO 67985840, which the second-named author acknowledges with thanks.

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