



INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

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inside a density dependent
incompressible fluid**

Šárka Nečasová

Mythily Ramaswamy

Archab Roy

Anja Schlömerkemper

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SELF-PROPELLED MOTION OF A RIGID BODY INSIDE A DENSITY DEPENDENT INCOMPRESSIBLE FLUID

Š. NEČASOVÁ, M. RAMASWAMY, A. ROY, AND A. SCHLÖMERKEMPER

ABSTRACT. This paper is devoted to the existence of a weak solution to a system describing a self-propelled motion of a rigid body in a viscous fluid in the whole \mathbb{R}^3 . The fluid is modelled by the incompressible nonhomogeneous Navier-Stokes system with a nonnegative density. The motion of the rigid body is described by the balance of linear and angular momentum. We consider the case where slip is allowed at the fluid-solid interface through Navier condition and prove the global existence of a weak solution.

Key words. self-propelled motion, fluid-structure interaction system, Navier-Stokes equations, nonnegative density, Navier boundary conditions

AMS subject classifications. 35Q35, 35Q30, 76D05, 76Z10.

1. INTRODUCTION

Fluid-structure interaction (FSI) systems are systems which include a fluid and a solid component. The study of motion of small particles in fluids became one of the main focuses in research in the last 50 years. The presence of the particles has influence on the flow of the fluid and the fluid affects the motion of the particles. It implies that the problem of determining the flow characteristic is highly coupled. For such type of everyday phenomena with a wide range of applications we refer to, e.g., [3, 4, 17] and references therein. Systems that arise from modeling such phenomena are typically nonlinear systems of partial differential equations with a moving boundary or interface.

We investigate a system where the rigid body moves in an incompressible Newtonian fluid which fills the whole space. The position of the rigid body at any given time moment is determined by two vectors describing the translation of the center of the mass and the rotation around the center of the mass, respectively. A system of six ordinary differential equations (Euler equations) describing the conservation of linear and angular momentum describes the dynamics of a rigid body.

The fluid flow is governed by the three-dimensional incompressible Navier-Stokes equations. The fluid domain depends on the position of the rigid body. The Navier-Stokes equations are coupled with a system of Euler ODE's via a dynamic and a kinematic coupling condition: The dynamic coupling condition is given by the balance of forces acting on the rigid body. There are several possibilities for the kinematic coupling condition. The no-slip condition, which postulates equality of the velocity of the fluid and of the structure on the boundary of the rigid body, is the most commonly used in the literature since it is the simplest to analyze and it is a physically reasonable condition. However, in many situations, e.g., in close to contact dynamics (see, e.g., [19, 20]) or in the case of rough surfaces (see, e.g., [7, 21, 23]), Navier's slip coupling condition is more appropriate since it allows for a discontinuity of the velocity in the tangential component on the boundary of the rigid body. In this article we therefore assume the Navier slip condition.

Fluid-rigid body systems have been extensively studied in the last twenty years and some aspects of the well-posedness theory are now well established. In the case of no self-propulsion, of homogeneous mass densities and constant viscosities, the existence of the unique local-in-time (or small data) solution is known in both two and three dimensions, and for both the slip [2, 37] and the no-slip [11, 18, 25, 36] coupling.

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Further, it is known that a weak solution of Leray-Hopf type exists and is global in time or exists until the moment of contact between the boundary of the container and the rigid body for the slip [9, 19, 27] and the no-slip case [10, 13, 14, 22, 26].

One of the novelties of our article is that we will allow for self-propelled motion of the rigid body. Self-propulsion is a common means of locomotion of macroscopic objects. Examples of such motions are performed by birds, fishes, rockets, submarines, etc. In microscopic world many organisms like ciliates, flagellates move by self-propulsion see, e.g., [17].

The case of no-slip boundary conditions for a moving rigid body with self-propulsion was considered in [16, 35] for different boundary conditions and homogeneous densities. The problem of the self-propelled motion as a control problem was tackled in [1, 28, 30] in the case of low Reynolds number (the fluid equations are the Stokes equations) and in [6, 8] in the case of a potential fluid (high Reynolds number). Let us also mention the work of Silvestre who investigated the slow motion of steady self-propelled motion and attainability [31, 32].

Nonhomogeneous densities in viscous incompressible fluids were investigated in [24, 34]. To the best of our knowledge, nonhomogeneous densities were not treated in corresponding systems with structure interaction before.

In this article we consider the case of non-steady self-propelled motion in the case of a nonhomogeneous fluid (non-constant density) with Navier boundary conditions.

To tackle our system, we first apply a global transformation such that the system is posed in a fixed but unbounded domain. In order to prove existence of a weak solution we first establish such an existence on a bounded domain $B_R = \{y \in \mathbb{R}^3 : |y| < R\}$ by extending work of [34] to the setting with a self-propelled rigid body. This is based on a Galerkin method. Due to the fluid-rigid structure interaction our test function depends here on the position of the body. Furthermore, we need to deal with the extra terms that are due to the global change of variables. We prove that all a-priori estimates are independent of R . We point out that the renormalized continuity equation enters in our definition of weak solutions to the fluid-structure problem. Up to our knowledge it is the first time that the renormalized continuity equation is considered in a nonhomogeneous fluid with structure. This way we obtain better regularity for the density. In the case without structure, the renormalized continuity equation was introduced by DiPerna, Lions [24, 15]. The final step is then to show the existence of a weak solution in the unbounded domain by letting $R \rightarrow \infty$. Here we follow ideas by Planas and Sueur [27, Theorem 1].

The novelties of our paper are as follows: (1) we include self-propelled motion of the rigid body, (2) we allow for nonnegative and nonhomogeneous densities and (3) replace the no-slip condition at the interface of the solid and the fluid by the Navier slip condition. Furthermore, we consider (4) the case of a viscosity that depends on the mass density.

The outline of this article is as follows. The next section is devoted to an introduction of the mathematical problem, which is reformulated in a fixed domain. In Section 3, we prove the existence of weak solution in a bounded domain. Section 4 is devoted to the existence proof in an unbounded domain. In Section 5 we mention the case of positive density and the case of a viscosity depending on the density and some further remarks and open problems. In the appendix we present a derivation of the weak formulation of the problem.

2. THE MATHEMATICAL MODEL

For $T > 0$ given and for any time $t \in (0, T)$ let $\mathcal{S}(t) \subset \mathbb{R}^3$ denote a closed, bounded and simply connected rigid body. We assume that the rest of the space, i.e., $\mathbb{R}^3 \setminus \mathcal{S}(t) = \mathcal{F}(t)$ is filled with a viscous incompressible nonhomogeneous fluid. The initial domain of the rigid body is denoted by \mathcal{S}_0 and is assumed to have a smooth boundary. Correspondingly, $\mathcal{F}_0 = \mathbb{R}^3 \setminus \mathcal{S}_0$ is the initial fluid domain.

Our system of a rigid body moving in a fluid is in the first instance a moving domain problem in an inertial frame in which the velocity of the rigid body is described by

$$U_S(x, t) = h'(t) + R(t) \times (x - h(t)) \quad \text{for } (x, t) \in \mathcal{S}(t) \times (0, T). \quad (2.1)$$

Here, $h'(t)$ is the linear velocity of the centre of mass h and $R(t)$ is the angular velocity of the body. The solid domain at time t in the same inertial frame is given by

$$\mathcal{S}(t) = \{h(t) + Q(t)y \mid y \in \mathcal{S}_0\},$$

where $Q(t) \in SO(3)$ is associated to the rotation of the rigid body. Mathematically, it will turn out useful to define U_S for all $(x, t) \in \mathbb{R}^3 \times (0, T)$. Let ϱ , U and P be the density, velocity and pressure of the viscous incompressible nonhomogeneous fluid, respectively, which satisfy

$$\begin{aligned} \frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho U) &= 0, \quad \varrho(x) \geq 0 \quad \text{in } \mathcal{F}(t) \times (0, T), \\ \frac{\partial}{\partial t}(\varrho U) + \operatorname{div}(\varrho U \otimes U) + \nabla P &= \operatorname{div}(2\nu D(U)), \quad \operatorname{div} U = 0 \quad \text{in } \mathcal{F}(t) \times (0, T). \end{aligned} \quad (2.2)$$

Here we consider a self-propelled motion of the body $\mathcal{S}(t)$, which we describe by a vectorial flux W at $\partial\mathcal{S}(t)$. In this article we consider Navier type conditions at the fluid-structure interface; we prescribe the normal and tangential parts of the flux by:

$$\begin{aligned} W \cdot N &= 0 \quad \text{on } \partial\mathcal{S}(t) \times (0, T), \\ \alpha(W \times N) &= (D(U)N) \times N + \alpha(U - U_S) \times N \quad \text{on } \partial\mathcal{S}(t) \times (0, T), \end{aligned}$$

where N is the unit outward normal to the boundary of $\mathcal{F}(t)$, i.e., directed towards $\mathcal{S}(t)$ and α is a given constant. We refer to Remark 5.4 for a discussion of the assumption that the normal component is zero.

For a given viscosity coefficient $\nu > 0$, we set

$$\Sigma(U, P) = -PI + 2\nu D(U) \quad \text{with} \quad D(U) = \left[\frac{1}{2} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \right]_{i,j=1,2,3}.$$

Let ϱ_S denote the density of the rigid body. Then $m = \int_{\mathcal{S}(t)} \varrho_S(x, t) dx$ is its total mass, which is constant in time and does not change under the coordinate transformation below. Further, the moment of inertia $J(t)$ is defined by

$$J(t) = \int_{\mathcal{S}(t)} \varrho_S(x, t) \left(|x - h(t)|^2 \mathbf{I} - (x - h(t)) \otimes (x - h(t)) \right) dx.$$

The ordinary differential equations modeling the dynamics of the rigid body then read

$$\begin{aligned} mh'' &= - \int_{\partial\mathcal{S}(t)} \Sigma(U, P) N \, d\Gamma, \\ (JR)' &= - \int_{\partial\mathcal{S}(t)} (x - h) \times \Sigma(U, P) N \, d\Gamma, \end{aligned}$$

We suppose the density and the velocity of the fluid to satisfy

$$\varrho(x, t) \rightarrow 0, \quad U(x, t) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty,$$

as well as the initial conditions

$$\varrho(x, 0) = \rho_0(x) \geq 0, \quad \varrho U(x, 0) = q_0(x) = \rho_0(x)u_0(x), \quad h(0) = 0, \quad h'(0) = \ell_0, \quad r(0) = r_0. \quad (2.3)$$

Following [16, 27, 29], we apply a global change of variables that transforms the system in such a way that it is formulated in a frame which is attached to the rigid body. At time $t = 0$, the two frames are assumed

to coincide with $h(0) = 0$. Hence the transformed system is posed on the fixed domain $\mathcal{F}_0 \times (0, T)$ via the following change of variables:

$$\begin{aligned} u(y, t) &= Q(t)^\top U(Q(t)y + h(t), t), & w(y, t) &= Q(t)^\top W(Q(t)y + h(t), t), \\ \rho(y, t) &= \varrho(Q(t)y + h(t), t), & p(y, t) &= P(Q(t)y + h(t), t). \end{aligned}$$

Further, the moment of inertia transforms to

$$J_0 = Q(t)^\top J(t) Q(t).$$

By (2.1) and the extension of U_S to the whole space, we have that

$$u_S(y, t) = Q(t)^\top U_S(Q(t)y + h(t), t) = \ell(t) + r(t) \times y, \quad y \in \mathcal{F}_0 \times (0, T),$$

where $\ell(t) = Q(t)^\top h'(t)$ is the transformed linear velocity and $r(t) = Q(t)^\top R(t)$ the transformed angular velocity of the rigid body. The normal and tangential parts of the self-propulsion flux in the new frame are given by:

$$w \cdot n = 0 \text{ on } \partial\mathcal{S}_0 \times (0, T), \quad (2.4)$$

$$\alpha(w \times n) = (D(u)n) \times n + \alpha(u - u_S) \times n \text{ on } \partial\mathcal{S}_0 \times (0, T) \quad (2.5)$$

with $n(y, t) = Q(t)^\top N(Q(t)y + h(t))$ for $(y, t) \in \partial\mathcal{S}_0 \times (0, T)$ denoting the inward pointing normal to $\partial\mathcal{S}_0$. With all this at hand, equations (2.2)–(2.3) can be rewritten in the fixed domain as

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho(u - u_S)) = 0, \quad \rho(y) \geq 0 \quad \text{in } \mathcal{F}_0 \times (0, T), \quad (2.6)$$

$$\frac{\partial}{\partial t}(\rho u) + \operatorname{div}[u \otimes \rho(u - u_S)] + \rho r \times u + \nabla p = \operatorname{div}(2\nu D(u)), \quad \operatorname{div} u = 0 \quad \text{in } \mathcal{F}_0 \times (0, T), \quad (2.7)$$

$$u \cdot n = u_S \cdot n \text{ on } \partial\mathcal{S}_0 \times (0, T), \quad (2.8)$$

$$(D(u)n) \times n = -\alpha(u - u_S - w) \times n \text{ on } \partial\mathcal{S}_0 \times (0, T), \quad (2.9)$$

$$m\ell' = - \int_{\partial\mathcal{S}_0} \sigma(u, p)n \, d\Gamma + m\ell \times r, \quad (2.10)$$

$$J_0 r' = - \int_{\partial\mathcal{S}_0} y \times \sigma(u, p)n \, d\Gamma + J_0 r \times r, \quad (2.11)$$

$$\rho(y, t) \rightarrow 0, \quad u(y, t) \rightarrow 0 \text{ as } |y| \rightarrow \infty, \quad (2.12)$$

$$\rho(y, 0) = \rho_0(y) \geq 0, \quad \rho u(y, 0) = q_0(y) = \rho_0(y)u_0(y), \quad h(0) = 0, \quad \ell(0) = \ell_0, \quad r(0) = r_0, \quad (2.13)$$

where

$$\sigma(u, p) = -pI + 2\nu D(u) \text{ with } D(u) = \left[\frac{1}{2} \left(\frac{\partial u_i}{\partial y_j} + \frac{\partial u_j}{\partial y_i} \right) \right]_{i,j=1,2,3}.$$

The system of partial and ordinary differential equations that we investigate in this work consists of the equations (2.6)–(2.13).

2.1. Notations and functional framework. Before initiating our analysis, we collect here some basic notation that will be used throughout. The linear space $\mathfrak{D}(\Omega)$ consists of all infinitely differentiable functions that have compact support in Ω . The norm in a Lebesgue space L^p (resp. Sobolev space $W^{k,p}$) is denoted by $\|\cdot\|_p$ (resp. $\|\cdot\|_{k,p}$). We denote by $W^{-1,r}(\Omega)$ the dual space of $W_0^{1,r'}(\Omega)$, where $1/r + 1/r' = 1$. For convenience, we introduce

$$H^1(\Omega) = W^{1,2}(\Omega), \quad H^{-1}(\Omega) = W^{-1,2}(\Omega).$$

We want to give an appropriate notion of weak solution to system (2.6)–(2.13). In order to do so, following [27], we introduce the space of divergence free vector functions

$$\mathcal{H} = \{\phi \in L^2(\mathbb{R}^3) \mid \operatorname{div} \phi = 0 \text{ in } \mathbb{R}^3 \text{ and } D(\phi) = 0 \text{ in } \mathcal{S}_0\}.$$

For any $\phi \in \mathcal{H}$, there exist $\ell_\phi \in \mathbb{R}^3$ and $r_\phi \in \mathbb{R}^3$ such that

$$\phi(y) = \ell_\phi + r_\phi \times y =: \phi_{\mathcal{S}}(y) \quad \text{for all } y \in \mathcal{S}_0. \quad (2.14)$$

Note that the tangential component of $\phi \in \mathcal{H}$ is allowed to jump at $\partial\mathcal{S}_0$, while the normal component has a continuous representative. We remark that, in the integrals on $\partial\mathcal{S}_0$ below, ϕ denotes the trace from the fluid side whereas $\phi_{\mathcal{S}}$ denotes the trace from the solid side.

We consider the space $L^2(\mathbb{R}^3)$ with the following inner product:

$$(\phi, \psi)_{\mathcal{H}} = \int_{\mathcal{F}_0} \rho \phi \cdot \psi \, dy + \int_{\mathcal{S}_0} \rho_{\mathcal{S}_0} \phi \cdot \psi \, dy.$$

This inner product is equivalent to the usual inner product of $L^2(\mathbb{R}^3)$. Furthermore, when $\phi, \psi \in \mathcal{H}$, we have

$$(\phi, \psi)_{\mathcal{H}} = \int_{\mathcal{F}_0} \rho \phi \cdot \psi \, dy + m \ell_\phi \cdot \ell_\psi + J_0 r_\phi \cdot r_\psi.$$

The norm associated with the above inner product is denoted by $\|\cdot\| = (\cdot, \cdot)_{\mathcal{H}}$. Let us define the space

$$\mathcal{V} = \left\{ \phi \in \mathcal{H} \mid \int_{\mathcal{F}_0} |\nabla \phi(y)|^2 \, dy < \infty \right\} \quad \text{with norm } \|\phi\|_{\mathcal{V}} = \|\phi\| + \|\nabla \phi\|_{L^2(\mathcal{F}_0)}.$$

Next we introduce some notations of time-dependent functions that we need later to describe the compactness properties. For $h > 0$, the translated function of a function f denoted as $\tau_h f$ is given by

$$(\tau_h f)(t) = f(t + h) \text{ for any } t \in [0, T].$$

Let E be a Banach space. For $1 \leq q \leq \infty$, $0 < s < 1$, Nikolskii spaces are defined by

$$N^{s,q}(0, T; E) = \left\{ f \in L^q(0, T; E) \mid \sup_{h>0} h^{-s} \|\tau_h f - f\|_{L^q(0, T-h; E)} < \infty \right\}. \quad (2.15)$$

2.2. Energy inequality and definition of weak solution. Let u be a smooth solution of (2.6)–(2.13) and $\phi \in C^\infty([0, T]; \mathcal{H})$ such that $\phi|_{\overline{\mathcal{F}_0}} \in C^\infty([0, T] \times \overline{\mathcal{F}_0})$. In the appendix we derive the weak form of our

system. It reads

$$\begin{aligned}
& \int_{\mathcal{F}_0} \rho u(y, t) \cdot \phi(y, t) dy + m\ell(t) \cdot \ell_\phi(t) + J_0 r(t) \cdot r_\phi(t) - \int_{\mathcal{F}_0} q_0 \cdot \phi(y, 0) dy - m\ell(0) \cdot \ell_\phi(0) - J_0 r(0) \cdot r_\phi(0) \\
&= \int_0^t \int_{\mathcal{F}_0} \rho u \cdot \frac{\partial \phi}{\partial s} dy ds + \int_0^t m\ell \cdot \ell'_\phi ds + \int_0^t J_0 r \cdot r'_\phi ds + \int_0^t \int_{\mathcal{F}_0} [u \otimes \rho(u - u_S)] : \nabla \phi dy ds \\
&\quad - \int_0^t \int_{\mathcal{F}_0} \det(\rho r, u, \phi) dy ds + \int_0^t \det(m\ell, r, \ell_\phi) ds + \int_0^t \det(J_0 r, r, r_\phi) ds - 2\nu \int_0^t \int_{\mathcal{F}_0} D(u) : D(\phi) dy ds \\
&\quad - 2\nu\alpha \int_0^t \int_{\partial \mathcal{S}_0} (u - u_S - w) \cdot (\phi - \phi_S) d\Gamma ds. \tag{2.16}
\end{aligned}$$

This relation helps us to obtain an energy estimate for the system (2.6)–(2.13).

Proposition 2.1. *A solution (ρ, u, ℓ, r) of the system (2.6)–(2.13) with $\rho \in C^\infty([0, T] \times \overline{\mathcal{F}_0})$, $u \in C^\infty([0, T]; \mathcal{H})$ such that $u|_{\overline{\mathcal{F}_0}} \in C^\infty([0, T] \times \overline{\mathcal{F}_0})$ and $\ell, r \in C^\infty([0, T])$ satisfy the following energy inequality: for almost every $t \in [0, T]$,*

$$\begin{aligned}
& \int_{\mathcal{F}_0} \frac{1}{2} \rho |u|^2 dy + \frac{m}{2} |\ell|^2 + \frac{J_0}{2} |r|^2 + 2\nu \int_0^t \int_{\mathcal{F}_0} |D(u)|^2 dy ds + \nu\alpha \int_0^t \int_{\partial \mathcal{S}_0} |u - u_S|^2 d\Gamma ds \\
&\leq \int_{\mathcal{F}_0} \frac{1}{2} \rho_0 |u_0|^2 dy + \frac{m}{2} |\ell_0|^2 + \frac{J_0}{2} |r_0|^2 + \nu\alpha \int_0^t \int_{\partial \mathcal{S}_0} |w|^2 d\Gamma ds. \tag{2.17}
\end{aligned}$$

Proof. To establish the energy inequality, we use the test function $\phi = u$ in (2.16), to obtain

$$\begin{aligned}
& \int_{\mathcal{F}_0} \rho |u|^2 dy + m|\ell|^2 + J_0 |r|^2 - \int_{\mathcal{F}_0} \rho_0 |u_0|^2 dy - m|\ell_0|^2 - J_0 |r_0|^2 \\
&= \int_0^t \int_{\mathcal{F}_0} \rho u \cdot \frac{\partial u}{\partial s} dy ds + \int_0^t m\ell \cdot \ell' ds + \int_0^t J_0 r \cdot r' ds + \int_0^t \int_{\mathcal{F}_0} [u \otimes \rho(u - u_S)] : \nabla u dy ds \\
&\quad - 2\nu \int_0^t \int_{\mathcal{F}_0} |D(u)|^2 dy ds - 2\nu\alpha \int_0^t \int_{\partial \mathcal{S}_0} |u - u_S|^2 d\Gamma ds + 2\nu\alpha \int_0^t \int_{\partial \mathcal{S}_0} w \cdot (u - u_S) d\Gamma ds. \tag{2.18}
\end{aligned}$$

In the above calculation, we have used that $\det(\rho r, u, u) = \det(m\ell, r, \ell) = \det(J_0 r, r, r) = 0$. Moreover,

$$\begin{aligned}
 \int_0^t \int_{\mathcal{F}_0} [u \otimes \rho(u - u_S)] : \nabla u \, dy \, ds &= \int_0^t \int_{\mathcal{F}_0} [(\rho(u - u_S) \cdot \nabla)u] \cdot u \, dy \, ds \\
 &= -\frac{1}{2} \int_0^t \int_{\mathcal{F}_0} \operatorname{div}(\rho(u - u_S)) |u|^2 \, dy \, ds + \frac{1}{2} \int_0^t \int_{\partial \mathcal{S}_0} \rho(u - u_S) \cdot n |u|^2 \, d\Gamma \, ds \\
 &= \frac{1}{2} \int_0^t \int_{\mathcal{F}_0} \frac{\partial \rho}{\partial s} |u|^2 \, dy \, ds, \tag{2.19}
 \end{aligned}$$

where the last equality follows from (2.6) and (2.8). By combining (2.18) and (2.19), we obtain

$$\begin{aligned}
 &\int_{\mathcal{F}_0} \rho |u|^2 \, dy + m |\ell|^2 + J_0 |r|^2 - \int_{\mathcal{F}_0} \rho_0 |u_0|^2 \, dy - m |\ell_0|^2 - J_0 |r_0|^2 \\
 &= \int_{\mathcal{F}_0} \int_0^t \frac{1}{2} \frac{d}{ds} (\rho |u|^2) \, ds \, dy + \int_0^t \frac{m}{2} \frac{d}{ds} |\ell|^2 \, ds + \int_0^t \frac{J_0}{2} \frac{d}{ds} |r|^2 \, ds - 2\nu \int_0^t \int_{\mathcal{F}_0} |D(u)|^2 \, dy \, ds \\
 &\quad - 2\nu\alpha \int_0^t \int_{\partial \mathcal{S}_0} |u - u_S|^2 \, d\Gamma \, ds + 2\nu\alpha \int_0^t \int_{\partial \mathcal{S}_0} w \cdot (u - u_S) \, d\Gamma \, ds.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 &\int_{\mathcal{F}_0} \frac{1}{2} \rho |u|^2 \, dy + \frac{m}{2} |\ell|^2 + \frac{J_0}{2} |r|^2 + 2\nu \int_0^t \int_{\mathcal{F}_0} |D(u)|^2 \, dy \, ds + 2\nu\alpha \int_0^t \int_{\partial \mathcal{S}_0} |u - u_S|^2 \, d\Gamma \, ds \\
 &= \int_{\mathcal{F}_0} \frac{1}{2} \rho_0 |u_0|^2 \, dy + \frac{m}{2} |\ell_0|^2 + \frac{J_0}{2} |r_0|^2 + 2\nu\alpha \int_0^t \int_{\partial \mathcal{S}_0} w \cdot (u - u_S) \, d\Gamma \, ds \\
 &\leq \int_{\mathcal{F}_0} \frac{1}{2} \rho_0 |u_0|^2 \, dy + \frac{m}{2} |\ell_0|^2 + \frac{J_0}{2} |r_0|^2 + \nu\alpha \int_0^t \int_{\partial \mathcal{S}_0} |u - u_S|^2 \, d\Gamma \, ds + \nu\alpha \int_0^t \int_{\partial \mathcal{S}_0} |w|^2 \, d\Gamma \, ds
 \end{aligned}$$

and (2.17) follows. \square

The relation (2.16) motivates us to define weak solutions in the following way.

Definition 2.2. *Let $T > 0$. A pair (ρ, u) is a weak solution to system (2.6)–(2.13) if the following conditions hold true:*

- $\rho \geq 0$, $\rho \in L^\infty((0, T) \times \mathbb{R}^3)$.
- $u \in L^2(0, T; \mathcal{V})$, $\rho |u|^2 \in L^\infty(0, T; L^1(\mathbb{R}^3))$.
- The equation of continuity (2.6) is satisfied in the weak sense, i.e.,

$$\int_{\mathbb{R}^3} \left(\rho(y, T) \phi(y, T) - \rho_0 \phi(y, 0) \right) \, dy = \int_0^T \int_{\mathbb{R}^3} \left[\rho \frac{\partial \phi}{\partial s} + \rho(u - u_S) : \nabla \phi \right] \, dy \, ds,$$

for any test function $\phi \in \mathcal{D}([0, T] \times \mathbb{R}^3)$. Also, a renormalized continuity equation holds in a weak sense, i.e.,

$$\int_{\mathbb{R}^3} b(\rho)\phi dy \Big|_0^T = \int_0^T \int_{\mathbb{R}^3} b(\rho) \left[\frac{\partial \phi}{\partial s} + (u - u_S) : \nabla \phi \right] dy ds,$$

for any test function $\phi \in \mathcal{D}([0, T] \times \mathbb{R}^3)$ and for all $b \in C^1(\mathbb{R})$.

- Balance of linear momentum holds in a weak sense, i.e., for all $\phi \in C^\infty([0, T]; \mathcal{H})$ such that $\phi|_{\overline{\mathcal{F}_0}} \in C^\infty([0, T] \times \overline{\mathcal{F}_0})$ and for all $t \in [0, T]$, the relation (2.16) holds.

Remark 2.3. • There is no a priori reason that the momentum ρu is continuous in time. We only have that, for any $t_0 \in [0, T]$, the function

$$t \mapsto \int_{\mathbb{R}^3} (\rho u)(t, y) \cdot \phi(y) dy$$

is continuous in a certain neighbourhood of t_0 , provided $\phi = \phi(y) \in \mathcal{D}(\mathbb{R}^3)$ and $D(\phi) = 0$ on a neighbourhood of $\mathcal{S}(t_0)$.

- The introduction of the renormalized continuity equation in the definition of weak solutions to our system yields that $\rho \in C([0, T]; L_{\text{loc}}^q(\mathbb{R}^3))$ for all $q \in [1, \infty)$, see below for details.

In the following theorem we state the main result of our paper regarding the global existence of a weak solution to system (2.6)–(2.13).

Theorem 2.4. Let \mathcal{S}_0 be a bounded, closed, simply connected set with smooth boundary and $\mathcal{F}_0 = \mathbb{R}^3 \setminus \mathcal{S}_0$. Assume that w satisfies (2.4)–(2.5) and that there exist constants $c_1, c_2 > 0$ such that

$$\begin{aligned} \rho_0 &\geq 0, \quad \rho_0|_{\mathcal{S}_0} \in [c_1, c_2], \quad \rho_0 \in L^\infty(\mathbb{R}^3), \quad u_0 \in \mathcal{H}, \\ q_0 &= 0 \text{ a.e. on } \{\rho_0 = 0\}, \quad \frac{q_0^2}{\rho_0} \in L^1(\mathcal{F}_0). \end{aligned}$$

Then, for any $T > 0$ there exists a weak solution (ρ, u) to system (2.6)–(2.13). Moreover, we have

$$\inf_{\mathbb{R}^3} \rho_0 \leq \rho \leq \sup_{\mathbb{R}^3} \rho_0, \quad \rho \in C([0, T]; L_{\text{loc}}^q(\mathbb{R}^3)) \forall q \in [1, \infty), \quad p \in W^{-1, \infty}(0, T; L^2(\mathcal{F}_0)),$$

and for a.e. $t \in [0, T]$, the energy inequality (2.17) holds.

In order to prove the main result, as a first step, we will prove the existence of a weak solution in a bounded domain B_R , where $B_R = \{y \in \mathbb{R}^3 \mid |y| < R\}$, see Section 3. Thereafter, we take R to infinity to establish the existence of a weak solution for the whole space \mathbb{R}^3 in Section 4.

3. EXISTENCE OF A WEAK SOLUTION IN A BOUNDED DOMAIN

In this section we consider the system (2.6)–(2.13) in a smooth bounded domain $\Omega \subset \mathbb{R}^3$, along with the complementary boundary condition

$$u(y, t) = 0, \quad y \in \partial\Omega. \tag{3.1}$$

We assume $\mathcal{S}_0 \subset \Omega$ and set $\mathcal{F}_0 = \Omega \setminus \mathcal{S}_0$. Let us define a weak solution to system (2.6)–(2.13) in any bounded domain Ω . To do that, we introduce two spaces $\mathcal{H}_\Omega, \mathcal{V}_\Omega$ for the bounded domain Ω , which are analogous to \mathcal{H}, \mathcal{V} . We set

$$\begin{aligned} \mathcal{H}_\Omega &= \{ \phi \in L^2(\Omega) \mid \operatorname{div} \phi = 0 \text{ in } \Omega \text{ and } D(\phi) = 0 \text{ in } \mathcal{S}_0 \}, \\ \mathcal{V}_\Omega &= \left\{ \phi \in \mathcal{H}_\Omega \mid \phi = 0 \text{ on } \partial\Omega, \int_{\mathcal{F}_0} |\nabla \phi(y)|^2 dy < \infty \right\}. \end{aligned}$$

Definition 3.1. Let $T > 0$ and let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain. A pair (ρ, u) is a weak solution to system (2.6)–(2.13) with (3.1) if the following conditions hold true:

- $\rho \geq 0$, $\rho \in L^\infty((0, T) \times \Omega)$.
- $u \in L^2(0, T; \mathcal{V}_\Omega)$, $\rho|u|^2 \in L^\infty(0, T; L^1(\Omega))$.
- The equation of continuity (2.6) is satisfied in the weak sense, i.e.,

$$\int_{\Omega} \left(\rho(y, T)\phi(y, T) - \rho_0\phi(y, 0) \right) dy = \int_0^T \int_{\Omega} \left[\rho \frac{\partial \phi}{\partial s} + \rho(u - u_S) : \nabla \phi \right] dy ds,$$

for any test function $\phi \in \mathfrak{D}([0, T] \times \Omega)$. Also, a renormalized continuity equation holds in a weak sense, i.e.,

$$\int_{\Omega} b(\rho)\phi dy \Big|_0^T = \int_0^T \int_{\Omega} b(\rho) \left[\frac{\partial \phi}{\partial s} + (u - u_S) : \nabla \phi \right] dy ds,$$

for any test function $\phi \in \mathfrak{D}([0, T] \times \Omega)$ and for all $b \in C^1(\mathbb{R})$.

- Balance of linear momentum holds in a weak sense, i.e., for all $\phi \in C^\infty([0, T]; \mathcal{H}_\Omega)$ such that $\phi|_{\overline{\mathcal{F}_0}} \in C^\infty([0, T] \times \overline{\mathcal{F}_0})$ and for all $t \in [0, T]$, the relation (2.16) holds.

Next we assert the existence of a weak solution in a bounded domain.

Theorem 3.2. Let R be sufficiently large and $\Omega = B_R$. Set $\mathcal{F}_0 = \Omega \setminus \mathcal{S}_0$, where $\mathcal{S}_0 \subset B_R$ is a bounded, closed, simply connected set with smooth boundary. Assume that w satisfies (2.4)–(2.5) and that there exist constants $c_1, c_2 > 0$ such that

$$\begin{aligned} \rho_0 &\geq 0, \quad \rho_0|_{\mathcal{S}_0} \in [c_1, c_2], \quad \rho_0 \in L^\infty(\Omega), \quad u_0 \in \mathcal{H}_\Omega, \\ q_0 &= 0 \text{ a.e. on } \{\rho_0 = 0\}, \quad \frac{q_0^2}{\rho_0} \in L^1(\mathcal{F}_0). \end{aligned}$$

Then for any time $T > 0$ there exists a weak solution (ρ_R, u_R) to system (2.6)–(2.13) satisfying (3.1) on the time interval $(0, T)$. Moreover, we have

$$\begin{aligned} \inf_{\Omega} \rho_0 &\leq \rho_R \leq \sup_{\Omega} \rho_0, \quad \rho_R \in C([0, T]; W^{-1, \infty}(\Omega)) \cap C([0, T]; L^q(\Omega)) \quad \forall q \in [1, \infty), \\ \rho_R &\in W^{-1, \infty}(0, T; L^2(\mathcal{F}_0)), \end{aligned}$$

and for a.e. $t \in [0, T]$, the energy inequality (2.17) holds for (ρ_R, u_R) .

Proof. The proof is divided into several steps. At first we construct an approximate solution and then establish the existence of a weak solution to system (2.6)–(2.13) as the limit of this approximation.

Step 1: Construction of N -th level approximate solution

Since the set

$$\mathcal{Y} = \{ \xi \in C^1(\overline{\Omega}) \cap \mathcal{V}_\Omega \mid \operatorname{div} \xi = 0 \text{ in } \Omega \text{ and } D(\xi) = 0 \text{ in } \mathcal{S}_0 \}$$

is dense in \mathcal{V}_Ω , we can choose an orthonormal basis $z_i \in \mathcal{Y}$, for all $i \geq 1$ of the Hilbert space \mathcal{V}_Ω . Define the N -dimensional space X_N by

$$X_N = \operatorname{span}\{z_i\}_{1 \leq i \leq N}.$$

If $u_N(t), z_j \in X_N$, then there exist $\ell_N(t), \ell_{z_j} \in \mathbb{R}^3$ and $r_N(t), r_{z_j} \in \mathbb{R}^3$ such that

$$\begin{aligned} u_N(y, t) &= \ell_N(t) + r_N(t) \times y, \quad \text{for all } y \in \mathcal{S}_0, \\ z_j(y) &= \ell_{z_j} + r_{z_j} \times y, \quad \text{for all } y \in \mathcal{S}_0. \end{aligned}$$

Let us define the following quantities:

$$u_{\mathcal{S},N} = \ell_N(t) + r_N(t) \times y, \quad z_{\mathcal{S},j} = \ell_{z_j} + r_{z_j} \times y, \quad \text{for all } (y, t) \in \mathcal{F}_0 \times (0, T).$$

We are looking for u_N, ρ_N , the solution of the approximate problem, such that for some $T_N > 0$,

$$u_N \in C^1([0, T_N], X_N), \quad \rho_N \in C^1([0, T_N], C^1(\bar{\Omega})) \quad (3.2)$$

satisfy for all $z_j \in X_N$:

$$\begin{aligned} & \int_{\mathcal{F}_0} \rho_N \frac{\partial u_N}{\partial s} \cdot z_j \, dy + m \ell'_N \cdot \ell_{z_j} + J_0 r'_N \cdot r_{z_j} \\ &= - \int_{\mathcal{F}_0} [(\rho_N(u_N - u_{\mathcal{S},N}) \cdot \nabla) u_N] \cdot z_j \, dy - \int_{\mathcal{F}_0} \det(\rho_N r_N, u_N, z_j) \, dy + \det(m \ell_N, r_N, \ell_{z_j}) + \det(J_0 r_N, r_N, r_{z_j}) \\ & \quad - 2\nu \int_{\mathcal{F}_0} D(u_N) : D(z_j) \, dy - 2\nu\alpha \int_{\partial \mathcal{S}_0} (u_N - u_{\mathcal{S},N}) \cdot (z_j - z_{\mathcal{S},j}) \, d\Gamma + 2\nu\alpha \int_{\partial \mathcal{S}_0} w \cdot (z_j - z_{\mathcal{S},j}) \, d\Gamma, \end{aligned} \quad (3.3)$$

$$\frac{\partial \rho_N}{\partial t} + \operatorname{div}(\rho_N(u_N - u_{\mathcal{S},N})) = 0, \quad \rho_N(y) > 0 \text{ in } \mathcal{F}_0 \times (0, T_N), \quad (3.4)$$

$$u_N(0) = u_{0N}, \quad \rho_N(0) = \rho_{0N}. \quad (3.5)$$

Here u_{0N} and ρ_{0N} are functions satisfying

$$u_{0N} \in X_N, \quad u_{0N} \rightarrow u_0 \text{ in } L^2(\mathcal{F}_0) \text{ as } N \rightarrow \infty \quad (3.6)$$

$$\rho_{0N} \in C^1(\bar{\Omega}), \quad \rho_{0N} \rightarrow \rho_0 \text{ in } L^\infty(\Omega) \text{ weak-* as } N \rightarrow \infty, \quad (3.7)$$

$$\frac{1}{N} + \inf \rho_0 \leq \rho_{0N} \leq \frac{1}{N} + \sup \rho_0 \quad (3.8)$$

Further we have

$$\ell_{0N} \rightarrow \ell_0 \text{ in } \mathbb{R}^3, \quad \omega_{0N} \rightarrow \omega_0 \text{ in } \mathbb{R}^3 \text{ as } N \rightarrow \infty.$$

We study the local existence of u_N, ρ_N by similar arguments as in [34, Theorem 9], or in [5, Chapter VI, Theorem VI.2.1]. We define the Banach space

$$E_N = C([0, T_N], X_N).$$

We will construct a map

$$\mathcal{N} : E_N \rightarrow E_N$$

which allows to find a fixed point that is a solution to the approximate problem. We do so in three steps.

Firstly, let $v_N \in E_N$ be given. We consider

$$\frac{\partial \rho_N}{\partial t} + \operatorname{div}(\rho_N(v_N - v_{\mathcal{S},N})) = 0, \quad \rho_N(0) = \rho_{0N} \text{ in } \mathcal{F}_0 \times (0, T_N). \quad (3.9)$$

We define the trajectory $Y^N = Y_{x,t}^N(s)$ of a particle located at x at time t as

$$\frac{dY^N}{ds}(s) = v_N(Y^N(s), s), \quad \text{for all } s \geq 0, Y^N(t) = x.$$

Since $v_N \in C([0, T_N], X_N)$, we have $Y^N \in C^1([0, T_N]; C^1(\bar{\Omega}))$. For fixed v_N , (3.9) has a unique solution

$$\rho_N(x, t) = \rho_{N0}(Y_{x,t}^N(t)).$$

such that $\rho_N(0) = \rho_{N0}$.

Secondly, after the construction of ρ_N , we are looking for $u_N \in E_N$ satisfying

$$\begin{aligned} & \int_{\mathcal{F}_0} \rho_N \frac{\partial u_N}{\partial s} \cdot z_j \, dy + m \ell'_N \cdot \ell_{z_j} + J_0 r'_N \cdot r_{z_j} \\ &= - \int_{\mathcal{F}_0} [(\rho_N(v_N - v_{S,N}) \cdot \nabla) u_N] \cdot z_j \, dy - \int_{\mathcal{F}_0} \det(\rho_N r_N, u_N, z_j) \, dy + \det(m \ell_N, r_N, \ell_{z_j}) + \det(J_0 r_N, r_N, r_{z_j}) \\ & \quad - 2\nu \int_{\mathcal{F}_0} D(u_N) : D(z_j) \, dy - 2\nu\alpha \int_{\partial \mathcal{S}_0} (u_N - u_{S,N}) \cdot (z_j - z_{S,j}) \, d\Gamma + 2\nu\alpha \int_{\partial \mathcal{S}_0} w \cdot (z_j - z_{S,j}) \, d\Gamma \end{aligned} \quad (3.10)$$

for all $z_j \in X_N$. Now we seek the solution u_N of (3.10) in the form

$$u_N(x, t) = \sum_{i=1}^N \alpha_{jN}(t) z_i(x), \text{ for any } j = 1, 2, \dots, N.$$

Let us introduce the matrices and vectors

$$\mathcal{M}_N = ((z_i, z_j)_{\mathcal{H}})_{1 \leq i, j \leq N}, \quad \alpha_N = (\alpha_{jN})_{1 \leq j \leq N}, \quad \mathcal{A}_N = (a(z_i, z_j))_{1 \leq i, j \leq N}, \quad \mathcal{C} = (c(w, z_j))_{1 \leq j \leq N},$$

where

$$\begin{aligned} a(u_N, v_N) &= -2\nu \int_{\mathcal{F}_0} D(u_N) : D(v_N) \, dy - 2\nu\alpha \int_{\partial \mathcal{S}_0} (u_N - u_{S,N}) \cdot (v_N - v_{N,S}) \, d\Gamma, \\ c(w, v) &= 2\nu\alpha \int_{\partial \mathcal{S}_0} w \cdot (v - v_S) \, d\Gamma. \end{aligned}$$

Furthermore, for any $u, v \in \mathbb{R}^N$, $\mathcal{B}_N(u, v) = (\mathcal{B}_{Nj}(u, v))_{1 \leq j \leq N}$, where

$$\begin{aligned} \mathcal{B}_{Nj}(u, v) &= - \sum_{1 \leq k \leq N} v_k \int_{\mathcal{F}_0} [(\rho_N(v_N - v_{S,N}) \cdot \nabla) z_k] \cdot z_j \, dy \\ & \quad + \sum_{1 \leq i, k \leq N} u_i v_k \left(- \int_{\mathcal{F}_0} \det(\rho_N r_{z_i}, z_k, z_j) \, dy + \det(m \ell_{z_i}, r_{z_k}, \ell_{z_j}) + \det(J_0 r_{z_i}, r_{z_k}, r_{z_j}) \right). \end{aligned}$$

Thus, equation (3.10) can be viewed as

$$\mathcal{M}_N \frac{d\alpha_N}{dt} = \mathcal{A}_N \alpha_N + \mathcal{B}_N + \mathcal{C}, \quad (3.11)$$

where $\alpha_N \in \mathbb{R}^N$ is the unknown vector, $\mathcal{A}_N, \mathcal{M}_N$ are $N \times N$ matrices and $\mathcal{B}_N, \mathcal{C} \in \mathbb{R}^N$ are vectors defined as above. As $(\cdot, \cdot)_{\mathcal{H}}$ is a scalar product on $L^2(\mathbb{R}^3)$ and z_i is an orthonormal basis, we have that \mathcal{M}_N is invertible. Hence, by Cauchy-Lipschitz theory, equation (3.11) with the corresponding initial condition has a unique solution in some interval $[0, T_N]$. As a consequence, there exists a unique $u_N \in E_N$ which satisfies (3.10).

Thirdly, given $v_N \in E_N$, we have obtained a unique couple (ρ_N, u_N) that solves (3.9)–(3.10). Now we define the map $\mathcal{N} : E_N \rightarrow E_N$ by

$$\mathcal{N}(v_N) = u_N.$$

Following the steps of [5, Chapter VI, Theorem VI.2.1], we obtain that the map \mathcal{N} has a fixed point in a suitable subset of E_N . This fixed point (denoted by u_N) along with ρ_N is the solution of the nonlinear approximate problem (3.3)–(3.5).

Step 2: Global existence

Since the velocity is smooth, we obtain from (3.4) and (3.8) that

$$\frac{1}{N} + \inf \rho_0 \leq \rho_N \leq \frac{1}{N} + \sup \rho_0. \quad (3.12)$$

As in (2.19) we have

$$\int_{\mathcal{F}_0} [(\rho(u_N - u_{S,N}) \cdot \nabla) u_N] \cdot u_N dy = \frac{1}{2} \int_{\mathcal{F}_0} \frac{\partial \rho_N}{\partial s} |u_N|^2 dy. \quad (3.13)$$

We take $z_j = u_N(t) \in X_N$ as the test function in (3.3), and by using (3.13), we obtain that

$$\begin{aligned} & \frac{1}{2} \frac{d}{ds} \left(\int_{\mathcal{F}_0} \rho_N |u_N|^2 dy + m |\ell_N|^2 + J_0 |r_N|^2 \right) + 2\nu \int_{\mathcal{F}_0} |D(u_N)|^2 dy + 2\nu\alpha \int_{\partial S_0} |u_N - u_{S,N}|^2 d\Gamma \\ &= 2\nu\alpha \int_{\partial S_0} w \cdot (u_N - u_{S,N}) d\Gamma. \end{aligned}$$

Integrating the above equation from 0 to t and using Hölder's inequality, we get

$$\begin{aligned} & \int_{\mathcal{F}_0} \frac{1}{2} \rho_N |u_N|^2 dy + \frac{m}{2} |\ell_N|^2 + \frac{J_0}{2} |r_N|^2 + 2\nu \int_0^t \int_{\mathcal{F}_0} |D(u_N)|^2 dy ds + \nu\alpha \int_0^t \int_{\partial S_0} |u_N - u_{S,N}|^2 d\Gamma ds \\ & \leq \int_{\mathcal{F}_0} \frac{1}{2} \rho_{0,N} |u_{0,N}|^2 dy + \frac{m}{2} |\ell_{0,N}|^2 + \frac{J_0}{2} |r_{0,N}|^2 + \nu\alpha \int_0^t \int_{\partial S_0} |w|^2 d\Gamma ds. \end{aligned} \quad (3.14)$$

We already know that the system (3.2)–(3.5) has a maximal solution in some time interval $[0, T_N]$ with $T_N > 0$. If $T_N < T$, then $\|u_N\|_{\mathcal{H}}$ must tend to ∞ as $t \rightarrow T$. But the energy inequality for approximate solution (3.14) suggests that $\|u_N\|_{\mathcal{H}}$ is finite as $t \rightarrow T$. Thus, $T_N = T$ and with the help of (3.12) and (3.14), we have

- u_N is bounded in $L^2(0, T; \mathcal{V}_\Omega)$,
- ρ_N is bounded in $L^\infty((0, T) \times \Omega)$,
- $\sqrt{\rho_N} u_N$ is bounded in $L^\infty(0, T; L^2(\Omega))$,
- ℓ_N, r_N are bounded in $L^\infty(0, T)$.

Furthermore, by using [34, Proposition 7] and [34, Proposition 8], we obtain

- $\frac{\partial \rho_N}{\partial t}$ is bounded in $L^2(0, T; W^{-1,6}(\Omega)) \cap L^\infty(0, T; H^{-1}(\Omega))$,
- $\rho_N u_N$ is bounded in $L^2(0, T; L^6(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$,
- $u_N \otimes \rho_N u_N$ is bounded in $L^{4/3}(0, T; L^2(\Omega))$,
- $\rho_N u_N$ is bounded in $N^{1/4,2}(0, T; W^{-1,3/2}(\Omega))$ (Nikolskii spaces are introduced in (2.15)).

Step 3: Compactness argument and convergence properties

We show that the approximate solution (ρ_N, u_N) constructed in step 1 has a limit in suitable spaces and its limit is a solution to system (2.6)–(2.13) with (3.1).

Let $X \subset E \subset Y$ be Banach spaces and the imbedding $X \hookrightarrow E$ be compact.

We recall Aubin-Lions-Simon Lemma ([34, Lemma 4(ii)], [33, Corollary 4]): If F is bounded in $L^\infty(0, T; X)$ and $\frac{\partial F}{\partial t}$ is bounded in $L^r(0, T; Y)$ for some $r > 1$, then F is relatively compact in $C([0, T]; E)$. If we consider

$X = L^\infty(\Omega)$, $E = W^{-1,\infty}(\Omega)$, $Y = W^{-1,6}(\Omega)$ and $r = 2$, the properties of ρ_N enlisted above imply

$$\{\rho_N\}_{N \in \mathbb{N}} \text{ is relatively compact in } C([0, T]; W^{-1,\infty}(\Omega)).$$

Moreover, [34, Lemma 4(iv)] states that if $X \subset E \subset Y$, $X \hookrightarrow E$ is compact and $F \in L^q(0, T; X) \cap N^{s,q}(0, T; Y)$, then F is relatively compact in $L^q(0, T; E)$ for $s > 0$, $1 \leq q \leq \infty$. Hence, if we take $X = L^2(\Omega)$, $E = H^{-1}(\Omega)$, $Y = W^{-1,3/2}(\Omega)$, $q = 2$ and $s = 1/4$, the properties of ρ_N enlisted above imply

$$\{\rho_N u_N\}_{N \in \mathbb{N}} \text{ is relatively compact in } L^2(0, T; H^{-1}(\Omega)).$$

Therefore, we can extract a subsequence of $\{\rho_N, u_N\}_{N \in \mathbb{N}}$, relabelled the same, such that

- $u_N \rightharpoonup u$ in $L^2(0, T; \mathcal{V}_\Omega)$ weakly,
- $\rho_N \rightharpoonup \rho$ in $L^\infty((0, T) \times \Omega)$ weak-* and in $C([0, T]; W^{-1,\infty}(\Omega))$ strongly,
- $\rho_N u_N \rightharpoonup g$ in $L^\infty(0, T; L^2(\Omega))$ weak-* and in $L^2(0, T; H^{-1}(\Omega))$ strongly,
- $u_N \otimes \rho_N(u_N - u_{S,N}) \rightharpoonup k$ in $L^{4/3}(0, T; L^2(\Omega))$ weakly.

Let $\Omega \subset \mathbb{R}^3$ be a bounded set and let $1 \leq r < 3$, $1 \leq s \leq \infty$ with $\frac{1}{r} + \frac{1}{s} \leq 1$. According to [34, Lemma 3(iii)], the imbedding of the product of two Sobolev spaces

$$W^{1,r}(\Omega) \times W^{-1,s}(\Omega) \rightarrow W^{-1,t}(\Omega), \quad \frac{1}{t} = \frac{1}{r} - \frac{1}{3} + \frac{1}{s},$$

is continuous.

Thus, if we consider $r = 2$, $s = \infty$, then $t = 6$ and the product is continuous from $H^1(\Omega) \times W^{-1,\infty}(\Omega)$ into $W^{-1,6}(\Omega)$. The strong convergence of $\rho_N \rightarrow \rho$ in $C([0, T]; W^{-1,\infty}(\Omega))$ and the weak convergence of $u_N \rightharpoonup u$ in $L^2(0, T; \mathcal{V}_\Omega)$ imply that

$$\rho_N u_N \rightharpoonup \rho u \text{ in } L^2(0, T; W^{-1,6}(\Omega)) \text{ weakly.}$$

Hence applying $\rho_N u_N \rightharpoonup g \in L^\infty(0, T; L^2(\Omega))$ weak-* and in $L^2(0, T; H^{-1}(\Omega))$ strongly, we get

$$g = \rho u.$$

On the other hand, if we consider $r = 2$, $s = 2$, then $t = \frac{3}{2}$ and the product is continuous from $H^1(\Omega) \times H^{-1}(\Omega)$ into $W^{-1,3/2}(\Omega)$. The weak convergence of $u_N \rightharpoonup u$ in $L^2(0, T; \mathcal{V}_\Omega)$ and the strong convergence of $\rho_N u_N$ in $L^2(0, T; H^{-1}(\Omega))$ imply that

$$u_N \otimes \rho_N(u_N - u_{S,N}) \rightharpoonup u \otimes (g - \rho u_S) \text{ in } L^1(0, T; W^{-1,3/2}(\Omega)) \text{ weakly.}$$

Hence,

$$k = u \otimes \rho(u - u_S).$$

We recall the interpolation inequality: let p_1, p_2, q_1, q_2 be four numbers in $[1, \infty]$. If $f \in L^{p_1}(0, T; L^{q_1}(\Omega)) \cap L^{p_2}(0, T; L^{q_2}(\Omega))$, then for all $\theta \in (0, 1)$, the function $f \in L^p(0, T; L^q(\Omega))$ with the estimate

$$\|f\|_{L^p(0, T; L^q(\Omega))} \leq \|f\|_{L^{p_1}(0, T; L^{q_1}(\Omega))}^\theta \|f\|_{L^{p_2}(0, T; L^{q_2}(\Omega))}^{1-\theta},$$

where

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Hence, we obtain

$$\begin{aligned} \|\sqrt{\rho_N} u_N\|_{L^{\frac{8}{3}}(0, T; L^4(\Omega))} &\leq \|\sqrt{\rho_N} u_N\|_{L^\infty(0, T; L^2(\Omega))}^{1/4} \|\sqrt{\rho_N} u_N\|_{L^2(0, T; L^6(\Omega))}^{3/4} \\ &\leq C \|\sqrt{\rho_N} u_N\|_{L^\infty(0, T; L^2(\Omega))}^{1/4} \|u_N\|_{L^2(0, T; H^1(\Omega))}^{1/4}. \end{aligned}$$

Thus, the sequence $u_N \otimes \rho_N(u_N - u_{S,N})$ is bounded in the space $L^{\frac{4}{3}}(0, T; L^2(\Omega))$. We recall a result of weak convergence ([5, Chapter II, Proposition II.2.10]): let E, F be Banach spaces with F reflexive and $\{x_n\}$ be

a sequence in $E \cap F$ such that there exists $x \in E$ satisfying $\{x_n\}$ is bounded in F , $\{x_n\}$ converges weakly to x in E . Then, $\{x_n\}$ converges weakly to x in F .

In our case, if we consider $E = L^1(0, T; W^{-1,3/2}(\Omega))$ and $F = L^{\frac{4}{3}}(0, T; L^2(\Omega))$, then we have

$$u_N \otimes \rho_N(u_N - u_{S,N}) \rightharpoonup u \otimes \rho(u - u_S) \text{ in } L^{\frac{4}{3}}(0, T; L^2(\Omega)) \text{ weakly.}$$

Step 4: $N \rightarrow \infty$

As $\rho_N \rightarrow \rho$ in $\mathfrak{D}'((0, T) \times \Omega)$ and $\rho_N u_N \rightarrow \rho u$ in $\mathfrak{D}'((0, T) \times \Omega)$ as $N \rightarrow \infty$, we can pass to the limit in $\mathfrak{D}'((0, T) \times \Omega)$ in the equation (3.4) and obtain

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho(u - u_S)) = 0, \text{ in } \mathfrak{D}'((0, T) \times \Omega) \text{ and } \rho \geq 0. \quad (3.15)$$

The convergence properties of $\rho_N \rightarrow \rho$ and $u_N \rightarrow u$ as $N \rightarrow \infty$, (discussed in Step 2) yield

$$\rho \in L^\infty((0, T) \times \Omega), \quad u \in L^2(0, T; \mathcal{V}_\Omega).$$

Similarly, we can also check that the limiting variables (ρ, u) satisfy the renormalized continuity equation. Precisely, using the regularization lemma (see [24, Lemma 2.3, page 43]), we can deduce that as in [24, Theorem 2.4]: if $u \in L^2(0, T; \mathcal{V}_\Omega)$, $\rho \in L^\infty((0, T) \times \Omega)$, $\operatorname{div} u = 0$ a.e. and ρ satisfies (3.15), then for any $b \in C^1(\mathbb{R})$, $b(\rho)$ also satisfies (3.15). Thus, the continuity equation is also satisfied in the renormalized sense. This regularization lemma and renormalization arguments also help us to establish (see [5, Chapter VI, Theorem VI.1.9], [24, Theorem 2.4]) the fact that:

$$\rho_N \rightarrow \rho \text{ in } C([0, T]; L^q(\Omega)), \text{ for any } q \in [1, \infty).$$

In order to prove the existence of a weak solution (ρ, u) , it remains to verify the relation (2.16) for all $t \in [0, T]$ and $\phi \in C^\infty([0, T]; \mathcal{H}_\Omega)$ such that $\phi|_{\overline{\mathcal{F}_0}} \in C^\infty([0, T] \times \overline{\mathcal{F}_0})$. To this end, we pass to the limit in equation (3.3). We know from [5, Chapter V, Lemma V.1.2] that the set

$$\left\{ \xi_k(y) \psi_k(t) \mid \xi_k \in \mathcal{Y}, \psi_k \in C^\infty([0, T]) \right\} \text{ is dense in } C^\infty([0, T]; \mathcal{H}_\Omega).$$

Hence, it suffices to verify the relation (2.16) for all $\phi(y, t) = \xi(y) \psi(t)$, where $\xi \in \mathcal{Y}$, $\psi \in C^\infty([0, T])$.

We consider equation (3.3) with z_j replaced with $\phi(y, t) = \xi(y) \psi(t)$ and use the relation

$$\int_{\mathcal{F}_0} [(\rho_N(u_N - u_{S,N}) \cdot \nabla) u_N] \cdot \phi \, dy = \int_{\mathcal{F}_0} \frac{\partial \rho_N}{\partial t} u_N \cdot \phi \, dy - \int_{\mathcal{F}_0} [u_N \otimes \rho_N(u_N - u_{S,N})] : \nabla \phi \, dy,$$

which follows similarly as (2.19). We then obtain

$$\begin{aligned} & \int_{\mathcal{F}_0} \frac{d}{ds} (\rho_N u_N(y, s)) \cdot \phi(y, s) \, dy + m \ell'_N(s) \cdot \ell_\phi(s) + J_0 r'_N(s) \cdot r_\phi(s) \\ &= \int_{\mathcal{F}_0} [u_N \otimes \rho_N(u_N - u_{S,N})] : \nabla \phi \, dy - \int_{\mathcal{F}_0} \det(\rho_N r_N, u_N, \phi) \, dy + \det(m \ell_N, r_N, \ell_\phi) + \det(J_0 r_N, r_N, r_\phi) \\ & \quad - 2\nu \int_{\mathcal{F}_0} D(u_N) : D(\phi) \, dy - 2\nu \alpha \int_{\partial \mathcal{S}_0} (u_N - u_{S,N}) \cdot (\phi - \phi_S) \, d\Gamma + 2\nu \alpha \int_{\partial \mathcal{S}_0} w \cdot (\phi - \phi_S) \, d\Gamma. \end{aligned}$$

The main problematic term is the first one on the left hand side. To deal with this term, we integrate from 0 to t and apply the product rule so that the derivative is on the test function ϕ . In this term as well as

in the other terms, we can then pass to the limit as $N \rightarrow \infty$ by using the convergence results obtained in Step 3:

$$\begin{aligned} u_N &\rightharpoonup u \text{ in } L^2(0, T; \mathcal{V}_\Omega) \text{ weakly,} \\ \rho_N u_N &\rightharpoonup \rho u \text{ in } L^2(0, T; W^{-1,6}(\mathcal{F}_0)) \text{ weakly,} \\ u_N \otimes \rho_N(u_N - u_{\mathcal{S},N}) &\rightharpoonup u \otimes \rho(u - u_{\mathcal{S}}) \text{ in } L^{\frac{4}{3}}(0, T; L^2(\mathcal{F}_0)) \text{ weakly.} \end{aligned}$$

Finally, we obtain

$$\begin{aligned} &\int_0^t \frac{d}{ds} \left(\int_{\mathcal{F}_0} \rho u(y, s) \cdot \phi(y, s) dy + m\ell(s) \cdot \ell_\phi(s) + J_0 r(s) \cdot r_\phi(s) \right) ds \\ &= \int_0^t \left(\int_{\mathcal{F}_0} \rho u \cdot \frac{\partial \phi}{\partial s} dy + m\ell \cdot \ell'_\phi + J_0 r \cdot r'_\phi \right) ds + \int_0^t \int_{\mathcal{F}_0} [u \otimes \rho(u - u_{\mathcal{S}})] : \nabla \phi dy ds \\ &\quad - \int_0^t \int_{\mathcal{F}_0} \det(\rho r, u, \phi) dy ds + \int_0^t \det(m\ell, r, \ell_\phi) ds + \int_0^t \det(J_0 r, r, r_\phi) ds \\ &\quad - 2\nu \int_0^t \int_{\mathcal{F}_0} D(u) : D(\phi) dy ds - 2\nu\alpha \int_0^t \int_{\partial \mathcal{S}_0} (u - u_{\mathcal{S}}) \cdot (\phi - \phi_{\mathcal{S}}) d\Gamma ds + 2\nu\alpha \int_0^t \int_{\partial \mathcal{S}_0} w \cdot (\phi - \phi_{\mathcal{S}}) d\Gamma ds. \end{aligned}$$

This equality clearly yields (2.16).

Step 5: Existence of pressure

Let us introduce

$$\mathcal{V} = \{\mathfrak{D}(\mathcal{F}_0) \mid \operatorname{div} \phi = 0 \text{ in } \Omega \text{ and } D(\phi) = 0 \text{ in } \mathcal{S}_0\}.$$

We choose an appropriate test function $\phi \in C^\infty([0, T]; \mathcal{V})$ in the relation (2.16) (so that the boundary terms vanish after integrate by parts) to obtain

$$\left\langle \frac{\partial}{\partial t}(\rho u) + \operatorname{div} [u \otimes \rho(u - u_{\mathcal{S}})] + \rho r \times u - \operatorname{div}(2\nu D(u)), \phi \right\rangle = 0, \quad \forall \phi \in C^\infty([0, T]; \mathcal{V}).$$

Moreover, according to [34, Proposition 7(ii)],

$$\frac{\partial}{\partial t}(\rho u) + \operatorname{div} [u \otimes \rho(u - u_{\mathcal{S}})] + \rho r \times u - \operatorname{div}(2\nu D(u)) \in W^{-1,\infty}(0, T; H^{-1}(\mathcal{F}_0)).$$

Thus, by [34, Lemma 2], there exists $p \in W^{-1,\infty}(0, T; L^2(\mathcal{F}_0))$ such that

$$\frac{\partial}{\partial t}(\rho u) + \operatorname{div} [u \otimes \rho(u - u_{\mathcal{S}})] + \rho r \times u - \operatorname{div}(2\nu D(u)) = \nabla p, \text{ in } \mathcal{F}_0 \times (0, T).$$

□

4. PROOF OF THEOREM 2.4

We prove the existence of a weak solution in the unbounded domain by using Theorem 3.2 (existence of solution in a ball B_R) and letting $R \rightarrow \infty$.

Proof. Let R_0 be such that $\mathcal{S}_0 \subset B(0, R_0/2)$. Choose $R > R_0$ and consider a smooth function $\chi_R : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ as in [27, Theorem 1] defined by

$$\chi_R(y) = \begin{cases} y & \text{for } y \in B(0, R), \\ \frac{R}{|y|}y & \text{for } y \in \mathbb{R}^3 \setminus B(0, R) \end{cases}$$

and set

$$u_{\mathcal{S},R}(y, t) = \ell(t) + r(t) \times \chi_R(y).$$

Observe that for any $r \in \mathbb{R}^3$ and $\phi \in \mathcal{V}$:

$$(r \times \chi_R) \cdot \nabla \phi \rightarrow (r \times y) \cdot \nabla \phi \text{ in } L^2(\mathbb{R}^3) \text{ as } R \rightarrow \infty,$$

and

$$u_{\mathcal{S},R}(y, t) \rightarrow \ell(t) + r(t) \times y = u_{\mathcal{S}}(y, t) \text{ as } R \rightarrow \infty.$$

Theorem 3.2 suggests that the bounds obtained in (2.17) for (ρ_R, u_R) are uniform with respect to R . Thus the maximum principle and energy inequality (2.17) help us to conclude

$$\begin{aligned} \|\rho_R\|_{L^\infty((0,T) \times \mathbb{R}^3)} &\leq C, \\ \|\sqrt{\rho_R}u_R\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} &\leq C, \\ \|u_R\|_{L^2(0,T;\mathcal{V})} &\leq C. \end{aligned}$$

This implies that up to an extraction of a subsequence we have $\rho_R \rightharpoonup \rho$ in L^∞ weak-* in $C([0, T]; L^q_{\text{loc}}(\mathbb{R}^3))$ for any $q \in [1, \infty)$ and $u_R \rightharpoonup u$ in $L^2(0, T; \mathcal{V})$ weak. Following [12, Section 1], we can show that $\rho_R \rightarrow \rho$ in $C([0, T]; W^{-1,2}(\mathbb{R}^3))$, $\rho_R u_R \rightarrow \rho u$ in $\mathcal{D}'((0, T) \times \mathbb{R}^3)$ and ρ satisfies

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho(u - u_{\mathcal{S}})) = 0 \text{ in } \mathcal{D}'((0, T) \times \mathbb{R}^3).$$

By [12, Lemma 1], ρ also satisfies the renormalized continuity equation. On the other hand, regularization lemma [24, Lemma 2.3] and renormalization arguments help us to establish (see [5, Chapter VI, Theorem VI.1.9], [24, Theorem 2.4]) that

$$\|\rho_R - \rho\|_{L^\infty([0,T]; L^q_{\text{loc}}(\mathbb{R}^3))} \rightarrow 0 \text{ as } R \rightarrow \infty \text{ for any } q \in [1, \infty).$$

Thus, we have $\rho_R \rightarrow \rho$ strongly in $C([0, T]; L^q_{\text{loc}}(\mathbb{R}^3))$. It remains to establish identity (2.16). We already know from Theorem 3.2 that (ρ_R, u_R) satisfy (2.16), i.e.,

$$\begin{aligned} &\int_{\mathcal{F}_0} \rho_R u_R(y, t) \cdot \phi(y, t) dy + m\ell(t) \cdot \ell_\phi(t) + J_0 r(t) \cdot r_\phi(t) - \int_{\mathcal{F}_0} q_0 \cdot \phi(y, 0) dy - m\ell(0) \cdot \ell_\phi(0) - J_0 r(0) \cdot r_\phi(0) \\ &= \int_0^t \int_{\mathcal{F}_0} \rho_R u_R \cdot \frac{\partial \phi}{\partial s} dy ds + \int_0^t m\ell \cdot \ell'_\phi ds + \int_0^t J_0 r \cdot r'_\phi ds + \int_0^t \int_{\mathcal{F}_0} [u_R \otimes \rho_R (u_R - u_{\mathcal{S},R})] : \nabla \phi dy ds \\ &\quad - \int_0^t \int_{\mathcal{F}_0} \det(\rho_R r, u_R, \phi) dy ds + \int_0^t \det(m\ell, r, \ell_\phi) ds + \int_0^t \det(J_0 r, r, r_\phi) ds - 2\nu \int_0^t \int_{\mathcal{F}_0} D(u_R) : D(\phi) dy ds \\ &\quad - 2\nu\alpha \int_0^t \int_{\partial \mathcal{S}_0} (u_R - u_{\mathcal{S},R}) \cdot (\phi - \phi_{\mathcal{S}}) d\Gamma ds + 2\nu\alpha \int_0^t \int_{\partial \mathcal{S}_0} w \cdot (\phi - \phi_{\mathcal{S}}) d\Gamma ds. \end{aligned} \tag{4.1}$$

We have the following convergence results:

$$\begin{aligned} u_R &\rightharpoonup u \text{ in } L^2(0, T; \mathcal{V}) \text{ weakly (see step 3 of Theorem 3.2),} \\ u_R \otimes \rho_R(u_R - u_{S,R}) &\rightharpoonup u \otimes \rho(u - u_S) \text{ in } L^{\frac{4}{3}}(0, T; L^2(\mathbb{R}^3)) \text{ weakly (see step 3 of Theorem 3.2),} \\ \sqrt{\rho_R} u_R &\rightarrow \sqrt{\rho} u \text{ strongly in } L^2(0, T; L^2_{\text{loc}}(\mathbb{R}^3)) \text{ (see [12, Proposition 1, Section 2.4]),} \\ \rho_R &\rightarrow \rho \text{ strongly in } C([0, T]; L^q_{\text{loc}}(\mathbb{R}^3)), \forall q \in [1, \infty). \end{aligned}$$

These above mentioned convergence properties allow us to pass to the limit $R \rightarrow \infty$ in each term of (4.1) and to obtain the identity (2.16). \square

5. DISCUSSION

In this section, we discuss two variants of our system and give a few remarks. Firstly, in the case of a positive initial density we mention the stronger results that can be obtained. Then, we will concentrate on the case when the fluid viscosity depends on the density.

5.1. Positive initial density. We can improve the results when the initial density is away from zero ($\inf \rho_0 > 0$), i.e., when the fluid does not contain any vacuum regions. Actually, for the approximate solution in the bounded domain, we obtain

- For all N , $\inf \rho_N > 0$.
- $\{u_N\}$ is bounded in $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; \mathcal{V}_\Omega)$.
- Under the translation operator $\tau_h : f \mapsto f(\cdot + h)$, we can obtain from [5, Chapter VI, Lemma VI.2.5] that

$$\|\tau_h u_N - u_N\|_{L^2((0, T-h), L^2(\Omega))} \leq Ch^{1/4},$$

i.e., $\{u_N\}$ is bounded in $N^{1/4, 2}(0, T; L^2(\Omega))$.

- As $\mathcal{V}_\Omega \hookrightarrow L^2(\Omega)$ is compact and $u_N \in L^2(0, T; \mathcal{V}_\Omega) \cap N^{1/4, 2}(0, T; L^2(\Omega))$, we have that, according to [34, Lemma 4(iv)], u_N is relatively compact in $L^2(0, T; L^2(\Omega))$. This allows us to achieve the strong convergence of u_N to u in $L^2(0, T; L^2(\Omega))$ which is the same as in the case of a homogeneous fluid (i.e., when the fluid has constant density).

Thus, in this case it is easier to justify the passage of the approximate problem (3.3) and to obtain identity (2.16). Precisely, we obtain the following results:

- As $\inf \rho_0 > 0$, we obtain $1/\rho \in L^\infty((0, T) \times \mathbb{R}^3)$. By using $u = \rho u \frac{1}{\rho}$, we have $u \in L^\infty(0, T; L^2(\mathbb{R}^3))$.
- As in [34, Proposition 8(ii)], $u \in N^{1/4, 2}(0, T; L^2(\Omega))$.

5.2. Fluid viscosity depends on the density. We discuss how to deal with the case if the fluid viscosity depends on density. Here we consider the fluid viscosity ν as a C^1 function of the fluid density and it satisfies the following:

$$\text{there exists } \nu_1, \nu_2 > 0 \text{ such that } \nu_1 \leq \nu(\eta) \leq \nu_2 \text{ for all } \eta \in \mathbb{R} \text{ and } \nu' \text{ is bounded.} \quad (5.1)$$

In this case, we have the following relation analogous to (2.16):

$$\begin{aligned}
& \int_{\mathcal{F}_0} \rho u(y, t) \cdot \phi(y, t) dy + m\ell(t) \cdot \ell_\phi(t) + J_0 r(t) \cdot r_\phi(t) - \int_{\mathcal{F}_0} q_0 \cdot \phi(y, 0) dy - m\ell(0) \cdot \ell_\phi(0) - J_0 r(0) \cdot r_\phi(0) \\
&= \int_{\mathcal{F}_0} \int_0^t \rho u \cdot \frac{\partial \phi}{\partial s} ds dy + \int_0^t m\ell \cdot \ell'_\phi ds + \int_0^t J_0 r \cdot r'_\phi ds + \int_0^t \int_{\mathcal{F}_0} [u \otimes \rho(u - u_S)] : \nabla \phi dy ds \\
&\quad - \int_0^t \int_{\mathcal{F}_0} \det(\rho r, u, \phi) dy ds + \int_0^t \det(m\ell, r, \ell_\phi) ds + \int_0^t \det(J_0 r, r, r_\phi) ds - 2 \int_0^t \int_{\mathcal{F}_0} \nu(\rho) D(u) : D(\phi) dy ds \\
&\quad - 2\alpha \int_0^t \int_{\partial \mathcal{S}_0} \nu(\rho) (u - u_S) \cdot (\phi - \phi_S) d\Gamma ds + 2\alpha \int_0^t \int_{\partial \mathcal{S}_0} \nu(\rho) w \cdot (\phi - \phi_S) d\Gamma ds. \tag{5.2}
\end{aligned}$$

We can also define weak solutions in this case as previously:

Definition 5.1. *Let $T > 0$. A pair (ρ, u) is a weak solution to system (2.6)–(2.13) with $\nu = \nu(\rho)$ if the following conditions hold true:*

- $\rho \geq 0$, $\rho \in L^\infty((0, T) \times \mathbb{R}^3)$.
- $u \in L^2(0, T; \mathcal{V})$, $\rho|u|^2 \in L^\infty(0, T; L^1(\mathbb{R}^3))$.
- The equation of continuity (2.6) is satisfied in the weak sense. Also, a renormalized continuity equation holds in a weak sense.
- Balance of linear momentum holds in a weak sense, i.e., for all $\phi \in C^\infty([0, T]; \mathcal{H})$ such that $\phi|_{\overline{\mathcal{F}_0}} \in C^\infty([0, T] \times \overline{\mathcal{F}_0})$ and for all $t \in [0, T]$, the relation (5.2) holds.

Now we state the existence result of a weak solution, when viscosity is a function of fluid density:

Theorem 5.2. *Let \mathcal{S}_0 be a bounded, closed, simply connected set with smooth boundary and $\mathcal{F}_0 = \mathbb{R}^3 \setminus \mathcal{S}_0$. Assume that the self-propelled motion w satisfies (2.4)–(2.5), the fluid viscosity satisfies (5.1) and that there exist constants $c_1, c_2 > 0$ such that*

$$\begin{aligned}
\rho_0 &\geq 0, \quad \rho_0|_{\mathcal{S}_0} \in [c_1, c_2], \quad \rho_0 \in L^\infty(\mathbb{R}^3), \quad u_0 \in \mathcal{H}, \\
q_0 &= 0 \text{ a.e. on } \{\rho_0 = 0\}, \quad \frac{q_0^2}{\rho_0} \in L^1(\mathcal{F}_0).
\end{aligned}$$

Then for an arbitrary $T > 0$, there exists a weak solution (ρ, u) to system (2.6)–(2.13) with $\nu = \nu(\rho)$. Moreover, we have

$$\inf_{\mathbb{R}^3} \rho_0 \leq \rho \leq \sup_{\mathbb{R}^3} \rho_0, \quad \rho \in C([0, T]; L^q_{\text{loc}}(\mathbb{R}^3)) \forall q \in [1, \infty), \quad p \in W^{-1, \infty}(0, T; L^2(\mathcal{F}_0)),$$

and for a.e. $t \in [0, T]$, the energy inequality

$$\begin{aligned}
& \int_{\mathcal{F}_0} \frac{1}{2} \rho |u|^2 dy + \frac{m}{2} |\ell|^2 + \frac{J_0}{2} |r|^2 + 2\nu_1 \int_0^t \int_{\mathcal{F}_0} |D(u)|^2 dy ds + \alpha \nu_1 \int_0^t \int_{\partial \mathcal{S}_0} |u - u_S|^2 d\Gamma ds \\
& \leq \int_{\mathcal{F}_0} \frac{1}{2} \rho_0 |u_0|^2 dy + \frac{m}{2} |\ell_0|^2 + \frac{J_0}{2} |r_0|^2 + \alpha \nu_2 \int_0^t \int_{\partial \mathcal{S}_0} |w|^2 d\Gamma ds.
\end{aligned}$$

holds.

The proof of this theorem is similar as before. We start with an approximation for the system on a bounded domain; later we pass to the unbounded domain. Additionally to before, we have to justify the passing of the limits in the terms:

$$\begin{aligned}
 & -2 \int_0^t \int_{\mathcal{F}_0} \nu(\rho_N) D(u_N) : D(\phi) \, dy \, ds - 2\alpha \int_0^t \int_{\partial\mathcal{S}_0} \nu(\rho_N) (u_N - u_{\mathcal{S},N}) \cdot (\phi - \phi_{\mathcal{S}}) \, d\Gamma \, ds \\
 & + 2\alpha \int_0^t \int_{\partial\mathcal{S}_0} \nu(\rho_N) w \cdot (\phi - \phi_{\mathcal{S}}) \, d\Gamma \, ds.
 \end{aligned} \tag{5.3}$$

Now to do this, observe that we already have

$$\rho_N \rightarrow \rho \text{ strongly in } C([0, T]; L^q(\Omega)) \quad \forall q \in [1, \infty).$$

With the help of hypothesis (5.1) for the fluid viscosity, we have for all $q \in [1, \infty)$:

$$\|\nu(\rho_N) - \nu(\rho)\|_{L^\infty(0, T; L^q(\Omega))} \leq \|\nu'\|_\infty \|\rho_N - \rho\|_{L^\infty(0, T; L^q(\Omega))}.$$

Thus, we have strong convergence of the viscosity

$$\nu(\rho_N) \rightarrow \nu(\rho) \text{ strongly in } L^\infty(0, T; L^q(\Omega)) \quad \forall q \in [1, \infty).$$

The above strong convergence of viscosity and

$$u_N \rightharpoonup u \text{ in } L^2(0, T; \mathcal{V}_\Omega) \text{ weakly}$$

allow us to pass to the limit in the terms of (5.3) $N \rightarrow \infty$. Similarly, we can proceed with the case for unbounded domain.

5.3. Further remarks and open problems.

Remark 5.3. *Silvestre [31] proved the global existence of weak solution of a fluid-structure system in the case of Dirichlet boundary conditions. She applied a global transformation and extended the self-propelled motion from $\partial\mathcal{S}_0$ to the whole domain. It was not possible for us to adopt her approach to our setting with Navier slip boundary conditions in order to show the corresponding result. The reasons are that we could not recover the Navier-slip boundary condition and that the extension of the self-propelled motion w to the whole domain requires initial data for w .*

Remark 5.4. *In our work we require the normal component of the self-propelled velocity $W \cdot N$ and $w \cdot n$, respectively, to be zero at the boundary of the rigid body. Actually, if $w \cdot n \neq 0$, then the interface conditions (2.8)–(2.9) are replaced by*

$$\begin{aligned}
 & u \cdot n = (u_{\mathcal{S}} + w) \cdot n \text{ for } y \in \partial\mathcal{S}_0, \\
 & (D(u)n) \times n = -\alpha(u - u_{\mathcal{S}} - w) \times n \text{ for } y \in \partial\mathcal{S}_0.
 \end{aligned}$$

In that case, instead of energy inequality (2.17), we obtain

$$\begin{aligned}
 & \int_{\mathcal{F}_0} \frac{1}{2} \rho |u|^2 \, dy + \frac{m}{2} |\ell|^2 + \frac{J_0}{2} |r|^2 + 2\nu \int_0^t \int_{\mathcal{F}_0} |D(u)|^2 \, dy \, ds + \nu\alpha \int_0^t \int_{\partial\mathcal{S}_0} |u - u_{\mathcal{S}}|^2 \, d\Gamma \, ds \\
 & \leq \int_{\mathcal{F}_0} \frac{1}{2} \rho_0 |u_0|^2 \, dy + \frac{m}{2} |\ell_0|^2 + \frac{J_0}{2} |r_0|^2 + \nu\alpha \int_0^t \int_{\partial\mathcal{S}_0} |w|^2 \, d\Gamma \, ds + \frac{1}{2} \int_0^t \int_{\partial\mathcal{S}_0} \rho |u|^2 (w \cdot n) \, d\Gamma \, ds.
 \end{aligned}$$

Here, we do not know how to bound the right hand side of the above inequality by given quantities (initial conditions and self-propelled force) as it involves the unknowns ρ and u . That is why we impose the condition $w \cdot n = 0$.

Remark 5.5. *As in all the works mentioned above, the elastic setting is missing. That is, it would be desirable to consider a system with a body which may be elastic and not just rigid. The difficulty of that setting is that a transformation to a fixed domain is not possible since the body may change its shape as time evolves. Hence different methods need to be developed.*

6. APPENDIX

Derivation of the weak formulation (2.16): Let $\phi \in C^\infty([0, T]; \mathcal{H})$ such that $\phi|_{\overline{\mathcal{F}_0}} \in C^\infty([0, T] \times \overline{\mathcal{F}_0})$. We show that (2.16) is the weak form of our system (2.6)–(2.13). To this end we multiply the momentum equation (2.7) formally by ϕ and integrate over $\mathcal{F}_0 \times (0, t)$, which yields

$$\int_0^t \int_{\mathcal{F}_0} \left[\frac{\partial}{\partial s} (\rho u) + \operatorname{div} [u \otimes \rho(u - u_S)] + \rho r \times u + \nabla p - \operatorname{div}(2\nu D(u)) \right] \cdot \phi \, dy \, ds = 0. \quad (6.1)$$

In the following we will consider each term of this identity separately; in particular we will integrate by parts several times and will apply the other equations of our system.

First term.

$$\int_{\mathcal{F}_0} \int_0^t \frac{\partial}{\partial s} (\rho u) \cdot \phi \, ds \, dy = - \int_{\mathcal{F}_0} \int_0^t \rho u \cdot \frac{\partial \phi}{\partial s} \, ds \, dy + \int_{\mathcal{F}_0} (\rho u(y, t) \cdot \phi(y, t) - q_0 \cdot \phi(y, 0)) \, dy. \quad (6.2)$$

Second term.

$$\int_0^t \int_{\mathcal{F}_0} \operatorname{div} [u \otimes \rho(u - u_S)] \cdot \phi \, dy \, ds = - \int_0^t \int_{\mathcal{F}_0} [u \otimes \rho(u - u_S)] : \nabla \phi \, dy \, ds + \int_0^t \int_{\partial \mathcal{S}_0} \rho((u - u_S) \cdot n)(u \cdot \phi) \, d\Gamma \, ds,$$

where for the boundary term, we have used the relation

$$(u \otimes v)w = (v \cdot w)u, \quad \text{for } u, v, w \in \mathbb{R}^3.$$

As $u \cdot n = u_S \cdot n$ for $y \in \partial \mathcal{S}_0$, we obtain

$$\int_0^t \int_{\mathcal{F}_0} \operatorname{div} [u \otimes \rho(u - u_S)] \cdot \phi \, dy \, ds = - \int_0^t \int_{\mathcal{F}_0} [u \otimes \rho(u - u_S)] : \nabla \phi \, dy \, ds.$$

Third term.

$$\int_0^t \int_{\mathcal{F}_0} (\rho r \times u) \cdot \phi \, dy \, ds = \int_0^t \int_{\mathcal{F}_0} \det(\rho r, u, \phi) \, dy \, ds.$$

Fourth term.

$$\int_0^t \int_{\mathcal{F}_0} \nabla p \cdot \phi \, dy \, ds = \int_0^t \int_{\partial \mathcal{S}_0} pn \cdot \phi \, d\Gamma \, ds = \int_0^t \int_{\partial \mathcal{S}_0} pn \cdot \phi_S \, d\Gamma \, ds = \int_0^t \int_{\partial \mathcal{S}_0} pn \cdot (\ell_\phi + r_\phi \times y) \, d\Gamma \, ds,$$

with ϕ_S as in (2.14).

Fifth term. We analyze this term as in [27, Lemma 1] and obtain

$$\begin{aligned}
 & - \int_0^t \int_{\mathcal{F}_0} \operatorname{div}(2\nu D(u)) \cdot \phi \, dy \, ds \\
 & = 2\nu \int_0^t \int_{\mathcal{F}_0} D(u) : D(\phi) \, dy \, ds - 2\nu \int_0^t \int_{\partial\mathcal{S}_0} D(u)n \cdot \ell_\phi \, d\Gamma \, ds \\
 & \quad - 2\nu \int_0^t \int_{\partial\mathcal{S}_0} (y \times D(u)n) \cdot r_\phi \, d\Gamma \, ds - 2\nu \int_0^t \int_{\partial\mathcal{S}_0} (D(u)n \times n) \cdot [(\phi - \phi_S) \times n] \, d\Gamma \, ds
 \end{aligned} \tag{6.3}$$

After plugging the terms (6.2)–(6.3) into the relation (6.1), we get

$$\begin{aligned}
 & - \int_0^t \int_{\mathcal{F}_0} \rho u \cdot \frac{\partial \phi}{\partial s} \, dy \, ds + \int_{\mathcal{F}_0} (\rho u(y, t) \cdot \phi(y, t) - q_0 \cdot \phi(y, 0)) \, dy - \int_0^t \int_{\mathcal{F}_0} [u \otimes \rho(u - u_S)] : \nabla \phi \, dy \, ds \\
 & + \int_0^t \int_{\mathcal{F}_0} \det(\rho r, u, \phi) \, dy \, ds + 2\nu \int_0^t \int_{\mathcal{F}_0} D(u) : D(\phi) \, dy \, ds \\
 & = 2\nu \int_0^t \int_{\partial\mathcal{S}_0} (D(u)n \times n) \cdot [(\phi - \phi_S) \times n] \, d\Gamma \, ds - \int_0^t \int_{\partial\mathcal{S}_0} p n \cdot (\ell_\phi + r_\phi \times y) \, d\Gamma \, ds \\
 & + 2\nu \int_0^t \int_{\partial\mathcal{S}_0} D(u)n \cdot \ell_\phi \, d\Gamma \, ds + 2\nu \int_0^t \int_{\partial\mathcal{S}_0} (y \times D(u)n) \cdot r_\phi \, d\Gamma \, ds.
 \end{aligned}$$

We use equations (2.10)–(2.11) to rewrite the preceding relation as

$$\begin{aligned}
 & - \int_0^t \int_{\mathcal{F}_0} \rho u \cdot \frac{\partial \phi}{\partial s} \, dy \, ds + \int_{\mathcal{F}_0} (\rho u(y, t) \cdot \phi(y, t) - q_0 \cdot \phi(y, 0)) \, dy - \int_0^t \int_{\mathcal{F}_0} [u \otimes \rho(u - u_S)] : \nabla \phi \, dy \, ds \\
 & + \int_0^t \int_{\mathcal{F}_0} \det(\rho r, u, \phi) \, dy \, ds + 2\nu \int_0^t \int_{\mathcal{F}_0} D(u) : D(\phi) \, dy \, ds \\
 & = 2\nu \int_0^t \int_{\partial\mathcal{S}_0} (D(u)n \times n) \cdot [(\phi - \phi_S) \times n] \, d\Gamma \, ds \\
 & - \int_0^t m \ell' \cdot \ell_\phi \, ds + \int_0^t (m \ell \times r) \cdot \ell_\phi \, ds - \int_0^t J_0 r' \cdot r_\phi \, ds + \int_0^t (J_0 r \times r) \cdot r_\phi \, ds.
 \end{aligned} \tag{6.4}$$

Now, integration by parts with respect to time yields

$$\begin{aligned} - \int_0^t m\ell' \cdot \ell_\phi ds - \int_0^t J_0 r' \cdot r_\phi ds &= \int_0^t m\ell \cdot \ell'_\phi ds - m\ell(t) \cdot \ell_\phi(t) ds + m\ell(0) \cdot \ell_\phi(0) \\ &\quad + \int_0^t J_0 r \cdot r'_\phi ds - J_0 r(t) \cdot r_\phi(t) + J_0 r(0) \cdot r_\phi(0). \end{aligned}$$

Regarding the first term on the right hand side of (6.4), we observe that by (2.9)

$$\int_{\partial S_0} (D(u)n \times n) \cdot [(\phi - \phi_S) \times n] d\Gamma = -\alpha \int_{\partial S_0} [(u - u_S - w) \times n] \cdot [(\phi - \phi_S) \times n] d\Gamma.$$

The identity $(A \times B) \cdot (C \times D) = (A \cdot C)(B \cdot D) - (A \cdot D)(B \cdot C)$ for any $A, B, C, D \in \mathbb{R}^3$ together with $(u - u_S - w) \cdot n = 0$ by (2.4) and (2.8) yield

$$\int_{\partial S_0} (D(u)n \times n) \cdot [(\phi - \phi_S) \times n] d\Gamma = -\alpha \int_{\partial S_0} (u - u_S - w) \cdot (\phi - \phi_S) d\Gamma. \quad (6.5)$$

Putting (6.4)–(6.5) together, we obtain the weak form (2.16) of our system.

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REFERENCES

- [1] F. ALOUGES, A. DESIMONE, AND A. LEFEBVRE, *Optimal strokes for low Reynolds number swimmers: an example*, J. Nonlinear Sci., 18 (2008), pp. 277–302.
- [2] H. A. BABA, N. V. CHEMETOV, Š. NEČASOVÁ, AND B. MUHA, *Strong solutions in L^2 framework for fluid-rigid body interaction problem. Mixed case*, Topol. Methods Nonlinear Anal., 52 (2018), pp. 337–350.
- [3] T. BODNÁR, G. P. GALDI, AND Š. NEČASOVÁ, *Fluid-Structure Interaction and Biomedical Applications*, Birkhäuser/Springer, Basel, 2014.
- [4] ———, *Particles in flows*, Advances in Mathematical Fluid Mechanics, Birkhäuser/Springer, Cham, 2017.
- [5] F. BOYER AND P. FABRIE, *Mathematical tools for the study of the incompressible Navier-Stokes equations and related models*, vol. 183 of Applied Mathematical Sciences, Springer, New York, 2013.
- [6] A. BRESSAN, *Impulsive control of Lagrangian systems and locomotion in fluids*, Discrete Contin. Dyn. Syst., 20 (2008), pp. 1–35.
- [7] D. BUCUR, E. FEIREISL, AND Š. NEČASOVÁ, *Boundary behavior of viscous fluids: influence of wall roughness and friction-driven boundary conditions*, Arch. Ration. Mech. Anal., 197 (2010), pp. 117–138.
- [8] T. CHAMBRION AND A. MUNNIER, *On the locomotion and control of a self-propelled shape-changing body in a fluid*, J. of Nonlinear Science, (to appear).
- [9] N. V. CHEMETOV AND Š. NEČASOVÁ, *The motion of the rigid body in the viscous fluid including collisions. Global solvability result*, Nonlinear Anal. Real World Appl., 34 (2017), pp. 416–445.
- [10] C. CONCA, J. S. M. H., AND M. TUCSNAK, *Existence of solutions for the equations modelling the motion of a rigid body in a viscous fluid*, Comm. Partial Differential Equations, 25 (2000), pp. 1019–1042.
- [11] P. CUMSILLE AND T. TAKAHASHI, *Wellposedness for the system modelling the motion of a rigid body of arbitrary form in an incompressible viscous fluid*, Czechoslovak Math. J., 58 (2008), pp. 961–992.
- [12] B. DESJARDINS, *Global existence results for the incompressible density-dependent Navier-Stokes equations in the whole space*, Differential Integral Equations, 10 (1997), pp. 587–598.
- [13] B. DESJARDINS AND M. J. ESTEBAN, *Existence of weak solutions for the motion of rigid bodies in a viscous fluid*, Arch. Ration. Mech. Anal., 146 (1999), pp. 59–71.

- [14] ———, *On weak solutions for fluid-rigid structure interaction: compressible and incompressible models*, Comm. Partial Differential Equations, 25 (2000), pp. 1399–1413.
- [15] R. J. DiPERNA AND P. L. LIONS, *Ordinary differential equations, transport theory and sobolev spaces*, Invent. Math., 98 (1989), pp. 511–547.
- [16] G. GALDI, *On the steady self-propelled motion of a body in a viscous incompressible fluid*, Arch. Rat. Mech. Anal., 148 (1999), pp. 53–88.
- [17] G. P. GALDI, *On the motion of a rigid body in a viscous liquid: a mathematical analysis with applications*, in Handbook of mathematical fluid dynamics, Vol. I, North-Holland, Amsterdam, 2002, pp. 653–791.
- [18] M. GEISSERT, K. GÖTZE, AND M. HIEBER, *L^p -theory for strong solutions to fluid-rigid body interaction in Newtonian and generalized Newtonian fluids*, Trans. Amer. Math. Soc., 365 (2013), pp. 1393–1439.
- [19] D. GÉRARD-VARET AND M. HILLAIRET, *Existence of weak solutions up to collision for viscous fluid-solid systems with slip*, Comm. Pure Appl. Math., 67 (2014), pp. 2022–2075.
- [20] D. GÉRARD-VARET, M. HILLAIRET, AND C. WANG, *The influence of boundary conditions on the contact problem in a 3D Navier-Stokes flow*, J. Math. Pures Appl., 9 (2015), pp. 1–38.
- [21] D. GÉRARD-VARET AND N. MASMOUDI, *Relevance of the slip condition for fluid flows near an irregular boundary*, Comm. Math. Phys., 295 (2010), pp. 99–137.
- [22] M. D. GUNZBURGER, H.-C. LEE, AND G. A. SEREGIN, *Global existence of weak solutions for viscous incompressible flows around a moving rigid body in three dimensions*, J. Math. Fluid Mech., 2 (2000), pp. 219–266.
- [23] W. JÄGER AND A. MIKELIĆ, *On the roughness-induced effective boundary conditions for an incompressible viscous flow*, J. Differential Equations, 170 (2001), pp. 96–122.
- [24] P.-L. LIONS, *Mathematical topics in fluid mechanics. Vol. 1*, vol. 3 of Oxford Lecture Series in Mathematics and its Applications, The Clarendon Press, Oxford University Press, New York, 1996. Incompressible models, Oxford Science Publications.
- [25] D. MAITY AND M. TUCSNAK, *L^p - L^q maximal regularity for some operators associated with linearized incompressible fluid-rigid body problems*, 710 (2018), pp. 175–201.
- [26] J. A. S. MARTÍN, V. STAROVOITOV, AND M. TUCSNAK, *Global weak solutions for the two-dimensional motion of several rigid bodies in an incompressible viscous fluid*, Arch. Ration. Mech. Anal., 161 (2002), pp. 113–147.
- [27] G. PLANAS AND F. SUEUR, *On the “viscous incompressible fluid + rigid body” system with Navier conditions*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 31 (2014), pp. 55–80.
- [28] J. SAN MARTÍN, T. TAKAHASHI, AND M. TUCSNAK, *A control theoretic approach to the swimming of microscopic organisms*, Quart. Appl. Math., 65 (2007), pp. 405–424.
- [29] D. SERRE, *Chute libre d’un solide dans un fluide visqueux incompressible*, Japan J. Appl. Math., 4 (1987), pp. 99–110.
- [30] M. SIGALOTTI AND J.-C. VIVALDA, *Controllability properties of a class of systems modelling swimming microscopic organisms*, ESAIM Control Optim. Calc. Var. to appear.
- [31] A. L. SILVESTRE, *On the self-propelled motion of a rigid body in a viscous liquid and on the attainability of steady symmetric self-propelled motions*, J. Math. Fluid Mech., 4 (2002), pp. 285–326.
- [32] ———, *On the slow motion of a self-propelled rigid body in a viscous incompressible fluid*, J. Math. Anal. Appl., 274 (2002), p. 203227.
- [33] J. SIMON, *Compact sets in the space $L^p(0, T; B)$* , Ann. Mat. Pura Appl. (4), 146 (1987), pp. 65–96.
- [34] ———, *Nonhomogeneous viscous incompressible fluids: existence of velocity, density, and pressure*, SIAM J. Math. Anal., 21 (1990), pp. 1093–1117.
- [35] V. N. STAROVOITOV, *Solvability of the problem of the self-propelled motion of several rigid bodies in a viscous incompressible fluid*, Comput. Math. Appl., 53 (2007), pp. 413–435.
- [36] T. TAKAHASHI, *Analysis of strong solutions for the equations modeling the motion of a rigid-fluid system in a bounded domain*, Adv. Differential Equations, 8 (2003), pp. 1499–1532.
- [37] C. WANG, *Strong solutions for the fluid-solid systems in a 2-D domain*, Asymptot. Anal., 89 (2014), pp. 263–306.

INSTITUTE OF MATHEMATICS, CZECH ACADEMY OF SCIENCES, ŽITNÁ 25, 11567 PRAHA 1, CZECH REPUBLIC
E-mail address: `matus@math.cas.cz`

CHENNAI MATHEMATICAL INSTITUTE, H1, SITCOT IT PARK, SIRUSERI, 603103
E-mail address: `mythily@cmi.ac.in`

INRIA NANCY - GRAND EST, 615, RUE DU JARDIN BOTANIQUE. 54600 VILLERS-LÈS-NANCY, FRANCE
E-mail address: `royarnab244@gmail.com`

INSTITUTE OF MATHEMATICS, UNIVERSITY OF WÜRZBURG, EMIL-FISCHER-STR. 40, 97074 WÜRZBURG, GERMANY
E-mail address: `anja.schloemerkerper@mathematik.uni-wuerzburg.de`