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# Embeddings between weighted local Morrey-type spaces

*Amiran Gogatishvili*
*Tuğçe Ünver*

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# EMBEDDINGS BETWEEN WEIGHTED LOCAL MORREY-TYPE SPACES

AMIRAN GOGATISHVILI AND TUĞÇE ÜNVER

*Dedicated to the $75^{th}$ birthday of Professor Lars Erik Persson*

Abstract. In this paper certain $n$-dimensional inequalities are shown to be equivalent to the inequalities in the one-dimensional setting. By this means, embeddings between weighted local Morrey-type spaces are characterized for some ranges of parameters.

## 1. Introduction

In this paper, we study some inequalities involving one-dimensional and $n$-dimensional Hardy-type operators. Our goal is to find the weight characterizations of inequality

$$\left(\int_0^\infty \left(\int_{B(0,t)} f(s)^{p_2} v_2(s)^{p_2} ds\right)^{\frac{q_2}{p_2}} w_2(t)^{q_2} dt\right)^{\frac{1}{q_2}} \leq c\left(\int_0^\infty \left(\int_{B(0,t)} f(s)^{p_1} v_1(s)^{p_1} ds\right)^{\frac{q_1}{p_1}} w_1(t)^{q_1} dt\right)^{\frac{1}{q_1}}, \tag{1.1}$$

where $0 < p_1, p_2, q_1, q_2 < \infty$ and $v_1, v_2, w_1, w_2$ are non-negative measurable functions. In order to characterize (1.1) we will use the solutions of the corresponding one-dimensional inequality.

Let us first present some notations used in this paper.

Let $A$ be a nonempty measurable subset of $\mathbb{R}^n$, $n \geq 1$. By $\mathcal{M}(A)$, we denote the set of all measurable functions on $A$. We denote by $\mathcal{M}^+(A)$, the set of all nonnegative measurable functions on $A$. A weight is a measurable, positive and finite a.e function on $A$ and we will denote the set of weights by $\mathcal{W}(A)$.

For $p \in [0,\infty)$, we define the functional $\|\cdot\|_{p,A}$ on $\mathcal{M}(A)$ by

$$\|f\|_{p,A} := \begin{cases} \left(\int_A |f(x)|^p dx\right)^{\frac{1}{p}} & if \quad p < \infty, \\ \operatorname*{ess\,sup}_{x\in A} |f(x)| & if \quad p = \infty. \end{cases}$$

If $w \in \mathcal{W}(A)$, then the weighted Lebesgue space $L_{p,w}(A)$ is given by

$$L_{p,w}(A) := \{f \in \mathcal{M}(A) : \|fw\|_{p,A} < \infty\}.$$

If $A = (a,b)$, $0 \leq a < b \leq \infty$, then we write $\mathcal{M}(a,b)$, $\mathcal{M}^+(a,b)$, $\mathcal{W}^+(a,b)$, $L_p(a,b)$ and $L_\infty(a,b)$, correspondingly.

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ be the open ball centered at $x$ of radius $r$ and ${}^{\mathbf{c}}B(x,r) = \mathbb{R}^n \backslash B(x,r)$.

Let $X$ and $Y$ be quasi normed vector spaces. If $X \subset Y$ and the identity operator is continuous from $X$ to $Y$, that is, there exists a positive constant $C$ such that $\|\,\mathrm{I}(z)\|_Y \leq C\|z\|_X$ for all $z \in X$, we say that $X$ is embedded into $Y$ and write $X \hookrightarrow Y$. Throughout the paper, we always denote by $c$ and $C$ a positive constant which is independent of main parameters but it may vary from line to line. However, a constant with subscript such as $c_1$ does not change in different occurrences. The standard notation $A \lesssim B$ means that there exists a constant $\lambda > 0$ depending only on inessential

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parameters such that $A \le \lambda B$ and we write $A \approx B$ if both $A \lesssim B$ and $B \lesssim A$. Since the expressions in our main results are too long, to make the formulas simpler we sometimes omit the differential element $dx$. We will denote the left-hand side (right-hand side) of an inequality $*$ by $LHS(*)$ $(RHS(*))$.

We denote by $LM_{p,q}(v,w)$, the weighted local Morrey-type spaces, the collection of all functions $f \in L^{loc}_{p,v}(\mathbb{R}^n)$ such that

$$\|f\|_{LM_{p,q}(v,w)} = \left( \int_0^\infty \left( \int_{B(0,t)} f(s)^p v(s)^p ds \right)^{\frac{q}{p}} w(t)^q dt \right)^{\frac{1}{q}} < \infty,$$

and ${}^{\mathfrak{c}}LM_{p,q}(v,w)$, the weighted complementary local Morrey-type spaces, the collection of all functions $f \in L_{p,v}({}^{\mathfrak{c}}B(0,t))$ such that

$$\|f\|_{{}^{\mathfrak{c}}LM_{p,q}(v,w)} = \left( \int_0^\infty \left( \int_{{}^{\mathfrak{c}}B(0,t)} f(s)^p v(s)^p ds \right)^{\frac{q}{p}} w(t)^q dt \right)^{\frac{1}{q}} < \infty,$$

where $p, q \in (0,\infty)$, $w \in \mathcal{M}^+(0,\infty)$ and $v \in \mathcal{W}(\mathbb{R}^n)$

Under these notations, (1.1) is equivalent to the continuous embedding between weighted local Morrey-type spaces, that is,

$$LM_{p_1,q_1}(v_1,w_1) \hookrightarrow LM_{p_2,q_2}(v_2,w_2). \tag{1.2}$$

The properties of local Morrey-type spaces and boundedness of the classical operators in these spaces have been studied intensively (for the detailed history and related results of these spaces we refer to the survey papers [1] and [2]). Some necessary and sufficient conditions for the boundedness of the maximal operator, fractional maximal operator, Riesz potential and singular integral operator in the local Morrey-type spaces $LM_{p,q}(1,w)({}^{\mathfrak{c}}LM_{p,q}(1,w))$ were given in [3, 5–9].

The main tool in the literature to investigate the boundedness of the aforementioned operators in local Morrey-type spaces is to reduce these problems to the boundedness of the Hardy operator in weighted Lebesgue spaces on the cone of nonnegative, nonincreasing functions with the help of Hölder's inequality. But in [4] authors characterized the boundedness of maximal operator from $L_{p_1}$ to $LM_{p_2,q}(1,w)$ by using the characterization of the embeddings $L_{p_1} \hookrightarrow LM_{p_2,q}(1,w)$ which is multidimensional weighted Hardy's inequality.

Since Hölder's inequality is strict, it is possible to obtain better results for the boundedness problem in local Morrey-type spaces, using the characterization of the embeddings between local Morrey-type spaces. With this motivation we started investigating the embedding problem between various function spaces. In [18], characterizations of the embeddings between weighted Lebesgue spaces and weighted (complementary) local Morrey-type spaces are given with the help of solutions of the multidimensional direct and reverse weighted Hardy inequalities. In [15], embeddings between weighted complementary local Morrey-type spaces and weighted local Morrey-type spaces are investigated. The method was based on the duality approach which allowed us to reduce this problem to the characterizations of some iterated Hardy inequalities. It was not possible to solve the embedding (1.2), because it required the solutions of some iterated Hardy inequalities which were unknown at that time. These problems have been solved in [17], recently.

Although the same method works in our case, in this paper we will use a different approach to give the characterization of the embeddings (1.2). We will reduce this problem to the one-dimensional case and use the solutions of the embeddings between weighted Cesàro function spaces from [20] which can be seen as the one-dimensional case of our problem.

Let us briefly describe the structure of our paper. In the next section we give some information about the equivalency of some one-dimensional and $n$-dimensional inequalities. In the last section, we will provide some background information on the properties of weighted local Morrey-type spaces and characterize the embeddings between these spaces.

## 2. Equivalent Inequalities

The weight characterizations for one-dimensional Hardy inequality, that is,

$$\Big(\int_0^\infty \Big(\int_0^x f(t)dt\Big)^q w(x)dx\Big)^{\frac{1}{q}} \le C\Big(\int_0^\infty f(t)^p v(t)dt\Big)^{\frac{1}{p}}$$

have been studied for all $f \in \mathcal{M}^+(0,\infty)$, throughly, where $0 < q \le \infty$ and $1 \le p \le \infty$. For a detailed history, see [16].

In [12], the complete weight characterizations for so-called reverse Hardy inequality, that is,

$$\Big(\int_0^\infty \Big(\int_0^x f(t)dt\Big)^q w(x)dx\Big)^{\frac{1}{q}} \ge C\Big(\int_0^\infty f(t)^p v(t)dt\Big)^{\frac{1}{p}}$$

were given for all $f \in \mathcal{M}^+(0,\infty)$ using the discretization method, where $0 < q \le \infty$ and $0 < p \le 1$.

In [11], using polar coordinates, authors extended the one-dimensional Hardy inequality to $n$-dimensional case and gave the characterizations of weights for which

$$\Big(\int_{\mathbb{R}^n} \Big(\int_{B(0,|x|)} f(t)dt\Big)^q w(x)dx\Big)^{\frac{1}{q}} \le C\Big(\int_{\mathbb{R}^n} f(t)^p v(t)dt\Big)^{\frac{1}{p}}$$

holds for all $f \in \mathcal{M}^+(\mathbb{R}^n)$, where $0 < q < \infty$ and $1 < p < \infty$. Note that in [10], this problem has been considered for a special case.

In [14], authors dealt with the multidimensional analogue of reverse Hardy inequality,

$$\Big(\int_0^\infty \Big(\int_{B(0,x)} f(t)dt\Big)^q w(x)dx\Big)^{\frac{1}{q}} \ge C\Big(\int_{\mathbb{R}^n} f(t)^p v(t)dt\Big)^{\frac{1}{p}} \tag{2.1}$$

for all $f \in \mathcal{M}^+(\mathbb{R}^n)$, where $0 < q \le \infty$ and $0 < p \le 1$. The approach they used here is the multidimensional analogues of the discretization technique given in [12].

Although these problems are in higher dimensions, it seems that with some simple modifications in one-dimensional theory, they can also be analysed using the similar steps. On that account it would be better to reduce the problems in the $n$-dimensional cases to the problems in one-dimesional cases. In [19], Sinnamon handled the $n$-dimensional Hardy inequality by reducing it to one-dimensional Hardy inequality and extended the results on $n$-dimensional Hardy inequality to more general star shape domains. We will use ideas from [19] in this paper.

Before proceeding to the theorems, let us recall the following integration in polar coordinates formula. By $S^{n-1}$, we denote the unit sphere $\{x \in \mathbb{R}^n : |x| = 1\}$. If $x \in \mathbb{R}^n \backslash \{0\}$, the polar coordinates of $x$ are

$$r = |x| \in (0,\infty), \quad x' = \frac{x}{|x|} \in S^{n-1}.$$

There is a unique Borel measure $\sigma = \sigma_{n-1}$ on $S^{n-1}$ such that if $f$ is Borel measurable on $\mathbb{R}^n$ and $f \ge 0$ or $f \in L_1(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} f(x)dx = \int_0^\infty \int_{S^{n-1}} f(\tau x')\tau^{n-1} d\sigma(x')d\tau$$

(see, for instance, [13]).

Let us now present our equivalency results.

**Theorem 2.1.** *Let $0 < p < 1$ and $0 < q, \theta < \infty$. Assume that $u, w \in \mathcal{W}(0,\infty)$ and $v \in \mathcal{W}(\mathbb{R}^n)$. Then the following two statements are equivalent.*

(i) *There exists a constant $C$ such that*

$$\Big(\int_0^\infty \Big(\int_{B(0,t)} f(s)^p v(s)ds\Big)^{\frac{q}{p}} u(t)dt\Big)^{\frac{1}{q}} \le C\Big(\int_0^\infty \Big(\int_{B(0,t)} f(s)ds\Big)^{\theta} w(t)dt\Big)^{\frac{1}{\theta}} \tag{2.2}$$

*holds for every $f \in \mathcal{M}^+(\mathbb{R}^n)$.*

(ii) *There exists a constant* $C'$ *such that*

$$\bigg(\int_0^\infty \bigg(\int_0^t g(s)^p \tilde{v}(s)ds\bigg)^{\frac{q}{p}} u(t)dt\bigg)^{\frac{1}{q}} \le C'\bigg(\int_0^\infty \bigg(\int_0^t g(s)ds\bigg)^{\theta} w(t)dt\bigg)^{\frac{1}{\theta}} \tag{2.3}$$

*holds for every* $g \in \mathcal{M}^+(0,\infty)$, *where*

$$\tilde{v}(t) := \bigg(\int_{S^{n-1}} v(ts')^{\frac{1}{1-p}} d\sigma(s')\bigg)^{1-p} t^{(n-1)(1-p)}.$$

*Moreover the best constants of inequalities* (2.2) *and* (2.3) *satisfies* $C = C'$.

*Proof.* Assume first that (2.2) holds for all $f \in \mathcal{M}^+(\mathbb{R}^n)$. Set

$$f(x) = g(|x|)\bigg(\int_{S^{n-1}} v(|x|\tau')^{\frac{1}{1-p}} d\sigma(\tau')\bigg)^{-1} v(x)^{\frac{1}{1-p}} |x|^{1-n}.$$

Using spherical coordinates, we have that

$$\begin{aligned}
LHS(2.2) &= \bigg(\int_0^\infty \bigg(\int_0^t \int_{S^{n-1}} f(rs')^p v(rs') d\sigma(s') r^{n-1} dr\bigg)^{\frac{q}{p}} u(t)dt\bigg)^{\frac{1}{q}} \\
&= \bigg(\int_0^\infty \bigg(\int_0^t \int_{S^{n-1}} g(|rs'|)^p \bigg(\int_{S^{n-1}} v(|rs'|\tau')^{\frac{1}{1-p}} d\sigma(\tau')\bigg)^{-p} \\
&\qquad\qquad \times v(rs')^{\frac{p}{1-p}+1} |rs'|^{p(1-n)} d\sigma(s') r^{n-1} dr\bigg)^{\frac{q}{p}} u(t)dt\bigg)^{\frac{1}{q}} \\
&= \bigg(\int_0^\infty \bigg(\int_0^t g(r)^p \bigg(\int_{S^{n-1}} v(r\tau')^{\frac{1}{1-p}} d\sigma(\tau')\bigg)^{1-p} r^{(n-1)(1-p)} dr\bigg)^{\frac{q}{p}} u(t)dt\bigg)^{\frac{1}{q}} \\
&= \bigg(\int_0^\infty \bigg(\int_0^t g(s)^p \tilde{v}(s) ds\bigg)^{\frac{q}{p}} u(t)dt\bigg)^{\frac{1}{q}} \\
&= LHS(2.3).
\end{aligned}$$

Moreover,

$$\begin{aligned}
RHS(2.2) &= C\bigg(\int_0^\infty \bigg(\int_0^t \bigg[\int_{S^{n-1}} f(rs') d\sigma(s')\bigg] r^{n-1} dr\bigg)^{\theta} w(t)dt\bigg)^{\frac{1}{\theta}} \\
&= C\bigg(\int_0^\infty \bigg(\int_0^t g(r)dr\bigg)^{\theta} w(t)dt\bigg)^{\frac{1}{\theta}}.
\end{aligned}$$

Therefore, (2.3) holds with the best constant $C'$ such that $C' \le C$.

Conversely, assume now that (2.3) holds for all $g \in \mathcal{M}^+(0,\infty)$. Applying Hölder's inequality with exponents $(\frac{1}{p}, \frac{1}{1-p})$, we get that

$$\begin{aligned}
LHS(2.2) &\le \bigg(\int_0^\infty \bigg(\int_0^t \bigg[\int_{S^{n-1}} f(rs') d\sigma(s')\bigg]^p \bigg[\int_{S^{n-1}} v(rs')^{\frac{1}{1-p}} d\sigma(s')\bigg]^{1-p} r^{n-1} dr\bigg)^{\frac{q}{p}} u(t)dt\bigg)^{\frac{1}{q}} \\
&= \bigg(\int_0^\infty \bigg(\int_0^t \tilde{v}(r) \bigg[\int_{S^{n-1}} f(rs')^p d\sigma(s')\bigg]^p r^{p(n-1)} dr\bigg)^{\frac{q}{p}} u(t)dt\bigg)^{\frac{1}{q}}.
\end{aligned}$$

Applying inequality (2.3), we obtain that

$$\begin{aligned}
LHS(2.2) &\le C'\bigg(\int_0^\infty \bigg(\int_0^t \bigg[\int_{S^{n-1}} f(rs') d\sigma(s')\bigg] r^{(n-1)} dr\bigg)^{\theta} w(t)dt\bigg)^{\frac{1}{\theta}} \\
&= C'\bigg(\int_0^\infty \bigg(\int_{B(0,t)} f(s)ds\bigg)^{\theta} w(t)dt\bigg)^{\frac{1}{\theta}}
\end{aligned}$$

holds. Hence, (2.2) holds. Moreover the best constant of inequality (2.2) satisfy $C \le C'$. □

Let us now consider the case when $p = 1$.

**Theorem 2.2.** *Let $0 < q, \theta < \infty$. Assume that $u, w \in \mathcal{W}(0,\infty)$ and $v \in \mathcal{W}(\mathbb{R}^n)$. Then the following two statements are equivalent.*

(i) *There exists a constant $C$ such that*

$$\left(\int_0^\infty \left(\int_{B(0,t)} f(s)v(s)ds\right)^q u(t)dt\right)^{\frac{1}{q}} \le C\left(\int_0^\infty \left(\int_{B(0,t)} f(s)ds\right)^\theta w(t)dt\right)^{\frac{1}{\theta}} \tag{2.4}$$

*holds for every $f \in \mathcal{M}^+(\mathbb{R}^n)$.*

(i) *There exists a constant $C'$ such that*

$$\left(\int_0^\infty \left(\int_0^t g(s)\tilde{v}(s)ds\right)^q u(t)dt\right)^{\frac{1}{q}} \le C'\left(\int_0^\infty \left(\int_0^t g(s)ds\right)^\theta w(t)dt\right)^{\frac{1}{\theta}} \tag{2.5}$$

*holds for every $g \in \mathcal{M}^+(0,\infty)$, where*

$$\tilde{v}(t) := \operatorname*{ess\,sup}_{s' \in S^{n-1}} v(ts').$$

*Moreover the best constants of inequalities (2.4) and (2.5) satisfy $C \approx C'$.*

*Proof.* Assume first that (2.4) holds for all $f \in \mathcal{M}^+(\mathbb{R}^n)$. Let $h \in \mathcal{M}^+(\mathbb{R}^n)$ be a function that saturates Hölder's inequality, that is, function satisfying

$$\operatorname{supp} h \subset S^{n-1}, \quad \int_{S^{n-1}} h(rs')d\sigma(s') = 1 \tag{2.6}$$

and

$$\int_{S^{n-1}} h(rs')v(rs')d\sigma(s') \gtrsim \tilde{v}. \tag{2.7}$$

Then we define

$$f(x) = h(x)g(|x|)|x|^{1-n}. \tag{2.8}$$

Using spherical coordinates, we have that

$$\begin{aligned} LHS(2.4) &= \left(\int_0^\infty \left(\int_0^t \int_{S^{n-1}} f(rs')v(rs')d\sigma(s')r^{n-1}dr\right)^q u(t)dt\right)^{\frac{1}{q}} \\ &= \left(\int_0^\infty \left(\int_0^t \int_{S^{n-1}} h(rs')g(r)r^{1-n}v(rs')d\sigma(s')r^{n-1}dr\right)^q u(t)dt\right)^{\frac{1}{q}} \\ &= \left(\int_0^\infty \left(\int_0^t g(r)\left[\int_{S^{n-1}} h(rs')v(rs')d\sigma(s')\right]dr\right)^q u(t)dt\right)^{\frac{1}{q}} \end{aligned}$$

Applying (2.7), we obtain that

$$LHS(2.4) \gtrsim \left(\int_0^\infty \left(\int_0^t g(r)\tilde{v}(r)dr\right)^q u(t)dt\right)^{\frac{1}{q}} = LHS(2.5).$$

On the other hand, using spherical coordinates and (2.6),

$$\begin{aligned} RHS(2.4) &= C\left(\int_0^\infty \left(\int_0^t \int_{S^{n-1}} h(rs')g(r)r^{1-n}d\sigma(s')r^{n-1}dr\right)^\theta w(t)dt\right)^{\frac{1}{\theta}} \\ &= C\left(\int_0^\infty \left(\int_0^t g(r)\left[\int_{S^{n-1}} h(rs')d\sigma(s')\right]dr\right)^\theta w(t)dt\right)^{\frac{1}{\theta}} \\ &= C\left(\int_0^\infty \left(\int_0^t g(r)dr\right)^\theta w(t)dt\right)^{\frac{1}{\theta}} \end{aligned}$$

holds. Therefore, the following chain of relations is true.

$$LHS(2.5) \lesssim LHS(2.4) \le CRHS(2.4) = C\left(\int_0^\infty \left(\int_0^t g(r)dr\right)^\theta w(t)dt\right)^{\frac{1}{\theta}}.$$

As a result, (2.5) holds and the best constant of inequality (2.5) satisfies $C' \lesssim C$.

Assume now that (2.5) holds for all $g \in \mathcal{M}^+(0,\infty)$. Using spherical coordinates again, we have that

$$\begin{aligned} LHS(2.4) &= \left(\int_0^\infty \left(\int_0^t \int_{S^{n-1}} f(rs')v(rs')d\sigma(s')r^{n-1}dr\right)^q u(t)dt\right)^{\frac{1}{q}} \\ &\le \left(\int_0^\infty \left(\int_0^t \operatorname*{ess\,sup}_{s'\in S^{n-1}} v(rs')\left[\int_{S^{n-1}} f(rs')d\sigma(s')\right]r^{n-1}dr\right)^q u(t)dt\right)^{\frac{1}{q}} \\ &= \left(\int_0^\infty \left(\int_0^t \tilde{v}(r)\left[\int_{S^{n-1}} f(rs')d\sigma(s')\right]r^{n-1}dr\right)^q u(t)dt\right)^{\frac{1}{q}}. \end{aligned}$$

Applying inequality (2.5),

$$\begin{aligned} LHS(2.4) &\le C'\left(\int_0^\infty \left(\int_0^t \left[\int_{S^{n-1}} f(rs')d\sigma(s')\right]r^{(n-1)}dr\right)^\theta w(t)dt\right)^{\frac{1}{\theta}} \\ &= C'\left(\int_0^\infty \left(\int_{B(0,t)} f(s)ds\right)^\theta w(t)dt\right)^{\frac{1}{\theta}} \end{aligned}$$

holds. Moreover, the best constant $C$ of inequality (2.4) satisfies $C \le C'$. □

**Remark 2.3.** We should note that when $\theta = 1$ or $p = q$, using Fubini's Theorem, inequality (2.2) coincides with some $n$-dimensional Hardy inequality and $n$-dimensional reverse Hardy inequality, respectively.

## 3. Embeddings

The aim of this section is to characterize the embeddings between weighted local Morrey-type spaces, that is, (1.2).

We will begin with the formulation of some properties of these spaces.

**Remark 3.1.** In [5], it was proved that for $0 < p, q \le \infty$ and $w \in \mathcal{M}^+(0,\infty)$, if $\|w\|_{q,(t,\infty)} = \infty$ for all $t > 0$, then $LM_{p,q}(1,w)$ consists only of function equivalent to 0 on $\mathbb{R}^n$. The same conclusion is true for $LM_{p,q}(v,w)$ for any $v \in \mathcal{W}(\mathbb{R}^n)$. Therefore we will always assume that $\|w\|_{q,(t,\infty)} < \infty$.

**Remark 3.2.** We can formulate Remark 2.3 also in the following way: Let $0 < p \le \infty$ and $v \in \mathcal{W}(\mathbb{R}^n)$. Then $LM_{p,p}(v,w) = L_p(u)$, where $u(x) := v(x)\,\|w\|_{p,(|x|,\infty)}$, $x \in \mathbb{R}^n$. With this result it is clear that when $p_1 = q_1$ or $p_2 = q_2$, (1.2) coincides with the embeddings between weighted Lebesgue spaces and weighted local Morrey-type spaces, which were given in [18].

Unfortunately, we will solve this embedding problem under the restriction $p_2 < q_2$, we will deal with the remaining cases in a future paper.

In order to shorten the expressions we will use the following notation:

$$LM_i := LM_{p_i,q_i}(v_i,w_i), \quad i = 1,2.$$

**Theorem 3.3.** *Let $0 < q_1 \le p_2 < \min\{p_1,q_2\}$. Assume that $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n)$ and $w_1, w_2 \in \mathcal{W}(0,\infty)$ such that $\int_t^\infty w_i^{q_i} < \infty$, $i = 1,2$ for all $t \in (0,\infty)$.*

(i) *If $p_1 \le q_2 < \infty$, then $LM_1 \hookrightarrow LM_2$ for all $f \in \mathcal{M}^+(\mathbb{R}^n)$ if and only if $I_1 < \infty$, where*

$$I_1 := \sup_{x\in(0,\infty)} \left(\int_x^\infty w_1^{q_1}\right)^{-\frac{1}{q_1}} \sup_{t\in(x,\infty)} \left(\int_{B(x,t)} v_1^{-\frac{p_1p_2}{p_1-p_2}} v_2^{\frac{p_1p_2}{p_1-p_2}}\right)^{\frac{p_1-p_2}{p_1p_2}} \left(\int_t^\infty w_2^{q_2}\right)^{\frac{1}{q_2}}.$$

*Moreover, $\|\,\mathrm{I}\,\|_{LM_1\to LM_2} \approx I_1$.*

(ii) *If $q_2 < p_1 < \infty$, then $LM_1 \hookrightarrow LM_2$ for all $f \in \mathcal{M}^+(\mathbb{R}^n)$ if and only if $I_2 < \infty$, where*

$$\begin{aligned} I_2 := \sup_{x\in(0,\infty)} &\left(\int_x^\infty w_1^{q_1}\right)^{-\frac{1}{q_1}} \left(\int_x^\infty \left(\int_{B(x,t)} v_1^{-\frac{p_1p_2}{p_1-p_2}} v_2^{\frac{p_1p_2}{p_1-p_2}}\right)^{\frac{q_2(p_1-p_2)}{p_2(p_1-q_2)}}\right. \\ &\left.\times \left(\int_t^\infty w_2^{q_2}\right)^{\frac{q_2}{p_1-q_2}} w_2^{q_2}(t)dt\right)^{\frac{p_1-q_2}{p_1q_2}}. \end{aligned}$$

*Moreover,* $\|\,\mathrm{I}\,\|_{LM_1\to LM_2} \approx I_2$.

*Proof.* Since

$$\begin{aligned}
\|\,\mathrm{I}\,\|_{LM_1\to LM_2} &= \sup_{f\in\mathcal{M}^+(\mathbb{R}^n)} \frac{\|f\|_{LM_2}}{\|f\|_{LM_1}} \\
&= \sup_{f\in\mathcal{M}^+(\mathbb{R}^n)} \frac{\left(\int_0^\infty \left(\int_{B(0,t)} f(s)^{p_2} v_2(s)^{p_2} ds\right)^{\frac{q_2}{p_2}} w_2(t)^{q_2} dt\right)^{\frac{1}{q_2}}}{\left(\int_0^\infty \left(\int_{B(0,t)} f(s)^{p_1} v_1(s)^{p_1} ds\right)^{\frac{q_1}{p_1}} w_1(t)^{q_1} dt\right)^{\frac{1}{q_1}}} \\
&= \sup_{g\in\mathcal{M}^+(\mathbb{R}^n)} \frac{\left(\int_0^\infty \left(\int_{B(0,t)} g(s)^{\frac{p_2}{p_1}} v_1(s)^{-p_2} v_2(s)^{p_2} ds\right)^{\frac{q_2}{p_2}} w_2(t)^{q_2} dt\right)^{\frac{1}{q_2}}}{\left(\int_0^\infty \left(\int_{B(0,t)} g(s) ds\right)^{\frac{q_1}{p_1}} w_1(t)^{q_1} dt\right)^{\frac{1}{q_1}}}.
\end{aligned}$$

Taking parameters

$$p := \frac{p_2}{p_1}, \quad q := \frac{q_2}{p_1}, \quad q := \frac{q_1}{p_1},$$

and weights,

$$u := w_2^{q_2}, \quad v := v_1^{-p_2} v_2^{p_2}, \quad w := w_1^{q_1},$$

we obtain that

$$\|\,\mathrm{I}\,\|_{LM_1\to LM_2}^{p_1} = \sup_{g\in\mathcal{M}^+(\mathbb{R}^n)} \frac{\left(\int_0^\infty \left(\int_{B(0,t)} g(s)^p v(s) ds\right)^{\frac{q}{p}} u(t) dt\right)^{\frac{1}{q}}}{\left(\int_0^\infty \left(\int_{B(0,t)} g(s) ds\right)^{q} w(t) dt\right)^{\frac{1}{q}}}.$$

By Theorem 2.1, we have that

$$\|\,\mathrm{I}\,\|_{LM_1\to LM_2}^{p_1} = \sup_{h\in\mathcal{M}^+(0,\infty)} \frac{\left(\int_0^\infty \left(\int_0^t h(s)^p \tilde{v}(s) ds\right)^{\frac{q}{p}} u(t) dt\right)^{\frac{1}{q}}}{\left(\int_0^\infty \left(\int_0^t h(s) ds\right)^{\theta} w(t) dt\right)^{\frac{1}{\theta}}},$$

where

$$\tilde{v}(\tau) := \left(\int_{S^{n-1}} v(\tau s')^{\frac{1}{1-p}} d\sigma(s')\right)^{1-p} \tau^{(n-1)(1-p)}.$$

(i) If $p_1 \le q_2 < \infty$, then $1 \le q < \infty$. Therefore, applying [20, Theorem 1.1, (i)], we have that

$$\begin{aligned}
\|\,\mathrm{I}\,\|_{LM_1\to LM_2}^{p_1} &\approx \sup_{x\in(0,\infty)} \left(\int_x^\infty w\right)^{-\frac{1}{\theta}} \sup_{t\in(x,\infty)} \left(\int_x^t \tilde{v}^{\frac{1}{1-p}}\right)^{\frac{1-p}{p}} \left(\int_t^\infty u\right)^{\frac{1}{q}} \\
&= \sup_{x\in(0,\infty)} \left(\int_x^\infty w\right)^{-\frac{1}{\theta}} \sup_{t\in(x,\infty)} \left(\int_x^t \tau^{n-1} \int_{S^{n-1}} v(\tau s')^{\frac{1}{1-p}} d\sigma(s') d\tau\right)^{\frac{1-p}{p}} \left(\int_t^\infty u\right)^{\frac{1}{q}} \\
&= \sup_{x\in(0,\infty)} \left(\int_x^\infty w\right)^{-\frac{1}{\theta}} \sup_{t\in(x,\infty)} \left(\int_{B(x,t)} v(y)^{\frac{1}{1-p}} dy\right)^{\frac{1-p}{p}} \left(\int_t^\infty u\right)^{\frac{1}{q}}.
\end{aligned}$$

Now, by substituting parameters $p, q, \theta$ and weights $u, v, w$ into the last expression, we arrive at

$$\|\,\mathrm{I}\,\|_{LM_1\to LM_2} \approx \sup_{x\in(0,\infty)} \left(\int_x^\infty w_1^{q_1}\right)^{-\frac{1}{q_1}} \sup_{t\in(x,\infty)} \left(\int_{B(x,t)} v_1^{-\frac{p_1 p_2}{p_1-p_2}} v_2^{\frac{p_1 p_2}{p_1-p_2}}\right)^{\frac{p_1-p_2}{p_1 p_2}} \left(\int_t^\infty w_2^{q_2}\right)^{\frac{1}{q_2}}.$$

(ii) If $q_2 < p_1$, then $q < 1$. Therefore, applying [20, Theorem 1.1, (ii)], we have that

$$\|\mathrm{I}\|_{LM_1\to LM_2}^{p_1} \approx \sup_{x\in(0,\infty)} \left(\int_x^\infty w\right)^{-\frac{1}{\theta}} \left(\int_x^\infty \left(\int_x^t \tilde{v}^{\frac{1}{1-p}}\right)^{\frac{q(1-p)}{p(1-q)}} \left(\int_t^\infty u\right)^{\frac{q}{1-q}} u(t)dt\right)^{\frac{1-q}{q}}$$

$$= \sup_{x\in(0,\infty)} \left(\int_x^\infty w\right)^{-\frac{1}{\theta}} \left(\int_x^\infty \left(\int_x^t \tau^{n-1}\int_{S^{n-1}} v(\tau s')^{\frac{1}{1-p}} d\sigma(s')d\tau\right)^{\frac{q(1-p)}{p(1-q)}} \times \left(\int_t^\infty u\right)^{\frac{q}{1-q}} u(t)dt\right)^{\frac{1-q}{q}}$$

$$= \sup_{x\in(0,\infty)} \left(\int_x^\infty w\right)^{-\frac{1}{\theta}} \left(\int_x^\infty \left(\int_{B(x,t)} v^{\frac{1}{1-p}}\right)^{\frac{q(1-p)}{p(1-q)}} \left(\int_t^\infty u\right)^{\frac{q}{1-q}} u(t)dt\right)^{\frac{1-q}{q}}.$$

Similarly, substituting parameters $p, q, \theta$ and weights $u, v, w$ gives

$$\|\mathrm{I}\|_{LM_1\to LM_2} \approx \sup_{x\in(0,\infty)} \left(\int_x^\infty w_1^{q_1}\right)^{-\frac{1}{q_1}} \left(\int_x^\infty \left(\int_{B(x,t)} v_1^{-\frac{p_1p_2}{p_1-p_2}} v_2^{\frac{p_1p_2}{p_1-p_2}}\right)^{\frac{q_2(p_1-p_2)}{p_2(p_1-q_2)}} \times \left(\int_t^\infty w_2^{q_2}\right)^{\frac{q_2}{p_1-q_2}} w_2^{q_2}(t)dt\right)^{\frac{p_1-q_2}{p_1q_2}}.$$

□

**Theorem 3.4.** *Let $0 < p_2 < \min\{p_1, q_1, q_2\}$. Assume that $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n)$ and $w_1, w_2 \in \mathcal{W}(0,\infty)$ such that $\int_t^\infty w_i^{q_i} < \infty$, $i = 1, 2$ for all $t \in (0,\infty)$ . Suppose that*

$$0 < \int_0^t \left(\int_{B(s,t)} v_1^{-\frac{p_1p_2}{p_1-p_2}} v_2^{\frac{p_1p_2}{p_1-p_2}}\right)^{\frac{q_1(p_1-p_2)}{p_1(q_1-p_2)}} \left(\int_s^\infty w_1^{q_1}\right)^{-\frac{q_1}{q_1-p_2}} w_1(s)^{q_1}\, ds < \infty$$

*holds for all $t \in (0,\infty)$.*

(i) *If $\max\{p_1, q_1\} \le q_2 < \infty$, then $LM_1 \hookrightarrow LM_2$ for all $f \in \mathcal{M}^+(\mathbb{R}^n)$ if and only if $I_3 < \infty$ and $I_4 < \infty$, where*

$$I_3 := \left(\int_0^\infty w_1^{q_1}\right)^{-\frac{1}{q_1}} \sup_{t\in(0,\infty)} \left(\int_{B(0,t)} v_1^{-\frac{p_1p_2}{p_1-p_2}} v_2^{\frac{p_1p_2}{p_1-p_2}}\right)^{\frac{p_1-p_2}{p_1p_2}} \left(\int_t^\infty w_2^{q_2}\right)^{\frac{1}{q_2}} \tag{3.1}$$

*and*

$$I_4 := \sup_{t\in(0,\infty)} \left(\int_t^\infty w_2^{q_2}\right)^{\frac{1}{q_2}} \left(\int_0^t \left(\int_{B(s,t)} v_1^{-\frac{p_1p_2}{p_1-p_2}} v_2^{\frac{p_1p_2}{p_1-p_2}}\right)^{\frac{q_1(p_1-p_2)}{p_1(q_1-p_2)}} \times \left(\int_s^\infty w_1^{q_1}\right)^{-\frac{q_1}{q_1-p_2}} w_1(s)^{q_1}\, ds\right)^{\frac{p_1-q_2}{q_1q_2}}. \tag{3.2}$$

*Moreover, $\|\mathrm{I}\|_{LM_1\to LM_2} \approx I_3 + I_4$.*

(ii) *If $p_1 \le q_2 < q_1 < \infty$, then $LM_1 \hookrightarrow LM_2$ for all $f \in \mathcal{M}^+(\mathbb{R}^n)$ if and only if $I_3 < \infty$, $I_5 < \infty$ and $I_6 < \infty$, where $I_3$ is defined in (3.1),*

$$I_5 := \left(\int_0^\infty \left(\int_0^t \left(\int_s^\infty w_1^{q_1}\right)^{-\frac{q_1}{q_1-p_2}} w_1(s)^{q_1} ds\right)^{\frac{q_1(q_2-p_2)}{p_2(q_1-q_2)}} \left(\int_t^\infty w_1^{q_1}\right)^{-\frac{q_1}{q_1-p_2}} w_1(t)^{q_1} \times \sup_{z\in(t,\infty)} \left(\int_{B(t,z)} v_1^{-\frac{p_1p_2}{p_1-p_2}} v_2^{\frac{p_1p_2}{p_1-p_2}}\right)^{\frac{q_1q_2(p_1-p_2)}{p_1p_2(q_1-q_2)}} \left(\int_z^\infty w_2^{q_2}\right)^{\frac{q_1}{q_1-q_2}} dt\right)^{\frac{q_1-q_2}{q_1q_2}}$$

*and*

$$I_6 := \left(\int_0^\infty \left(\int_0^t \left(\int_s^\infty w_1^{q_1}\right)^{-\frac{q_1}{q_1-p_2}} w_1(s)^{q_1} \left(\int_{B(s,t)} v_1^{-\frac{p_1p_2}{p_1-p_2}} v_2^{\frac{p_1p_2}{p_1-p_2}}\right)^{\frac{q_1(q_2-p_2)}{p_1(q_1-p_2)}} ds\right)^{\frac{q_1(q_2-p_2)}{p_2(q_1-q_2)}}\right.$$

$$\times \sup_{z\in(t,\infty)} \left( \int\limits_{B(t,z)} v_1^{-\frac{p_1p_2}{p_1-p_2}} v_2^{\frac{p_1p_2}{p_1-p_2}} \right)^{\frac{q_1(p_1-p_2)}{p_1(q_1-p_2)}} \left( \int\limits_z^\infty w_2^{q_2} \right)^{\frac{q_1}{q_1-q_2}} \left( \int_t^\infty w_1^{q_1} \right)^{-\frac{q_1}{q_1-p_2}} w_1(t)^{q_1} dt \Bigg)^{\frac{q_1-q_2}{q_1q_2}}. \tag{3.3}$$

*Moreover,* $\|\mathrm{I}\|_{LM_1\to LM_2} \approx I_3 + I_5 + I_6$.

(iii) *If* $q_1 \le q_2 < p_1 < \infty$, *then* $LM_1 \hookrightarrow LM_2$ *for all* $f \in \mathcal{M}^+(\mathbb{R}^n)$ *if and only if* $I_4 < \infty$, $I_7 < \infty$ *and* $I_8 < \infty$, *where* $I_4$ *is defined in* (3.2),

$$I_7 := \left( \int\limits_0^\infty w_1^{q_1} \right)^{-\frac{1}{q_1}} \left( \int_0^\infty \left( \int_{B(0,t)} v_1^{-\frac{p_1p_2}{p_1-p_2}} v_2^{\frac{p_1p_2}{p_1-p_2}} \right)^{\frac{q_2(p_1-p_2)}{p_2(p_1-q_2)}} \left( \int_t^\infty w_2^{q_2} \right)^{\frac{q_2}{p_1-q_2}} w_2(t)^{q_2} dt \right)^{\frac{p_1-q_2}{p_1q_2}} \tag{3.4}$$

*and*

$$I_8 := \sup_{t\in(0,\infty)} \left( \int_0^t \left( \int_s^\infty w_1^{q_1} \right)^{-\frac{q_1}{q_1-p_2}} w_1(s)^{q_1} ds \right)^{\frac{q_1-p_2}{q_1p_2}}$$
$$\times \left( \int_t^\infty \left( \int_{B(t,s)} v_1^{-\frac{p_1p_2}{p_1-p_2}} v_2^{\frac{p_1p_2}{p_1-p_2}} \right)^{\frac{q_2(p_1-p_2)}{p_2(p_1-q_2)}} \left( \int_s^\infty w_2^{q_2} \right)^{\frac{q_2}{p_1-q_2}} w_2(s)^{q_2} ds \right)^{\frac{p_1-q_2}{p_1q_2}}.$$

*Moreover,* $\|\mathrm{I}\|_{LM_1\to LM_2} \approx I_4 + I_7 + I_8$.

(iv) *If* $p_1 < \infty$, $q_1 < \infty$ *and* $q_2 < \min\{p_1, q_1\}$, *then* $LM_1 \hookrightarrow LM_2$ *for all* $f \in \mathcal{M}^+(\mathbb{R}^n)$ *if and only if* $I_6 < \infty$, $I_7 < \infty$ *and* $I_9 < \infty$, *where* $I_6$ *and* $I_7$ *are defined in* (3.3) *and* (3.4), *respectively, and*

$$I_9 := \Bigg( \int_0^\infty \left( \int_0^t \left( \int_s^\infty w_1^{q_1} \right)^{-\frac{q_1}{q_1-p_2}} w_1(s)^{q_1} ds \right)^{\frac{q_1(q_2-p_2)}{p_2(q_1-q_2)}} \left( \int_t^\infty w_1^{q_1} \right)^{-\frac{q_1}{q_1-p_2}} w_1(t)^{q_1}$$
$$\times \left( \int_t^\infty \left( \int_{B(t,s)} v_1^{-\frac{p_1p_2}{p_1-p_2}} v_2^{\frac{p_1p_2}{p_1-p_2}} \right)^{\frac{q_2(p_1-p_2)}{p_2(p_1-q_2)}} \left( \int_s^\infty w_2^{q_2} \right)^{\frac{q_1}{p_1-q_2}} w_2(s)^{q_2} ds \right)^{\frac{q_1(p_1-q_2)}{q_1-q_2}} dt \Bigg)^{\frac{q_1-q_2}{q_1q_2}}.$$

*Moreover,* $\|\mathrm{I}\|_{LM_1\to LM_2} \approx I_6 + I_7 + I_9$.

*Proof.* As in the proof of Theorem 3.3, using Theorem 2.1 and applying [20, Theorem 1.3] the result follows. □

**Theorem 3.5.** *Let* $0 < q_1 \le p_1 = p_2 < q_2 < \infty$. *Assume that* $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n)$ *and* $w_1, w_2 \in \mathcal{W}(0,\infty)$ *such that* $\int_t^\infty w_i^{q_i} < \infty$, $i = 1,2$ *for all* $t \in (0,\infty)$. *Then,* $LM_1 \hookrightarrow LM_2$ *for all* $f \in \mathcal{M}^+(\mathbb{R}^n)$ *if and only if* $I_{10} < \infty$, *where*

$$I_{10} := \sup_{x\in(0,\infty)} \left( \int_x^\infty w_2(s)^{q_2} \right)^{\frac{1}{q_2}} \operatorname*{ess\,sup}_{y\in B(0,x)} v_1(y)^{-p_1} v_2(y)^{p_1} \left( \int_{|y|}^\infty w_1(s)^{q_1} ds \right)^{-\frac{1}{q_1}}.$$

*Moreover,* $\|\mathrm{I}\|_{LM_1\to LM_2} \approx I_{10}$.

*Proof.* As in the proof of Theorem 3.3, since

$$\|\mathrm{I}\|_{LM_1\to LM_2} = \sup_{g\in\mathcal{M}^+(\mathbb{R}^n)} \frac{\left( \int_0^\infty \left( \int_{B(0,t)} g(s)v_1(s)^{-p_1} v_2(s)^{p_1} ds \right)^{\frac{q_2}{p_1}} w_2(t)^{q_2} dt \right)^{\frac{1}{q_2}}}{\left( \int_0^\infty \left( \int_{B(0,t)} g(s) ds \right)^{\frac{q_1}{p_1}} w_1(t)^{q_1} dt \right)^{\frac{1}{q_1}}}$$

taking parameters

$$p := 1, \quad q := \frac{q_2}{p_1}, \quad \theta := \frac{q_1}{p_1},$$

and weights,

$$u := w_2^{q_2}, \quad v := v_1^{-p_1} v_2^{p_1}, \quad w := w_1^{q_1},$$

we obtain that

$$\|\,\mathrm{I}\,\|_{LM_1\to LM_2}^{p_1} = \sup_{g\in\mathcal{M}^+(\mathbb{R}^n)} \frac{\left(\int_0^\infty \left(\int_{B(0,t)} g(s)v(s)ds\right)^q u(t)dt\right)^{\frac{1}{q}}}{\left(\int_0^\infty \left(\int_{B(0,t)} g(s)ds\right)^\theta w(t)dt\right)^{\frac{1}{\theta}}}.$$

By Theorem 2.2, we have that

$$\|\,\mathrm{I}\,\|_{LM_1\to LM_2}^{p_1} = \sup_{h\in\mathcal{M}^+(0,\infty)} \frac{\left(\int_0^\infty \left(\int_0^t h(s)\tilde{v}(s)ds\right)^q u(t)dt\right)^{\frac{1}{q}}}{\left(\int_0^\infty \left(\int_0^t h(s)ds\right)^\theta w(t)dt\right)^{\frac{1}{\theta}}}, \tag{3.5}$$

where

$$\tilde{v}(\tau) := \operatorname*{ess\,sup}_{s'\in S^{n-1}} v(\tau s').$$

If $0 < q_1 \le p_1 = p_2 < q_2 < \infty$, then $p = 1$, $0 < \theta \le 1 < q < \infty$. Therefore, applying [20, Theorem 1.2], we have that

$$\begin{aligned}
\|\,\mathrm{I}\,\|_{LM_1\to LM_2}^{p_1} &\approx \sup_{x\in(0,\infty)} \left(\int_x^\infty u\right)^{\frac{1}{q}} \operatorname*{ess\,sup}_{\tau\in(0,x)} \tilde{v}(\tau)\left(\int_\tau^\infty w\right)^{-\frac{1}{\theta}} \\
&= \sup_{x\in(0,\infty)} \left(\int_x^\infty u\right)^{\frac{1}{q}} \operatorname*{ess\,sup}_{\tau\in(0,x)} \left(\operatorname*{ess\,sup}_{s'\in S^{n-1}} v(\tau s')\right)\left(\int_\tau^\infty w\right)^{-\frac{1}{\theta}}.
\end{aligned}$$

Observe that

$$\operatorname*{ess\,sup}_{s'\in S^{n-1}} v(\tau s') = \operatorname*{ess\,sup}_{|x|=\tau} v(x). \tag{3.6}$$

Then,

$$\begin{aligned}
\|\,\mathrm{I}\,\|_{LM_1\to LM_2}^{p_1} &\approx \sup_{x\in(0,\infty)} \left(\int_x^\infty u\right)^{\frac{1}{q}} \operatorname*{ess\,sup}_{\tau\in(0,x)} \left(\operatorname*{ess\,sup}_{|x|=\tau} v(x)\right)\left(\int_\tau^\infty w\right)^{-\frac{1}{\theta}} \\
&= \sup_{x\in(0,\infty)} \left(\int_x^\infty u\right)^{\frac{1}{q}} \operatorname*{ess\,sup}_{\tau\in(0,x)} \operatorname*{ess\,sup}_{|x|=\tau} v(x)\left(\int_{|x|}^\infty w\right)^{-\frac{1}{\theta}} \\
&= \sup_{x\in(0,\infty)} \left(\int_x^\infty u\right)^{\frac{1}{q}} \operatorname*{ess\,sup}_{y\in B(0,x)} v(y)\left(\int_{|y|}^\infty w\right)^{-\frac{1}{\theta}}.
\end{aligned}$$

Now, substituting the powers $p, q, \theta$, and weights $u, v, w$, we obtain that

$$\|\,\mathrm{I}\,\|_{LM_1\to LM_2} \approx \sup_{x\in(0,\infty)} \left(\int_x^\infty w_2(s)^{q_2}\right)^{\frac{1}{q_2}} \operatorname*{ess\,sup}_{y\in B(0,x)} v_1(y)^{-p_1} v_2(y)^{p_1} \left(\int_{|y|}^\infty w_1(s)^{q_1} ds\right)^{-\frac{1}{q_1}}.$$

□

**Theorem 3.6.** *Let $0 < p_1 = p_2 < \min\{q_1, q_2\}$, $q_1, q_2 < \infty$. Assume that $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n)$ such that $v_1^{-1}v_2$ is continuous and $w_1, w_2 \in \mathcal{W}(0,\infty)$ such that $\int_t^\infty w_i^{q_i} < \infty$, $i = 1, 2$ for all $t \in (0,\infty)$. Suppose that*

- $0 < \int_0^t \operatorname{ess\,sup}_{|x|=\tau} v(x)^{\frac{q_1}{q_1-p_1}} d\tau < \infty$,
- $0 < \int_0^t \left(\int_x^\infty w_1(s)^{q_1} ds\right)^{-\frac{q_1}{q_1-p_1}} w_1(x)^{q_1} dx < \infty$,
- $0 < \int_0^t w_2(s)^{-\frac{q_2 p_1}{q_2-p_1}} ds < \infty$

*hold for all* $t \in (0,\infty)$.

(i) *If* $q_1 \le q_2$, *then* $LM_1 \hookrightarrow LM_2$ *for all* $f \in \mathcal{M}^+(\mathbb{R}^n)$ *if and only if* $I_{11} < \infty$ *and* $I_{12} < \infty$, *where*

$$I_{11} := \left( \int_0^\infty w_1^{q_1} \right)^{-\frac{1}{q_1}} \sup_{t \in (0,\infty)} \left( \int_t^\infty w_2^{q_2} \right)^{\frac{1}{q_2}} \operatorname*{ess\,sup}_{x \in B(0,t)} v_1(x)^{-1} v_2(x) \tag{3.7}$$

*and*

$$I_{12} := \sup_{t \in (0,\infty)} \left( \int_0^t \left( \int_x^\infty w_1^{q_1} \right)^{-\frac{q_1}{q_1 - p_1}} w_1(x)^{q_1} \sup_{\tau \in B(x,t)} v_1(\tau)^{-p_1} v_2(\tau)^{p_1} dx \right)^{\frac{q_1 - p_1}{q_1 p_1}} \left( \int_t^\infty w_2^{q_2} \right)^{\frac{1}{q_2}}.$$

*Moreover,* $\|\mathrm{I}\|_{LM_1 \to LM_2} \approx I_{11} + I_{12}$.

(ii) *If* $q_2 < q_1$, *then* $LM_1 \hookrightarrow LM_2$ *for all* $f \in \mathcal{M}^+(\mathbb{R}^n)$ *if and only if* $I_{11} < \infty$, $I_{13} < \infty$ *and* $I_{14} < \infty$, *where* $I_{11}$ *is defined in* (3.7),

$$I_{13} := \left( \int_0^\infty \left( \int_0^t \left( \int_x^\infty w_1^{q_1} \right)^{-\frac{q_1}{q_1 - p_1}} w_1(x)^{q_1} dx \right)^{\frac{q_1(q_2 - p_1)}{p_1(q_1 - q_2)}} \left( \int_t^\infty w_1^{q_1} \right)^{-\frac{q_1}{q_1 - p_1}} w_1(t)^{q_1} \right.$$
$$\left. \times \sup_{x \in {}^{\mathfrak{c}}B(0,t)} v_1(x)^{-\frac{q_2(q_1 - p_1)}{q_1 - q_2}} v_2(x)^{\frac{q_2(q_1 - p_1)}{q_1 - q_2}} \left( \int_{|x|}^\infty w_2^{q_2} \right)^{\frac{q_1}{q_1 - q_2}} dt \right)^{\frac{q_1 - q_2}{q_1 q_2}}$$

*and*

$$I_{14} := \left( \int_0^\infty \left( \int_0^t \left( \int_x^\infty w_1^{q_1} \right)^{-\frac{q_1}{q_1 - p_1}} w_1(x)^{q_1} \sup_{z \in B(x,t)} v_1(z)^{-p_1} v_2(z)^{p_1} dx \right)^{\frac{q_1(q_2 - p_1)}{p_1(q_1 - q_2)}} \right.$$
$$\left. \times \sup_{x \in {}^{\mathfrak{c}}B(0,t)} v_1(x)^{-p_1} v_2(x)^{p_1} \left( \int_{|x|}^\infty w_2^{q_2} \right)^{\frac{q_1}{q_1 - q_2}} \left( \int_t^\infty w_1^{q_1} \right)^{-\frac{q_1}{q_1 - p_1}} w_1(t)^{q_1} dt \right)^{\frac{q_1 - q_2}{q_1 q_2}}.$$

*Moreover,* $\|\mathrm{I}\|_{LM_1 \to LM_2} \approx I_{11} + I_{13} + I_{14}$.

*Proof.* As in the proof of Theorem 3.5, using Theorem 2.2, (3.5) holds.

(i) If $\theta_1 \le \theta_2$, then $\theta \le q$, applying [20, Theorem 1.4, (i)], we have that $\|\mathrm{I}\|_{LM_1 \to LM_2}^{p_1} \approx I + II$, where

$$I := \left( \int_0^\infty w \right)^{-\frac{1}{\theta}} \sup_{t \in (0,\infty)} \left( \int_t^\infty u \right)^{\frac{1}{q}} \operatorname*{ess\,sup}_{s \in (0,t)} \tilde{v}(s)$$

and

$$II := \sup_{t \in (0,\infty)} \left( \int_0^t \left( \int_x^\infty w \right)^{-\frac{\theta}{\theta - 1}} w(x) \sup_{z \in (x,t)} \tilde{v}(z) dx \right)^{\frac{\theta - 1}{\theta}} \left( \int_t^\infty u \right)^{\frac{1}{q}}.$$

Using (3.6) and substituting the powers $p, q, \theta$, and weights $u, v, w$ into $I$ and $II$, we obtain that $\|\mathrm{I}\|_{LM_1 \to LM_2} \approx I_{11} + I_{12}$.

(ii) If $\theta_2 < \theta_1$, then $q < \theta$, and applying [20, Theorem 1.4, (ii)], we have that $\|\mathrm{I}\|_{LM_1 \to LM_2}^{p_1} \approx I_{11} + III + IV$, where

$$III = \left( \int_0^\infty \left( \int_0^t \left( \int_x^\infty w \right)^{-\frac{\theta}{\theta - 1}} w(x) dx \right)^{\frac{\theta(q-1)}{\theta - q}} \left( \int_t^\infty w \right)^{-\frac{\theta}{\theta - 1}} w(t) \right.$$
$$\left. \times \sup_{z \in (t,\infty)} \tilde{v}(z)^{\frac{q(\theta - 1)}{\theta - q}} \left( \int_z^\infty u \right)^{\frac{\theta}{\theta - q}} dt \right)^{\frac{\theta - q}{\theta q}}$$

and

$$IV := \left( \int_0^\infty \left( \int_0^t \left( \int_x^\infty w \right)^{-\frac{\theta}{\theta - 1}} w(x) \sup_{z \in (x,t)} \tilde{v}(z) dx \right)^{\frac{\theta(q-1)}{\theta - q}} \left( \int_t^\infty w \right)^{-\frac{\theta}{\theta - 1}} w(t) \right.$$

$$\times \sup_{z\in(t,\infty)} \tilde{v}(z)\bigg(\int_z^\infty u\bigg)^{\frac{\theta}{\theta-q}} dt\bigg)^{\frac{\theta-q}{\theta q}}.$$

Since,

$$\begin{aligned}\sup_{z\in(t,\infty)} \tilde{v}(z)^{\frac{q(\theta-1)}{\theta-q}}\bigg(\int_z^\infty u\bigg)^{\frac{\theta}{\theta-q}} &= \sup_{z\in(t,\infty)}\bigg(\operatorname*{ess\,sup}_{|x|=z} v(x)\bigg)^{\frac{q(\theta-1)}{\theta-q}}\bigg(\int_z^\infty u\bigg)^{\frac{\theta}{\theta-q}}\\ &= \sup_{z\in(t,\infty)}\operatorname*{ess\,sup}_{|x|=z} v(x)^{\frac{q(\theta-1)}{\theta-q}}\bigg(\int_{|x|}^\infty u\bigg)^{\frac{\theta}{\theta-q}}\\ &= \sup_{x\in {}^{\mathbf{c}}B(0,t)} v(x)^{\frac{q(\theta-1)}{\theta-q}}\bigg(\int_{|x|}^\infty u\bigg)^{\frac{\theta}{\theta-q}}\end{aligned}$$

and, similarly

$$\sup_{z\in(t,\infty)} \tilde{v}(z)\bigg(\int_z^\infty u\bigg)^{\frac{\theta}{\theta-q}} = \sup_{x\in {}^{\mathbf{c}}B(0,t)} v(x)\bigg(\int_{|x|}^\infty u\bigg)^{\frac{\theta}{\theta-q}},$$

substituting the powers $p, q, \theta$, and weights $u, v, w$ into $III$ and $IV$, the result follows. □

**Remark 3.7.** We should mention that by the change of variables $x = y/|y|^2$ and $t = 1/\tau$, the embedding

$$^{\mathbf{c}}LM_{p_1,q_1}(v_1, w_1) \hookrightarrow {}^{\mathbf{c}}LM_{p_2,q_2}(v_2, w_2)$$

is equivalent to the embedding

$$LM_{p_1,q_1}(\tilde{v}_1, \tilde{w}_1) \hookrightarrow LM_{p_2,q_2}(\tilde{v}_2, \tilde{w}_2),$$

where $\tilde{v}_i(y) = v_i(y/|y|^2)|y|^{-2n/p_i}$ $\tilde{w}_i(\tau) = \tau^{-2/q_i} w_i(1/\tau)$. Therefore, by using the characterizations of (1.2), it is possible to give the characterizations of the embeddings between weighted complementary local Morrey-type spaces.

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A. Gogatishvili, Mathematical Institute, Czech Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic
*Email address*: gogatish@math.cas.cz

T. Ünver, Department of Mathematics, Faculty of Science and Arts, Kirikkale University, 71450 Yahsihan, Kirikkale, Turkey
*Email address*: tugceunver@kku.edu.tr