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Proofs and surfaces

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PROOFS AND SURFACES

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ABSTRACT. A formal sequent system dealing with Menelaus' configurations is introduced in this paper. The axiomatic sequents of the system stem from 2-cycles of Δ -complexes. The Euclidean and projective interpretations of the sequents are defined and a soundness result is proved. This system is decidable and its provable sequents deliver incidence results. A cyclic operad structure tied to this system is presented by generators and relations.

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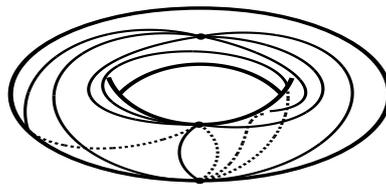
Keywords: Ceva-Menelaus proof, connected sum, cyclic operad, decidability, Δ -complex, incidence theorem, sequent system, simplicial homology

1. INTRODUCTION

As a part of his program of mechanical theorem proving in projective geometry, Jürgen Richter-Gebert, partly in collaboration with Susanne Jasmin Apel, investigated so-called Ceva-Menelaus proofs of incidence theorems (see [18], [19], [1] and [2]). In the paper [18], he gave a proof-theoretical analysis of the method, which served as the starting point for our investigation. For the sake of clarity, we restrict ourselves only to the Menelaus proofs in this paper.

Roughly speaking, a Ceva-Menelaus proof transforms a triangulation of a surface into an incidence result in projective geometry. At the first glance, this is something that connects geometry with geometry. However, the triangulation in question could be envisaged purely combinatorially through a type of abstract cell complexes, like we do in this paper. Even one step further is possible: triangulations of surfaces may be considered as a special kind of syntax built out of two symbols (a dot and a dash) written on various writing pads in the way explained below.

It is convenient to think about syntax as something built out of primitive symbols combined together in words and written on a piece of paper or a blackboard. These writing pads, for the sake of uniformity, could be taken as parts of the two-dimensional sphere. The same writing pads could be used in proof theory for trees, which are not linear syntactical forms. The syntax appropriate for Ceva-Menelaus proofs requires some other surfaces, not just the sphere. For example, one can start with a piece of paper (or some other material) in the shape of a torus and produce “words” consisting of dots and line segments. A “word” is considered to be correct if it triangulates the torus. For example, the following word consisting of three dots and nine line segments is correct.



However, it is difficult to “read” directly such a syntax (we will see, in Section 8, that the above “word” could be read as a proof of Pappus theorem). That is the main reason for us to present our results within a sequent system, which is, at least from the proof-theoretical standpoint, a more convenient syntax. The axiomatic sequents of this system are obtained by translating, via simplicial homology, the triangulations of surfaces. The required interpretation forces, quite naturally, one-sided sequents (the formulae of a sequent are placed at one side of the symbol \vdash). Besides the structural rule of cut, which is implicit in [18], we consider two propositional connectives and their rules of introduction. Also, an action of the octahedral group on the set of atomic formulae, which stems from possibility to organise a set of six points into twenty four Menelaus’ configurations, is included in our system.

It is proved that the system is sound with respect to both Euclidean and projective interpretation. This system is also decidable. An analysis of possible generalizations of the method, which relies on a constructive solution of Steenrod’s problem in dimension 2, shows that there is no way of extending the system by means of homological arguments. Also, a few examples of incidence results delivered from provable sequents of our system are given.

As it is natural to relate the single-conclusion sequent systems with multicategories and operads, this one-sided sequent system gives rise to a cyclic operad, which may serve as an initial framework for a general proof-theoretical study of the matters. This cyclic operad is based on the operation of connected sum on the abstract cell complexes that we use to generate the axiomatic sequents. A presentation of this operad by generators and relations is given in the last section of the paper.

2. THE MENELAUS THEOREM

By the Menelaus theorem we consider here a lemma used for the Sector theorem of spherical trigonometry, which appears in Al-Harawī’s version of Menelaus’ *Spherics*. The Greek original of the Menelaus text written at the end of the 1st century A.D. is lost and the Arabic version mentioned above is from the 9th or 10th century A.D. For historical remarks see [17] and [20]. The same lemma appears in Ptolemy’s *Almagest* (see [24, Book I.13]).

For three mutually distinct colinear points X , Y and Z in the Euclidean plane \mathbf{R}^2 let

$$(X, Y; Z) =_{df} \begin{cases} \frac{XZ}{YZ}, & \text{if } Z \text{ is between } X \text{ and } Y, \\ -\frac{XZ}{YZ}, & \text{otherwise.} \end{cases}$$

Theorem 2.1 (Menelaus). *For a triangle ABC and points P , Q and R (different from the vertices) on the lines BC , CA and AB respectively, it holds that*

$$P, Q, R \text{ are colinear} \quad \text{iff} \quad (B, C; P) \cdot (C, A; Q) \cdot (A, B; R) = -1.$$

(The cardinality of the set of proofs of this theorem is not known. This is mostly because the question: *What is a proof?*, i.e. when two proofs are equal, is still open.)

3. 2-CYCLES AND MENELAUS CONFIGURATIONS

Our intention is to formalise and extend, within the proof theory, the Richter-Gebert’s idea (see [18, Section 2.2]), which could be paraphrased as follows.

We consider compact, orientable 2-manifolds without boundary and subdivisions by CW-complexes whose faces are triangles. Consider such a cycle as being interpreted by flat triangles (it does not matter

if these triangles intersect, coincide or are coplanar as long as they represent the combinatorial structure of the cycle). The presence of Menelaus configurations on all but one of the faces will imply automatically the existence of a Menelaus configuration on the final face.

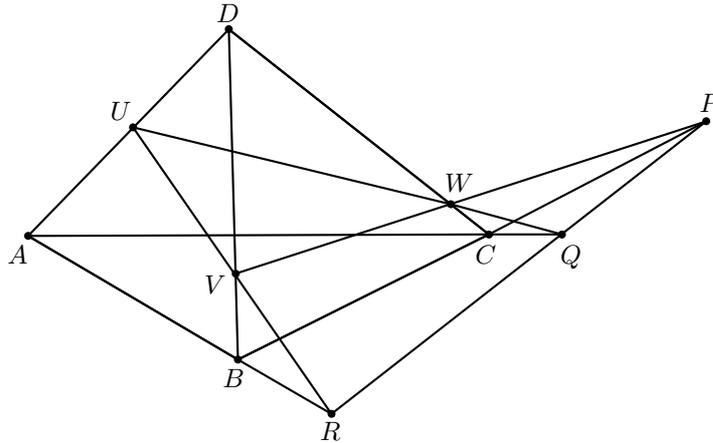
For example, consider the sphere S^2 triangulated in four triangles arranged as the facets of a tetrahedron. Suppose that the vertices of this triangulation as well as its six 1-faces are interpreted as points $A, B, C, D, P, Q, R, U, V$ and W in the Euclidean plane. Assume that the triangles BCD, CAD and ABD together with the lines WVP, WUQ and VUR make Menelaus configurations. Hence, by the Menelaus theorem, the following holds

$$\begin{aligned} (C, D; W) \cdot (D, B; V) \cdot (B, C; P) &= -1 \\ (D, C; W) \cdot (A, D; U) \cdot (C, A; Q) &= -1 \\ (B, D; V) \cdot (D, A; U) \cdot (A, B; R) &= -1, \end{aligned}$$

which, after multiplication and cancelation, delivers

$$(B, C; P) \cdot (C, A; Q) \cdot (A, B; R) = -1.$$

Again, by the Menelaus theorem, this means that we have the Menelaus configuration on the final triangle ABC .



In order to formalise these arguments, we start with some definitions. Note that for X, Y and Z not mutually distinct colinear points, we consider $(X, Y; Z)$ *undefined*. A sextuple (A, B, C, P, Q, R) of points in \mathbf{R}^2 makes a *Menelaus configuration* when $(B, C; P), (C, A; Q)$ and $(A, B; R)$ are defined and their product is -1 .

We note that if (A, B, C, P, Q, R) makes a Menelaus configuration, then P, Q and R are colinear. This follows from Menelaus theorem if A, B and C are not colinear, while if A, B and C are colinear, then all A, B, C, P, Q and R are colinear. On the other hand, if A, B and C are not colinear and P, Q and R , different from the points A, B and C , are colinear and lie on the lines BC, CA and AB , respectively, then the sextuple (A, B, C, P, Q, R) , by the Menelaus theorem, makes a Menelaus configuration.

In the example above, the elements of the union of sets of vertices and edges of the triangulation are interpreted as points in \mathbf{R}^2 and it is assumed that the three sextuples of points obtained by interpreting the vertices and the edges of three facets of the tetrahedron make Menelaus configurations. This suffices to conclude that the sextuple obtained by interpreting the vertices and the edges of the final

facet makes a Menelaus configuration. Our formalisation of these matters is given in terms of Δ -complexes (cf. [14, Section 2.1]). (These combinatorial objects were introduced in [8] under the name *semi-simplicial* complexes, and are also called *Delta sets*, see [9].)

An (abstract) Δ -complex K consists of mutually disjoint sets K_0, K_1, \dots and functions $d_i^n: K_n \rightarrow K_{n-1}$, $n \geq 1$, $0 \leq i \leq n$, which for $l-1 \geq j$ satisfy

$$d_j^{n-1} \circ d_i^n = d_{i-1}^{n-1} \circ d_j^n.$$

The elements of K_n are the n -cells of K , and the functions d_i^n are called *faces*. The intuitive meaning of K_n is that this is the set of (ordered) n -dimensional simplices, and a face d_i^n maps such a simplex to the facet opposite to its i th vertex.

For the standard ordered simplices

$$\Delta^n = \{(t_0, \dots, t_n) \mid t_0, \dots, t_n \geq 0, \sum_{i=0}^n t_i = 1\},$$

in the Euclidean space, and the maps $\delta_i^n: \Delta^{n-1} \rightarrow \Delta^n$ defined by

$$\delta_i^n(t_0, \dots, t_i, \dots, t_{n-1}) = (t_0, \dots, 0, t_i, \dots, t_{n-1}),$$

the *geometric realisation* of a Δ -complex K is the following quotient space (a tensor product)

$$|K| = \left(\prod_n K_n \times \Delta^n \right) / \sim,$$

where the equivalence relation \sim is generated by

$$(d_i^n x, t) \sim (x, \delta_i^n t).$$

In order to relax the notation, we omit the superscripts from d_i^n when they are clear from the context. For every $n \geq 0$, let C_n be the free abelian group generated by K_n and let the *boundary* homomorphism $\partial_n: C_n \rightarrow C_{n-1}$ be defined on generators by

$$\partial_n x = \sum_{i=0}^n (-1)^i d_i x.$$

Since we are interested just in the boundary homomorphism ∂_2 , we omit the subscript 2. A 2-cycle c is an element of C_2 such that $\partial c = 0$. For $n \geq 1$, let

$$c = \varepsilon_1 x_1 + \varepsilon_2 x_2 + \dots + \varepsilon_{n-1} x_{n-1} - x_n$$

be a 2-cycle, where $\varepsilon_i \in \{-1, 1\}$, and x_i, x_j could be equal when $i \neq j$. (If the occurrence of x_n in c is positive, then one can replace c by the 2-cycle $-c$.)

Remark 3.1. The n is even.

Proof. Since c is a 2-cycle, and every boundary ∂x_i is of the form $y_{3i-2} - y_{3i-1} + y_{3i}$, for $y_j \in K_1$, we have that

$$0 = \partial c = \sum_{j=1}^{3n} \tau_j y_j,$$

where $\tau_j \in \{-1, 1\}$. Here y 's must repeat, since C_1 is free, and the cancellation happens only when two identical y 's with the opposite signs occur in the sum. Therefore the number $3n$, and hence n , must be even. \square

For an arbitrary function $v: K_0 \cup K_1 \rightarrow \mathbf{R}^2$, consider the operator $\mu: K_2 \rightarrow (\mathbf{R}^2)^6$ defined by

$$\mu x = (vd_1 d_2 x, vd_0 d_2 x, vd_0 d_0 x, vd_0 x, vd_1 x, vd_2 x).$$

Therefore, the operator μ maps an oriented triangle ABC , whose sides are a , b and c respectively, into the sextuple (vA, vB, vC, va, vb, vc) . We claim the following.

Proposition 3.2. *If $\mu x_1, \dots, \mu x_{n-1}$ make Menelaus configurations, then μx_n makes a Menelaus configuration, too.*

For the proof of this proposition we introduce the following partial function (“partial homomorphism”) $h: (C_1, +, 0) \rightarrow (\mathbf{R} - \{0\}, \cdot, 1)$. Note that every element a of $C_1 - \{0\}$ could be written uniquely (up to associativity and commutativity) as $\alpha_1 y_1 + \dots + \alpha_m y_m$, where $\alpha_i \in \mathbf{Z} - \{0\}$ and the y_i ’s are mutually distinct elements of K_1 . If for every $i \in \{1, \dots, m\}$ we have that

$$hy_i = (vd_0 y_i, vd_1 y_i; v y_i)$$

is defined, then $ha =_{df} (hy_1)^{\alpha_1} \cdot \dots \cdot (hy_m)^{\alpha_m}$, otherwise, ha is undefined. To complete the definition, let $h0 = 1$.

Remark 3.3. If ha_1 and ha_2 are defined, then $h(a_1 + a_2)$ is defined and equal to $ha_1 \cdot ha_2$.

Remark 3.4. A sextuple μx makes a Menelaus configuration iff $h\partial x$ is defined and equal to -1 .

Proof of Proposition 3.2. Since c is a 2-cycle, we have that

$$\partial x_n = \sum_{i=1}^{n-1} \varepsilon_i \partial x_i.$$

By Remark 3.4, for every $i \in \{1, \dots, n-1\}$, we have that $h(\varepsilon_i \partial x_i)$, which is either $h(\partial x_i)$ or its reciprocal value, is equal to -1 . By Remark 3.1, the number $n-1$ is odd, and by Remark 3.3, we have $h\partial x_n = -1$, which means, again by Remark 3.4, that μx_n makes a Menelaus configuration. \square

Remark 3.5. If the 2-dimensional Δ -complex structure involved in the 2-cycle c does not satisfy that for every $i \in \{1, \dots, n\}$ and every $j \in \{1, \dots, 3n\}$ two different faces map x_i (y_j) to two different elements of K_1 (K_0), then the implication of Proposition 3.2 holds vacuously since its antecedent is false.

Proposition 3.2 shows that Richter-Gebert’s idea could be generalized from closed orientable triangulated surfaces to arbitrary 2-cycles of Δ -complexes. However, we will show that this generalisation is not essential. What follows may serve as a light introduction to Steenrod’s problem (see [7, Section 7, Problem 25] for the formulation, and [23] for the solution).

Let K be a Δ -complex and $c = \sum_{i=1}^n \varepsilon_i x_i$, where $\varepsilon_i \in \{-1, 1\}$ and $x_i \in K_2$, be a 2-cycle as above. Moreover, assume that the 2-dimensional Δ -complex structure involved in the 2-cycle c satisfies the conditions of Remark 3.5, which is the only interesting case. For $\partial x_i = y_{3i-2} - y_{3i-1} + y_{3i}$, we have that

$$0 = \partial c = \sum_{j=1}^{3n} \tau_j y_j,$$

where $\tau_j \in \{-1, 1\}$, and $3n = 2m$ for some $m \geq 1$. Choose a partition of the set $\{1, \dots, 2m\}$ with all classes containing exactly two elements, such that i and j belong to the same class when $y_i = y_j$ and $\tau_i = -\tau_j$. Denote these classes by s_1, \dots, s_m .

Let L be the Δ -complex constructed as follows. For $m \geq 3$, we set $L_m = \emptyset$, $L_2 = \{u_1, \dots, u_n\}$ (for genuine u_i ’s), and $L_1 = \{s_1, \dots, s_m\}$. For $k \in \{0, 1, 2\}$, let $d_k: L_2 \rightarrow L_1$ be the functions defined so that

$$d_k(u_i) = s_j \quad \text{when} \quad 3i - 2 + k \in s_j.$$

Let L_0 be the quotient set $\{(s_1, 0), (s_1, 1), \dots, (s_m, 0), (s_m, 1)\} / \approx$, where \approx is the smallest equivalence relation satisfying for every $i \in \{1, \dots, n\}$:

$$(d_1 u_i, 0) \approx (d_0 u_i, 0), \quad (d_2 u_i, 0) \approx (d_0 u_i, 1), \quad (d_2 u_i, 1) \approx (d_1 u_i, 1).$$

Finally, let $d_0 s_j = (s_j, 0)_{\approx}$ and $d_1 s_j = (s_j, 1)_{\approx}$.

Remark 3.6. Note that $c' = \sum_{i=1}^n \varepsilon_i u_i$ is a 2-cycle of the Δ -complex L . This complex depends on the 2-cycle c of the Δ -complex K , and the choice of an appropriate pairing of occurrences of 1-cells in the expression of ∂c given above.

A *morphism* $f: L \rightarrow K$ between Δ -complexes is a family of functions $\{f_i: L_i \rightarrow K_i \mid i \in \mathbf{N}\}$ that commute with the faces. For K and L as above, if f_2, f_1 and f_0 are defined so that $f_2(u_i) = x_i, f_1(s_j) = y_k$, for $k \in s_j$, and for $l \in \{0, 1\}$, $f_0((s_j, l)_{\approx}) = d_l f_1(s_j)$, then $f = (f_0, f_1, f_2, \emptyset, \dots)$ is a morphism from L to K .

Every function $v: K_0 \cup K_1 \rightarrow \mathbf{R}^2$, as above, is lifted by f to a function $v': L_0 \cup L_1 \rightarrow \mathbf{R}^2$ (roughly $v' = v \circ (f_0 \cup f_1)$). If the operator $\mu': L_2 \rightarrow \mathbf{R}^2$ is defined as the operator μ from above, save that v is now replaced by v' , then we have that $\mu' u_i$ makes a Menelaus configuration iff μx_i makes a Menelaus configuration. Hence, if an incidence result follows from an interpretation of the 2-cycle $c = \sum_{i=1}^n \varepsilon_i x_i$ of K , then it follows from an interpretation of the 2-cycle $c' = \sum_{i=1}^n \varepsilon_i u_i$ of L .

The complex L that we just constructed satisfies special properties, which we now list. We first introduce some terminology. For a 2-cell u , we call a face of u an *edge* of u and a face of a face of u a *vertex* of u . Also, a face of a 1-cell is called its *vertex*. A Δ -complex is *connected* when for each pair of mutually distinct 0-cells w and w' , there is a sequence $w = w_0, \dots, w_n = w'$ of 0-cells, such that every pair of consecutive elements in it is the pair of vertices of a 1-cell. A *connected component* of a Δ -complex is defined as expected. We say that two 2-cells are *w-neighbours* when they share an edge having w as a vertex.

The Δ -complex L satisfies:

- (0) L is *finite*, i.e. it has a finite number of cells;
- (1) L is *homogeneous* 2-dimensional, i.e. for every $m \geq 3$ the set L_m is empty and every element of $L_0 \cup L_1$ is a face of some element of $L_1 \cup L_2$;
- (2) L is *regular*, i.e. two different faces map an element of L_2 (L_1) to two different elements of L_1 (L_0);
- (3) for every 1-cell of L there are exactly two 2-cells having this 1-cell as an edge;
- (4) for every $w \in L_0$, the set $L_w = \{u \in L_2 \mid w \text{ is a vertex of } u\}$ is *linked* in the sense that if $u, u' \in L_w$, then there is a sequence of 2-cells starting at u and ending at u' , such that every two consecutive 2-cells are w -neighbours;
- (5) L is *orientable*, i.e. the second homology group $H_2(L; \mathbf{Z})$, which consists of all 2-cycles of L , is isomorphic to the direct sum of k copies of \mathbf{Z} , where k is the number of connected components of L . If L is connected, then one may take a generator $\sum_{i=1}^n \varepsilon_i u_i$ of $H_2(L; \mathbf{Z})$ as an *orientation* of L .

The regularity condition follows from our assumption that c satisfies the conditions from Remark 3.5. The condition (4) follows from the definition of \approx . We call a connected Δ -complex that satisfies (0)-(5), an *\mathcal{M} -complex* (\mathcal{M} comes from Menelaus).

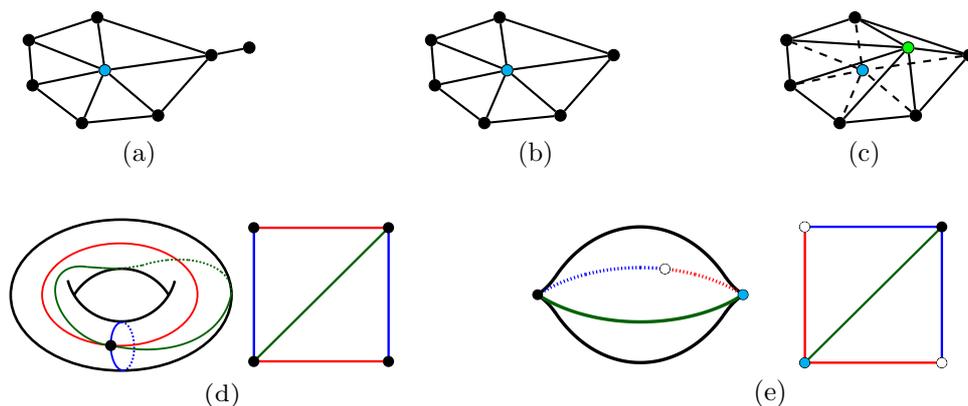
Remark 3.7. By the regularity property, every 2-cell u has three mutually distinct edges $d_0 u, d_1 u$ and $d_2 u$, and three mutually distinct *vertices*, the 0th, $d_1 d_2 u = d_1 d_1 u$, the 1st, $d_0 d_2 u = d_1 d_0 u$ and the 2nd, $d_0 d_0 u = d_0 d_1 u$. Every vertex of u is the common vertex of exactly two edges of u .

Remark 3.8. For a definition of orientability, one may consult textbooks in algebraic topology. According to [14, Theorem 3.25], a connected n -manifold M is

orientable iff $H_n(M; \mathbf{Z}) = \mathbf{Z}$. A generator of the infinite cyclic group is called the *fundamental class*. Since there are two possible choices of a generator, there are exactly two orientations of such a manifold. The fundamental class of a connected orientable n -manifold having a structure of CW complex is the sum of n -cells oriented in accordance with the local orientations of the manifold, see [5, Problem 5, Section 16.4], and this is the classical interpretation of the fundamental class of a triangulated manifold.

EXAMPLE 1. The “dunce hat” is a geometric realisation of a Δ -complex obtained by identifying all the three edges of a single triangle, preserving the orientations of these edges. By construction, this is not a regular Δ -complex. The “dunce hat” is not a manifold, and its second homology group contains just the trivial 2-cycle—hence, it cannot be triangulated in a manner interesting for the Menelaus reasoning.

Consider the Δ -complexes (a), (b), (c), (d) and (e). The Δ -complex (a) is not homogeneous, since it contains an “antenna” which is not an edge of any triangle. The Δ -complex (b), obtained from the Δ -complex (a) by removing the antenna, is homogeneous, but it does not satisfy the property (3) from the definition of \mathcal{M} -complexes. The Δ -complex (d), obtained by identifying (without twisting) the opposite sides of the square, triangulated by a diagonal, is not regular. The Δ -complexes (c) and (e) are \mathcal{M} -complexes.



Proposition 3.9. *The geometric realisation of any \mathcal{M} -complex L is a closed orientable surface.*

Proof. By (1) and (3) it is evident that $|L|$ is a manifold, locally homeomorphic to \mathbf{R}^2 , except at the realisations of 0-cells. Let $w \in L_0$, and let L_w be defined as above. If $u \in L_w$ and w is its i th vertex, then let Δ_u^2 be the intersection of Δ^2 and the open halfspace $t_i > 1/2$. For \sim obtained by restricting the equivalence relation that defines geometric realisation, we claim that

$$U_w = \left(\prod_{u \in L_w} \{u\} \times \Delta_u^2 \right) / \sim,$$

is an open neighbourhood of the realisation of w , which is homeomorphic to \mathbf{R}^2 .

Let $u \in L_w$ and let y and y' be the edges of u having w as a vertex. Let u' be the 2-cell sharing y' with u , and let y'' be the second edge of u' having w as a vertex. If $y = y''$, then $L_w = \{u, u'\}$, otherwise, L_w cannot be linked. If $y \neq y''$, then let u'' be the 2-cell sharing y'' with u' , and let y''' be the second edge of u'' having w as a vertex. We have that $y''' \neq y', y''$ and if $y = y'''$, then $L_w = \{u, u', u''\}$ by the same

reasons as above. If $y \neq y'''$, then it is evident how to proceed with this listing of L_w until we reach $L_w = \{u, u', u'', \dots, u^{(k)}\}$ such that every two consecutive members, as well as u and $u^{(k)}$, are w -neighbours. Then U_w is an open disc triangulated in k triangles. By (5), we have that $|L|$ is orientable. If it is connected, then $\sum_{i=1}^n \varepsilon_i u_i$ is one of its two possible orientations. \square

4. PERMUTATIONS AND SWITCHING OF TRIANGLES

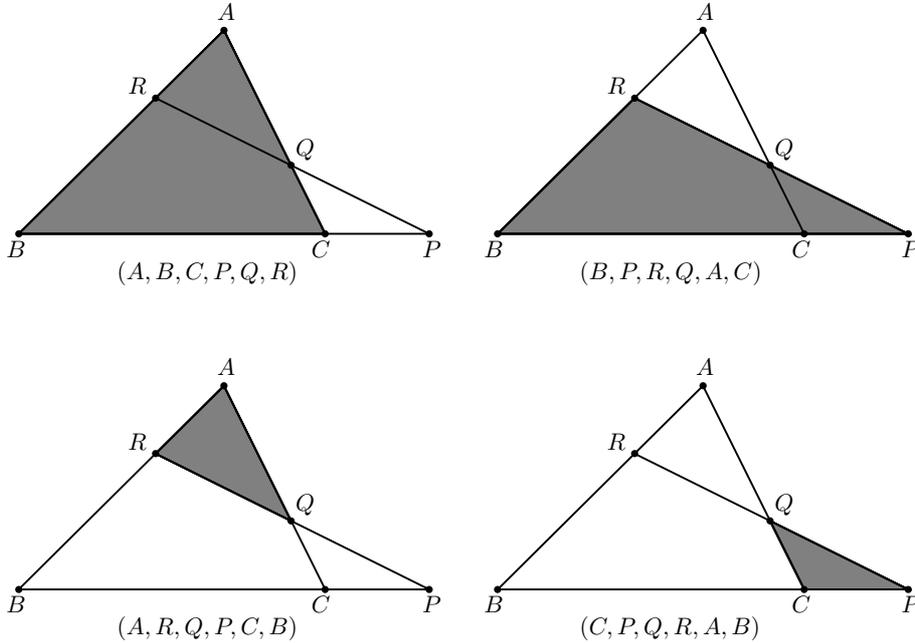
All the entailments that we have, up to now, are of the form: conclude a Menelaus configuration from several such configurations. Is the above “surface” reasoning the only one of such a form? We show that there are some other, quite elementary, reasonings with Menelaus configurations, which keep this form.

Remark 4.1. If $(A_1, A_2, A_3, B_1, B_2, B_3)$ makes a Menelaus configuration and π is a permutation of the set $\{1, 2, 3\}$, then it is easy to check that

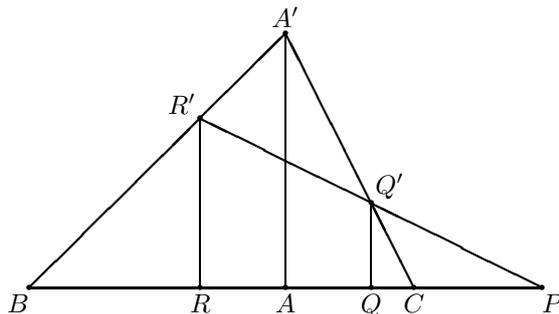
$$(A_{\pi(1)}, A_{\pi(2)}, A_{\pi(3)}, B_{\pi(1)}, B_{\pi(2)}, B_{\pi(3)})$$

makes a Menelaus configuration, too.

Remark 4.2. If (A, B, C, P, Q, R) makes a Menelaus configuration, then the sextuples (B, P, R, Q, A, C) , (A, R, Q, P, C, B) and (C, P, Q, R, A, B) make Menelaus configurations, too.



Proof. If A , B and C are not colinear, then this follows by using the Menelaus theorem in both directions. If A , B and C are colinear, then let A' be a point outside the line BC (for example, $A'A$ is perpendicular to BC , as below) and let $Q' \in A'C$ and $R' \in A'B$ be such that $A'A \parallel Q'Q \parallel R'R$. Then it remains to apply the Thales theorem, and the Menelaus theorem in both directions. \square



The symmetric group S_6 , which acts naturally on sextuples, contains a subgroup G generated by the permutations $s = (123)(456)$ and $t = (26)(35)$. The group G is of order 24 and it is isomorphic to the octahedral group (also isomorphic to S_4), presented by $\langle s, t \mid s^3, t^2, (st)^4 \rangle$. By Remarks 4.1 and 4.2, it follows that if a sextuple makes a Menelaus configuration, then every sextuple from its G -orbit makes a Menelaus configuration, too.

5. THE MENELAUS SYSTEM

The aim of this section is to introduce a one-sided sequent system, which deals with propositions of the form “this sextuple of points makes a Menelaus configuration”. An intuition behind the sequents of our system is that an arbitrary formula in a sequent is entailed by the remaining formulae of the sequent.

Probably the most prominent one-sided sequent system is the system of *proof nets* for the multiplicative fragment of *linear logic*, introduced by Girard, [13]. There are many criteria of correctness to ensure that a given derivation is actually a proof net. One geometric criterion that differs from but is comparable to what we have mentioned about reasoning with triangulations of surfaces in Section 3 is the *acyclic-connected correctness criterion* of Danos and Regnier, [4].

Some further examples of one-sided sequent systems are the Gentzen-Schütte-Tait system (see [21] and [22]), which is a standard one-sided formulation of the propositional fragment of Gentzen’s classical sequent calculus, and the one-sided sequent system called *minimal sequent calculus* introduced in [15].

In order to build the formal language, we introduce the following set whose elements take the role of atomic formulae. For an arbitrary countable set W , let

$$F^6(W) = W^6 - \{(X_1, \dots, X_6) \in W^6 \mid X_i = X_j \text{ for some } i \neq j\}.$$

The *atomic formulae* of our language are the elements of $F^6(W)$. (It would be more convenient to write a predicate symbol in front of a sextuple, but since we deal with one predicate only, we will use no symbol for it.) The *formulae* are built out of atomic formulae by using the connective \bowtie , which plays the role of conjunction and disjunction, simultaneously, and \leftrightarrow , which plays the role of two implications, simultaneously. The metavariables we use for formulae are $\varphi, \psi, \theta, \dots$, possibly with indices. A *sequent* is a finite multiset of formulae, and the sequent consisting of a multiset Γ is denoted by $\vdash \Gamma$.

The axiomatic sequents are formed in the following manner. For every \mathcal{M} -complex L such that $L_0 \cup L_1 \subseteq W$, let $\nu: L_2 \rightarrow F^6(W)$ be defined as

$$\nu x = (d_1 d_2 x, d_0 d_2 x, d_0 d_0 x, d_0 x, d_1 x, d_2 x)$$

(cf. the definition of the operator μ in Section 3).

The set of axiomatic sequents includes the sequents of the form

$$\vdash \{\nu x \mid x \in L_2\},$$

for every \mathcal{M} -complex L whose 0 and 1-cells belong to W . For example, let L be the sphere S^2 , together with the \mathcal{M} -complex structure given by two 2-cells having the same boundaries. If the 0-cells are A, B, C , and the corresponding 1-cells are P, Q, R respectively, then

$$\vdash (A, B, C, P, Q, R), (A, B, C, P, Q, R)$$

is an axiomatic sequent playing the role of the *identity* derivation.

Moreover, we have the following two axiom schemata corresponding to permutations of vertices and switching of triangles (cf. Section 4).

$$\begin{aligned} &\vdash (A, B, C, P, Q, R), (B, C, A, Q, R, P), \\ &\vdash (A, B, C, P, Q, R), (A, R, Q, P, C, B). \end{aligned}$$

The rules of inference of the system are introduced as follows. Besides the *cut* rules:

$$\frac{\vdash \Gamma, \varphi \quad \vdash \Delta, \varphi}{\vdash \Gamma, \Delta} \qquad \frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}$$

there are the following inference rules of \bowtie -*introduction* and \leftrightarrow -*introduction*:

$$\frac{\vdash \Gamma, \varphi \quad \vdash \Gamma, \psi}{\vdash \Gamma, \varphi \bowtie \psi} \qquad \frac{\vdash \Gamma, \varphi \quad \vdash \Delta, \psi}{\vdash \Gamma, \Delta, \varphi \leftrightarrow \psi}.$$

The formula φ in the first cut rule is called the *cut formula*, and we refer to the second cut rule as the one “whose cut formula is empty”. The second cut rule enables us to take into account just connected complexes while creating the family of \mathcal{M} -complexes.

Analogously, the first cut rule enables us to build the axiomatic sequents not with respect to arbitrary \mathcal{M} -complexes, but to restrict this family to those not expressible as *connected sums* of two simpler complexes. However, we do not take advantage of this opportunity in the present paper (see Section 8, Example 3).

The notion of reducible \mathcal{M} -complex and Proposition 10.7 given in Section 10.1 enable us to prove the following for an axiomatic sequent $\vdash \Gamma$ obtained from an \mathcal{M} -complex K .

Proposition 5.1. *K is reducible if and only if $\vdash \Gamma$ could be obtained by the first cut rule from two simpler (with respect to cardinality) axiomatic sequents.*

Note that our connective \bowtie corresponds to the *additive* connective $\&$, and \leftrightarrow corresponds to the *multiplicative* connective \otimes in linear logic (cf. [13]). Also, the cut rule of [13], which is of the form

$$\frac{\vdash \varphi, \Gamma \quad \vdash \varphi^\perp, \Delta}{\vdash \Gamma, \Delta},$$

where φ^\perp denotes the *linear negation* of φ , differs from ours. Informally, in the Menelaus system, a formula φ coincides with its linear negation φ^\perp .

Alternatively, we could introduce the system in a slightly less syntactical manner. Instead of taking $F^6(W)$ as the set of atomic formulae, we could start with the orbit set $\mathcal{W} = F^6(W)/G$ and omit the two axiom schemata from above. Similar “strictifications” are ubiquitous in logic, e.g. omitting parentheses and neglecting the order of conjuncts in purely conjunctive formulae. In such cases one works not with syntactical objects but with classes of equivalence. Our intention is to use that less syntactical, but more practical system with \mathcal{W} as the set of atomic formulae.

6. THE SOUNDNESS

An *Euclidean interpretation* is a function from W to \mathbf{R}^2 , and we abuse the notation by denoting by X the point which is the interpretation of $X \in W$. We say that an interpretation *satisfies* the atomic formula (A, B, C, P, Q, R) , when the sextuple (A, B, C, P, Q, R) of points in \mathbf{R}^2 makes a Menelaus configuration.

Let $\Gamma \models_E \varphi$ mean that every Euclidean interpretation that satisfies every formula in Γ also satisfies φ , where every occurrence of \bowtie in Γ is interpreted as disjunction \vee and every occurrence of \bowtie in φ is interpreted as conjunction \wedge . Concerning the connective \leftrightarrow , it is always interpreted as classical equivalence.

Proposition 6.1 (Soundness). *If $\vdash \Gamma, \varphi$ is derivable, then $\Gamma \models_E \varphi$.*

Proof. We proceed by induction on the complexity of a derivation of $\vdash \Gamma, \varphi$. Assume that an interpretation satisfies every formula in Γ .

If $\vdash \Gamma, \varphi$ is an axiomatic sequent derived from an \mathcal{M} -complex L , then we proceed as in the proof of Proposition 3.2, with c being an orientation of L . If $\vdash \Gamma, \varphi$ is an instance of one of the two axiomatic schemata, then we rely on Remarks 4.1 and 4.2.

If the last inference rule in the derivation of $\vdash \Gamma, \varphi$ is

$$\frac{\vdash \Gamma_1, \psi \quad \vdash \Gamma_2, \varphi, \psi}{\vdash \Gamma_1, \Gamma_2, \varphi},$$

then by the induction hypothesis applied to $\vdash \Gamma_1, \psi$, the interpretation satisfies ψ . Thence, by the induction hypothesis applied to $\vdash \Gamma_2, \varphi, \psi$, the interpretation satisfies φ . If the cut formula ψ is empty, then the induction hypothesis applied to the right premise $\vdash \Gamma_2, \varphi$ is sufficient. We proceed analogously when φ occurs in the left premise of the cut rule.

If the last inference rule is

$$\frac{\vdash \Gamma_1, \varphi, \psi \quad \vdash \Gamma_1, \varphi, \theta}{\vdash \Gamma_1, \varphi, \psi \bowtie \theta},$$

then the interpretation that satisfies $\Gamma_1, \psi \bowtie \theta$, by the definition of the interpretation of \bowtie in Γ , must satisfy Γ_1, ψ or Γ_1, θ , and by the induction hypothesis applied to the corresponding premise, one obtains that the interpretation satisfies φ too.

If the last inference rule is

$$\frac{\vdash \Gamma, \varphi_1 \quad \vdash \Gamma, \varphi_2}{\vdash \Gamma, \varphi_1 \bowtie \varphi_2},$$

then by the induction hypothesis applied to both premisses, one obtains that both φ_1 and φ_2 are satisfied by the interpretation. Hence, $\varphi = \varphi_1 \bowtie \varphi_2$ is satisfied by the definition of the interpretation of \bowtie in φ .

If the last inference rule is

$$\frac{\vdash \Gamma_1, \psi \quad \vdash \Gamma_2, \varphi, \theta}{\vdash \Gamma_1, \Gamma_2, \varphi, \psi \leftrightarrow \theta},$$

then by the induction hypothesis applied to the left premise, one obtains that ψ is satisfied by the interpretation, and since $\psi \leftrightarrow \theta$ is satisfied, θ must be satisfied too. Then by the induction hypothesis applied to the right premise, one obtains that φ is satisfied. We proceed analogously when φ occurs in the left premise of the rule.

Finally, if the last inference rule is

$$\frac{\vdash \Gamma_1, \varphi_1 \quad \vdash \Gamma_2, \varphi_2}{\vdash \Gamma_1, \Gamma_2, \varphi_1 \leftrightarrow \varphi_2},$$

then by the induction hypothesis applied to both premisses, one obtains that both φ_1 and φ_2 are satisfied by the interpretation. Hence, $\varphi = \varphi_1 \leftrightarrow \varphi_2$ is satisfied too. \square

7. THE PROJECTIVE INTERPRETATION

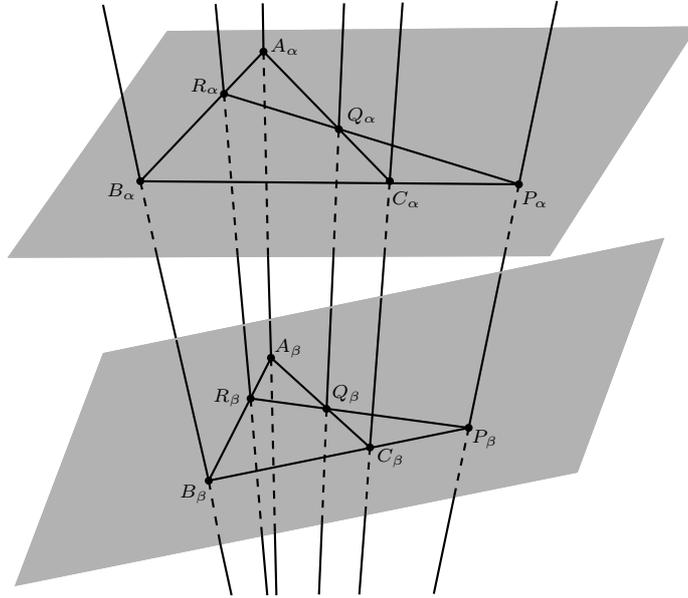
We define the projective interpretation and the satisfiability relation along the lines of the previous section. Then the corresponding soundness result is a corollary of Proposition 6.1.

A *projective interpretation* is a function from W to the *projective plane* \mathbf{RP}^2 . Again, we denote by X the point in \mathbf{RP}^2 , which is the interpretation of $X \in W$.

We consider the points of \mathbf{RP}^2 as lines through the origin in \mathbf{R}^3 . For a finite set \mathcal{S} of such points there is a plane α in \mathbf{R}^3 , not containing the origin, which intersects all the corresponding lines. (It is sufficient to choose a plane whose normal vector is not orthogonal to the direction vector of all the elements of this finite set, and such a plane exists since \mathbf{R}^3 cannot be covered by a finite number of planes.) In that case, we say that α *intersects properly* the points from \mathcal{S} , and for every $A \in \mathcal{S}$ we denote the intersection of A and α by A_α .

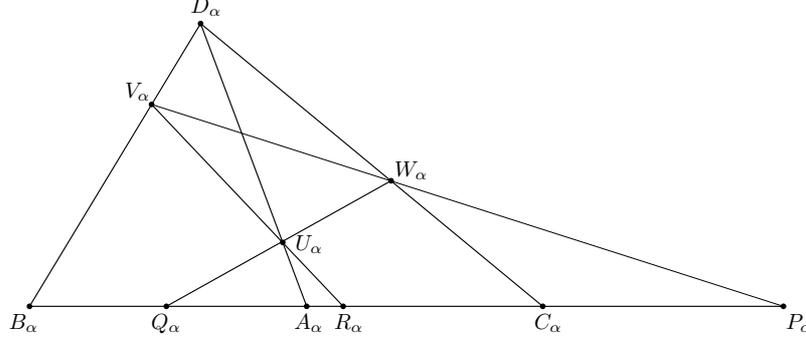
Lemma 7.1. *Let α and β be two planes in \mathbf{R}^3 that intersect properly the points A, B, C, P, Q and R from \mathbf{RP}^2 . If the sextuple $(A_\alpha, B_\alpha, C_\alpha, P_\alpha, Q_\alpha, R_\alpha)$ makes a Menelaus configuration, then $(A_\beta, B_\beta, C_\beta, P_\beta, Q_\beta, R_\beta)$ makes it too.*

Proof. If A, B and C are not colinear, then neither A_α, B_α and C_α , nor A_β, B_β and C_β are colinear and since P_α, Q_α and R_α are three colinear points lying on the lines $B_\alpha C_\alpha, C_\alpha A_\alpha$ and $A_\alpha B_\alpha$ respectively, we have that P_β, Q_β and R_β are three colinear points lying on the lines $B_\beta C_\beta, C_\beta A_\beta$ and $A_\beta B_\beta$ respectively, which means that $(A_\beta, B_\beta, C_\beta, P_\beta, Q_\beta, R_\beta)$ makes a Menelaus configuration.



If A, B and C are colinear, then for the plane β_0 such that $0 \in \beta_0$ and $\beta \parallel \beta_0$, denote by b the intersection of α and β_0 . Hence b is either a line in α or the empty set. In the sequel we assume that all the chosen points are outside b . Let D_α be a point in α outside the line $A_\alpha B_\alpha$, and let U_α be a point on the segment $A_\alpha D_\alpha$ such that the lines $R_\alpha U_\alpha$ and $B_\alpha D_\alpha$, as well as the lines $Q_\alpha U_\alpha$ and $C_\alpha D_\alpha$,

intersect (we can always choose such a point U_α). Let $\{V_\alpha\} = R_\alpha U_\alpha \cap B_\alpha D_\alpha$ and $\{W_\alpha\} = Q_\alpha U_\alpha \cap C_\alpha D_\alpha$. Note that the sextuples $(A_\alpha, B_\alpha, D_\alpha, V_\alpha, U_\alpha, R_\alpha)$ and $(A_\alpha, C_\alpha, D_\alpha, W_\alpha, U_\alpha, Q_\alpha)$ make Menelaus configurations.



Consider the following axiomatic sequent derived from a tetrahedral triangulation of the sphere S^2 (cf. Section 8, Example 1)

$$\vdash (A, B, C, P, Q, R), (A, B, D, V, U, R), (A, C, D, W, U, Q), (B, C, D, W, V, P),$$

and an Euclidean interpretation that interprets $X \in \{A, B, C, D, P, Q, R, U, V, W\}$ as X_α . This interpretation satisfies the first three sextuples, and by the soundness result for Euclidean interpretation the sextuple $(B_\alpha, C_\alpha, D_\alpha, W_\alpha, V_\alpha, P_\alpha)$ makes a Menelaus configuration.

By appealing to the case of non-colinear A, B and C , we know that an Euclidean interpretation that interprets $X \in \{A, B, C, D, P, Q, R, U, V, W\}$ as X_β satisfies the last three sextuples, and by the soundness result the sextuple $(A_\beta, B_\beta, C_\beta, P_\beta, Q_\beta, R_\beta)$ makes a Menelaus configuration. \square

After this lemma, we can say that a sextuple (A, B, C, P, Q, R) of points in \mathbf{RP}^2 makes a Menelaus configuration when for some (or every) plane α in \mathbf{R}^3 , which intersects properly all these points, the sextuple $(A_\alpha, B_\alpha, C_\alpha, P_\alpha, Q_\alpha, R_\alpha)$ makes a Menelaus configuration in the Euclidean sense. We say that a projective interpretation satisfies the atomic formula (A, B, C, P, Q, R) , when the sextuple (A, B, C, P, Q, R) of points in \mathbf{RP}^2 makes a Menelaus configuration.

Let $\Gamma \models_P \varphi$ mean that every projective interpretation that satisfies every formula in Γ also satisfies φ . As a corollary of Proposition 6.1 we have the following result.

Proposition 7.2 (Projective Soundness). *If $\vdash \Gamma, \varphi$ is derivable, then $\Gamma \models_P \varphi$.*

8. FROM DERIVABLE SEQUENTS TO INCIDENCE RESULTS

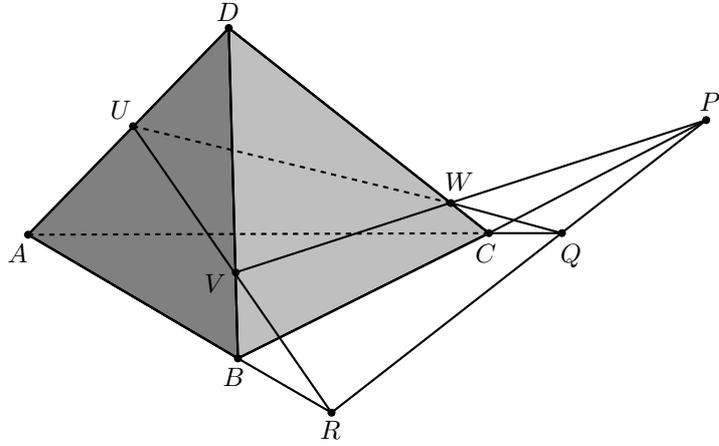
A general pattern for extracting an incidence result (its formulation and a proof) from derivable sequents of our system is the following. One has to use interpretations that satisfy all but one formulae in a derivable sequent. By the soundness result, such an interpretation satisfies the last formula too. For some results it is enough to use just one sequent and one interpretation. On the other hand, some results require several interpretations and one derivable sequent, and perhaps, there are cases when several derivable sequents are involved in one incidence result.

Our general position is such that we treat the Menelaus system as the syntax and the projective plane as the semantics. In order to use a provable sequent, one has to choose one of its formulae that should be treated as a consequence of the others. However, for proving that the other formulae of the sequent are satisfied by the assumptions used in the statement of the incidence result one cannot rely exclusively on the Menelaus system. This system does not deal with some negative

statements (expressing that some points are not colinear or that they are mutually distinct) or some positive statements (expressing that some lines coincide), which are left to (the axioms of) projective geometry. Hence, the Menelaus system is somehow treated as an assistant for a larger system staying behind the projective geometry.

In all the examples below, we behave as if the Menelaus system had been formulated with \mathcal{W} as the set of atomic formulae. This means that we make no distinction between a sextuple of elements of \mathcal{W} and any other member of its G -orbit.

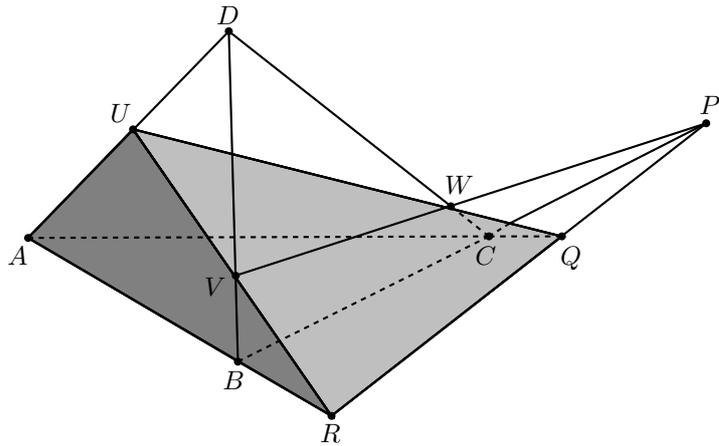
EXAMPLE 1. The following proof of the Desargues theorem uses two axiomatic sequents connected by \bowtie -introduction. Consider the following tetrahedral triangulation of S^2 .



This leads to the axiomatic sequent

$$\vdash (A, B, D, V, U, R), (B, C, D, W, V, P), (A, C, D, W, U, Q), (A, B, C, P, Q, R).$$

On the other hand, there is another tetrahedral triangulation of S^2 ,



which leads to the axiomatic sequent

$$\vdash (A, R, U, V, D, B), (A, R, Q, P, C, B), (U, R, Q, P, W, V), (A, Q, U, W, D, C).$$

Since these two sequents share the three elements

$$(A, B, D, V, U, R), (A, C, D, W, U, Q), (A, B, C, P, Q, R)$$

one may derive the sequent

$$\begin{aligned} \vdash (A, B, D, V, U, R), (A, C, D, W, U, Q), (A, B, C, P, Q, R), \\ (B, C, D, W, V, P) \bowtie (U, R, Q, P, W, V). \end{aligned}$$

We explain now a connection between this derivable sequent and the following famous incidence theorem.

Theorem 8.1 (Desargues). *Let ABC and UVW be two triangles in \mathbf{RP}^2 such that $A \neq U$, $B \neq V$ and $C \neq W$. Let $BC \cap VW = \{P\}$, $AC \cap UW = \{Q\}$ and $AB \cap UV = \{R\}$. Then the lines AU , BV and CW are concurrent if and only if the points P , Q and R are colinear.*

Proof. For the direction from left to right, consider a projective interpretation that maps $A, B, C, U, V, W, P, Q, R$ as indicated and D is interpreted as the common point of AU , BV and CW . This interpretation satisfies

$$(A, B, D, V, U, R), (A, C, D, W, U, Q), \text{ and } (B, C, D, W, V, P),$$

and hence

$$(A, B, D, V, U, R), (A, C, D, W, U, Q), \text{ and } (B, C, D, W, V, P) \vee (U, R, Q, P, W, V),$$

which, by the soundness result, guarantees that it satisfies (A, B, C, P, Q, R) . Therefore, P , Q and R are colinear.

For the other direction, consider a projective interpretation that maps again $A, B, C, U, V, W, P, Q, R$ as indicated, and D is now interpreted as the intersection point of AU and BV . This interpretation satisfies

$$(A, R, U, V, D, B), (A, R, Q, P, C, B), \text{ and } (U, R, Q, P, W, V),$$

and hence

$$(A, B, D, V, U, R), (A, B, C, P, Q, R), \text{ and } (B, C, D, W, V, P) \vee (U, R, Q, P, W, V),$$

which, by the soundness result, guarantees that it satisfies (A, C, D, W, U, Q) , i.e. (A, Q, U, W, D, C) . Therefore W , D and C are colinear, which means that the lines AU , BV and CW are concurrent. \square

EXAMPLE 2. This example contains a non-axiomatic proof of an incidence result. Consider the following two axiomatic sequents obtained from tetrahedral triangulations $ABCD$ and $BRPE$ of two spheres.

$$\vdash (A, B, D, V, U, R), (B, C, D, W, V, P), (A, C, D, W, U, Q), (A, B, C, P, Q, R)$$

$$\vdash (B, R, E, Y, X, A), (B, P, E, Z, X, C), (R, P, E, Z, Y, Q), (B, P, R, Q, A, C)$$

Since (A, B, C, P, Q, R) and (B, P, R, Q, A, C) are identified, one may apply the cut rule to these two sequents in order to obtain the sequent

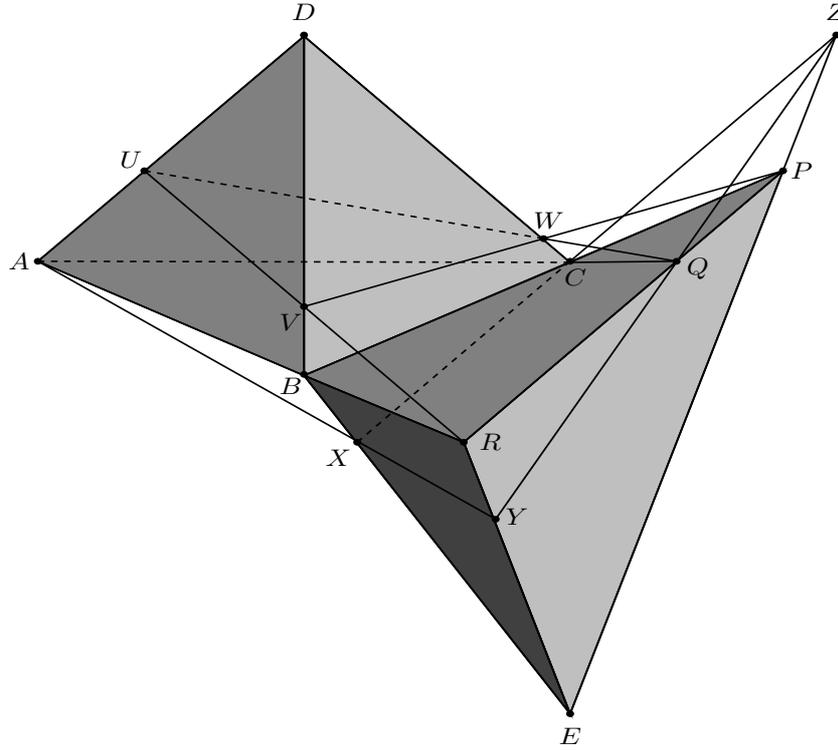
$$\begin{aligned} \vdash (A, B, D, V, U, R), (B, C, D, W, V, P), (A, C, D, W, U, Q), (B, R, E, Y, X, A), \\ (B, P, E, Z, X, C), (R, P, E, Z, Y, Q). \end{aligned}$$

which may serve for a proof of the following incidence result.

Let AU , BV and CW be three concurrent lines in \mathbf{RP}^2 , and let X and E be such that B , X and E are colinear. For

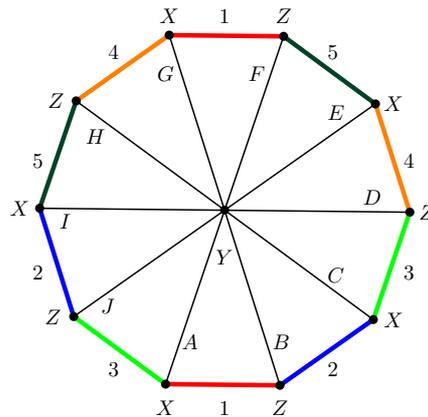
$$\begin{aligned} \{P\} = BC \cap VW, \quad \{Q\} = AC \cap UW, \quad \{R\} = AB \cap UV, \\ \{Y\} = AX \cap RE, \quad \{Z\} = XC \cap EP, \end{aligned}$$

the points Q , Y and Z are colinear.

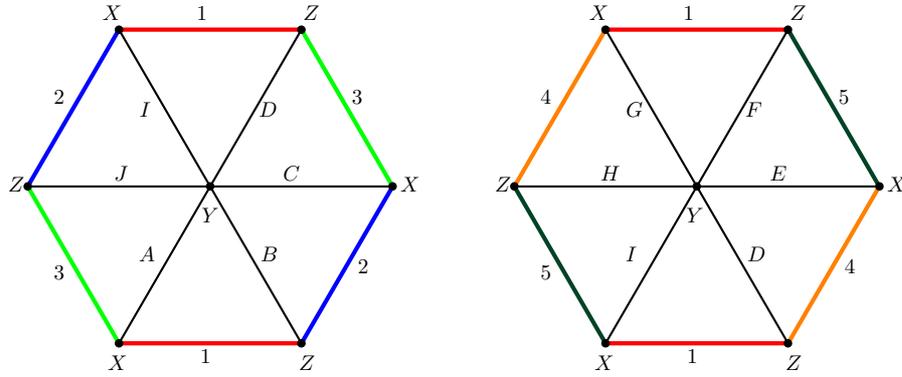


EXAMPLE 3. This example provides an incidence result tied to a more involved axiomatic sequent. Consider the following triangulation of the torus with two holes into ten triangles and three zero-cells in total. The corresponding axiomatic sequent is

$$\begin{aligned} \vdash & (X, Y, Z, B, 1, A), (X, Y, Z, B, 2, C), (X, Y, Z, D, 3, C), (X, Y, Z, D, 4, E) \\ & (X, Y, Z, F, 5, E), (X, Y, Z, F, 1, G), (X, Y, Z, H, 4, G), \\ & (X, Y, Z, H, 5, I), (X, Y, Z, J, 2, I), (X, Y, Z, J, 3, A). \end{aligned}$$

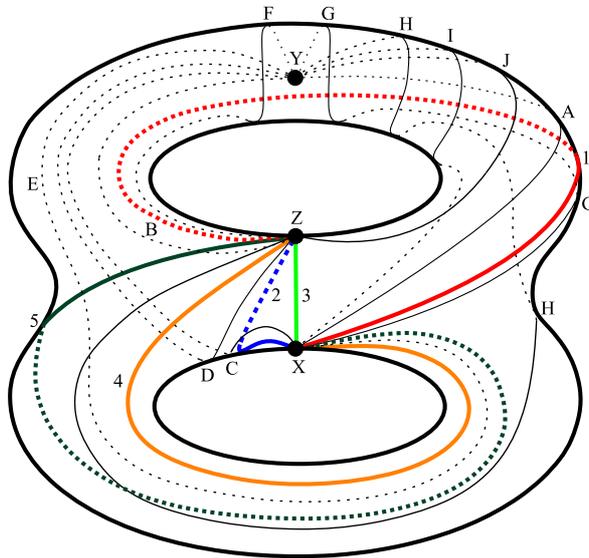


On the other hand, this sequent could be obtained by the cut rule applied to two axiomatic sequents corresponding to the Pappus theorem, which are derived from triangulations of the torus in six triangles (cf. the next example and see [18, Section 3.4] for more details).

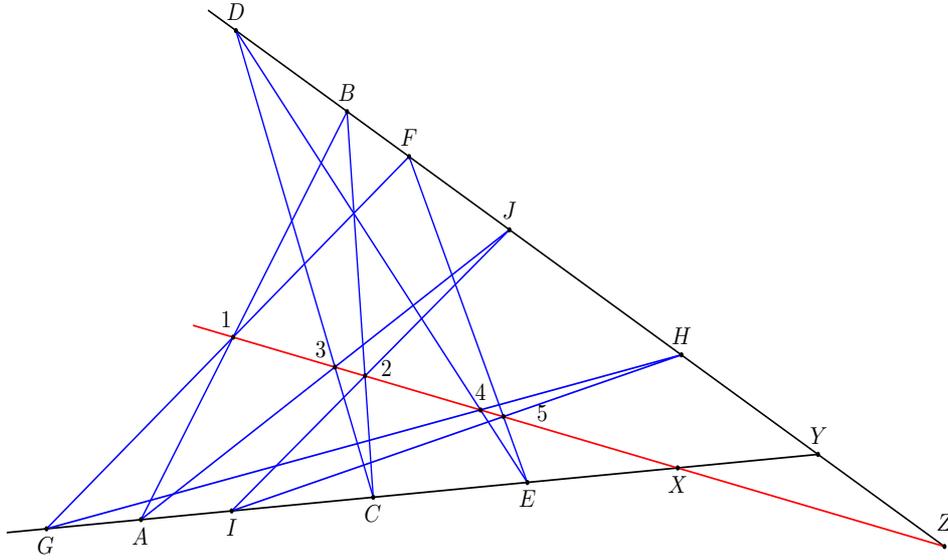


$$\begin{aligned} &\vdash (X, Y, Z, B, 1, A), (X, Y, Z, B, 2, C), (X, Y, Z, D, 3, C), (X, Y, Z, J, 2, I), \\ &\quad (X, Y, Z, J, 3, A), (X, Y, Z, D, 1, I), \\ &\vdash (X, Y, Z, D, 4, E), (X, Y, Z, F, 5, E), (X, Y, Z, F, 1, G), (X, Y, Z, H, 4, G), \\ &\quad (X, Y, Z, H, 5, I), (X, Y, Z, D, 1, I). \end{aligned}$$

Note that the torus with two holes corresponding to the decagon is obtained by the connected sum, with respect to the triangle whose sides are D , 1 and I , from the two tori corresponding to the hexagons. This shows that some axiomatic sequents could be derived from more primitive axiomatic sequents. The following “game” may be treated as an incidence result extracted from the sequent above.



Let p , q and s be three non-concurrent lines in the projective plane. Choose a point A on p , then choose B on q and let the line AB intersects s in 1 . Then choose C on p and let BC intersect s in 2 , and so on up to the point F on q and the intersection point 5 of EF and s . Continue with this zigzag game, save that the path now crosses s in the “old” points $1, 4, 5, 2, 3$, respectively, i.e. the line FG intersects s in 1 , GH intersects s in 4 , and so on. Eventually, the last segment, which starts in J and goes in the direction of 3 toward p , ends in the initial point A .



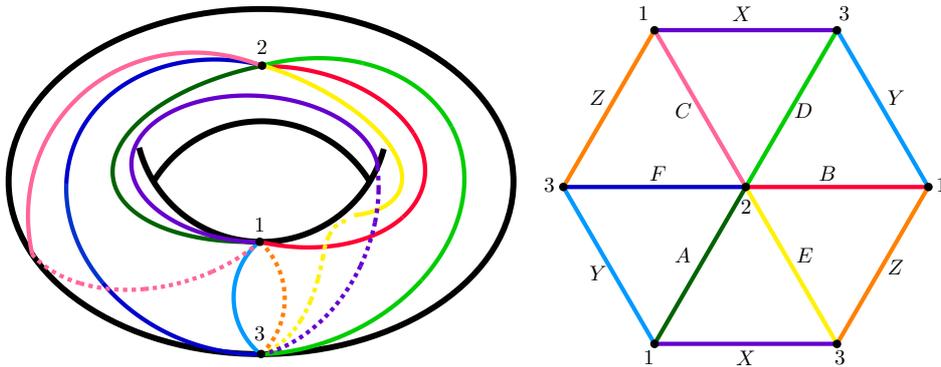
EXAMPLE 4. The proof of the following incidence result uses two axiomatic sequents connected by \leftrightarrow -introduction.

Proposition 8.2. Consider the Pappus configuration consisting of two triples (A, B, C) and (D, E, F) of colinear points, all mutually distinct. Assume that for $\{X\} = CD \cap AE$ and $\{Z\} = BE \cap CF$, the lines AB , DE and XZ are not concurrent. For

$$\begin{aligned} \{K\} &= BE \cap CD, & \{L\} &= AF \cap CD, & \{M\} &= AF \cap BE, \\ \{U\} &= AE \cap CF, & \{V\} &= AE \cap BD, & \{W\} &= CF \cap BD, \end{aligned}$$

the Pappus lines KU , LV and MW are concurrent.

Proof. Let $\{1\} = XZ \cap AB$, $\{2\} = AB \cap DE$, $\{3\} = XZ \cap DE$, $\{O\} = KU \cap LV$, and $\{Y\} = LW \cap BD$.

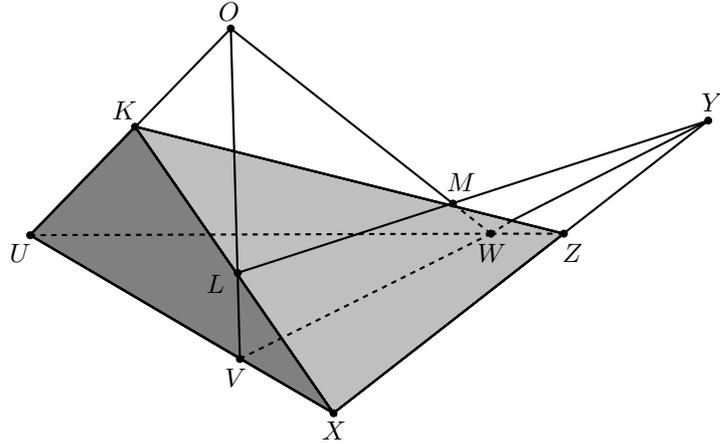


The following axiomatic sequent is obtained by a triangulation of the torus in six triangles all having vertices 1, 2 and 3.

$$\begin{aligned} \vdash (1, 2, 3, E, X, A), (1, 2, 3, E, Z, B), (1, 2, 3, D, Y, B), (1, 2, 3, D, X, C), \\ (1, 2, 3, F, Z, C), (1, 2, 3, F, Y, A). \end{aligned}$$

On the other hand, as in Example 1, from a tetrahedral triangulation of the sphere given by a tetrahedron $UXZK$, we obtain the axiomatic sequent

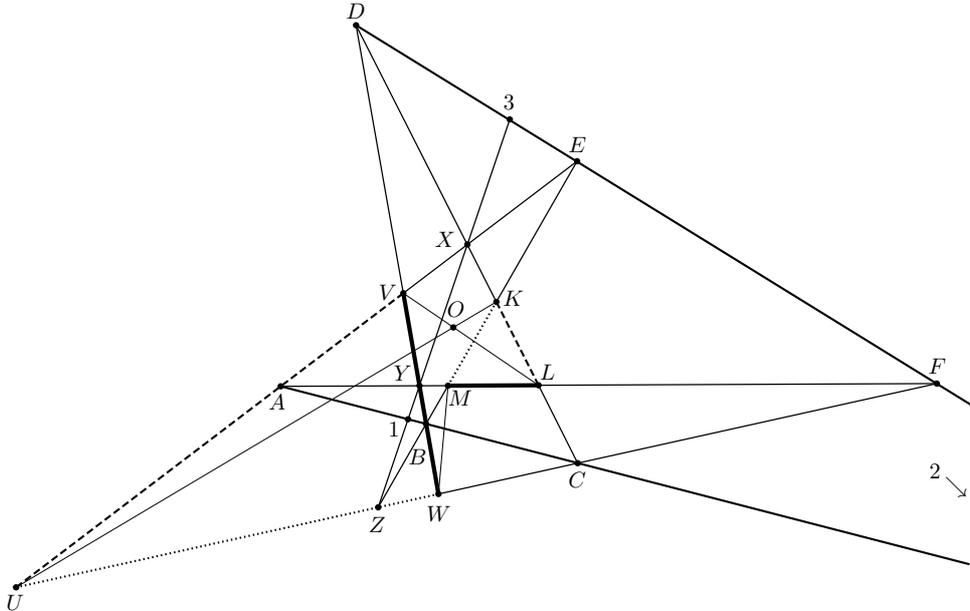
$$\vdash (U, X, K, L, O, V), (U, X, Z, Y, W, V), (K, X, Z, Y, M, L), (U, Z, K, M, O, W).$$



By applying \leftrightarrow -introduction, one obtains the following nine-element sequent.

$$\begin{aligned} \vdash (1, 2, 3, E, X, A), (1, 2, 3, E, Z, B), (1, 2, 3, D, Y, B), (1, 2, 3, D, X, C), \\ (1, 2, 3, F, Z, C), (1, 2, 3, F, Y, A) \leftrightarrow (K, X, Z, Y, M, L), \\ (U, X, K, L, O, V), (U, X, Z, Y, W, V), (U, Z, K, M, O, W). \end{aligned}$$

Consider now a projective interpretation that maps all points as indicated. (The pairs of corresponding sides of the triangles KLM and UVW are indicated in the picture.)



Let us prove that this interpretation satisfies the first eight formulae of this sequent. The proofs of the negative statements that we mention below require a lot of space and go beyond the scope of this paper. They are secondary for incidence

results, and are usually assumed as a kind of general position of points involved in such results. However, we pay attention to all the positive statements involved in this example. The Menelaus theorem is tacitly used during the proof.

In order to show that our interpretation satisfies $(1, 2, 3, E, X, A)$, note that 1, 2 and 3 are not colinear and that 1, 2, 3, E , X , A are mutually distinct. The points E , 2, 3 are colinear by the definition of 2 and 3, the points X , 1, 3 are colinear by the definition of 1 and 3, and the points A , 1, 2 are colinear by the definition of 1 and 2. Finally, the points E , X , A are colinear by the definition of X . We proceed analogously for the next four formulae of the sequent.

In order to show that $(1, 2, 3, F, Y, A) \leftrightarrow (K, X, Z, Y, M, L)$ is satisfied by the interpretation, note that, by reasoning as in the preceding paragraph, the left-hand side of this equivalence is satisfied iff the points F , Y , A are colinear. By the definition of L and M , we have that the lines AF and LM coincide. Hence, the above condition is equivalent to the statement that L , Y , M are colinear. By reasoning as in the preceding paragraph, it is evident that L , Y , M are colinear iff the interpretation satisfies (K, L, M, Y, Z, X) . This means that the interpretation satisfies the above equivalence.

In order to show that our interpretation satisfies (O, U, V, X, L, K) , note that O , U , V are not colinear and that O , U , V , X , L , K are mutually distinct. The lines UV and AE coincide, hence, the definition of X implies that X , U , V are colinear. The definition of O implies that K , O , U as well as L , O , V are colinear. Finally, since the lines CD and KL coincide, the definition of X implies that K , L , X are colinear.

In order to show that our interpretation satisfies (U, V, W, Y, Z, X) , note that U , V , W are not colinear and that U , V , W , Y , Z , X are mutually distinct. The lines VW and BD coincide, hence, the definition of Y implies that Y , V , W are colinear. The lines UW and CF coincide, hence, the definition of Z implies that Z , U , W are colinear. As in the preceding paragraph, we have that X , U , V are colinear. Finally, since the lines XZ and 13 coincide, the definition of Y implies that X , Y , Z are colinear.

By the soundness result, our interpretation satisfies (U, Z, K, M, O, W) , which means that the line WM contains the intersection point of KU and LV , and these three lines are concurrent. \square

9. DECIDABILITY

The aim of this section is to prove the following result.

Proposition 9.1. *The Menelaus system is decidable.*

We say that a derivation in the Menelaus system is *normal*, when neither \mathbb{X} -introduction, nor \leftrightarrow -introduction precedes an application of a cut rule in this derivation. By a formula in a derivation, we mean here a particular *occurrence* of this formula in the derivation.

Lemma 9.2. *If the last inference rule in a derivation is*

$$\frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}$$

and there are no other applications of cut in this derivation preceded by some \mathbb{X} -introduction or \leftrightarrow -introduction, then there is a normal derivation of the sequent $\vdash \Gamma, \Delta$

Proof. We proceed by induction on the number $n \geq 0$ of \mathbb{X} -introductions and \leftrightarrow -introductions in this derivation. If $n = 0$, then the derivation is already normal.

If $n > 0$, then by the assumption, one of the premisses is obtained by either \bowtie -introduction or \leftrightarrow -introduction. If the end of the derivation is of the form

$$\frac{\frac{\frac{\vdash \Gamma', \gamma_1 \quad \vdash \Gamma', \gamma_2}{\vdash \Gamma', \gamma_1 \bowtie \gamma_2} \quad \vdash \Delta}{\vdash \Gamma', \gamma_1 \bowtie \gamma_2, \Delta}},$$

then we transform it into the derivation ending as

$$\frac{\frac{\frac{\vdash \Gamma', \gamma_1 \quad \vdash \Delta}{\vdash \Gamma', \gamma_1, \Delta} \quad \frac{\vdash \Gamma', \gamma_2 \quad \vdash \Delta}{\vdash \Gamma', \gamma_2, \Delta}}{\vdash \Gamma', \gamma_1 \bowtie \gamma_2, \Delta}},$$

where we can apply the induction hypothesis to the subderivations ending with $\vdash \Gamma', \gamma_1, \Delta$ and $\vdash \Gamma', \gamma_2, \Delta$.

If the end of the derivation is of the form

$$\frac{\frac{\frac{\vdash \Gamma', \gamma_1 \quad \vdash \Gamma'', \gamma_2}{\vdash \Gamma', \Gamma'', \gamma_1 \leftrightarrow \gamma_2} \quad \vdash \Delta}{\vdash \Gamma', \Gamma'', \gamma_1 \leftrightarrow \gamma_2, \Delta}},$$

then we transform it into the derivation ending as

$$\frac{\frac{\frac{\vdash \Gamma', \gamma_1 \quad \vdash \Delta}{\vdash \Gamma', \gamma_1, \Delta} \quad \vdash \Gamma'', \gamma_2}{\vdash \Gamma', \Gamma'', \gamma_1 \leftrightarrow \gamma_2, \Delta}},$$

where we can apply the induction hypothesis to the subderivation ending with $\vdash \Gamma', \gamma_1, \Delta$. \square

The following definitions consider the cut whose cut formula is *not* empty. Let the *degree* of a cut be the number of occurrences of \bowtie and \leftrightarrow in the cut formula. For the cut rule and \leftrightarrow -introduction, every formula of the lower sequent, except the *principal* formula $\varphi \leftrightarrow \psi$ in the case of \leftrightarrow -introduction, has a unique *successor*, an occurrence of the same formula, in the upper sequent. In the case of \bowtie -introduction, every formula of the lower sequent, except the *principal* formula $\varphi \bowtie \psi$, has two *successors*, occurrences of the same formula, in the upper sequent. Let the *rank* of a formula in a derivation be the number of formulae that are related to this formula by the reflexive and transitive closure of the successor relation. Let the *rank* of a cut rule in a derivation be the sum of the ranks of the cut formulae in both premisses of this cut.

For the proof of the following lemma, we use a procedure akin to the cut-elimination procedure introduced by Gentzen, [10], which corresponds to cut-disintegration of Došen, [6, Section 1.8.1].

Lemma 9.3. *If the last inference rule in a derivation is*

$$\frac{\frac{\vdash \Gamma, \varphi \quad \vdash \Delta, \varphi}{\vdash \Gamma, \Delta}}$$

and there are no other applications of cut in this derivation preceded by some \bowtie -introduction or \leftrightarrow -introduction, then there is a normal derivation of the sequent $\vdash \Gamma, \Delta$.

Proof. We proceed by induction on the lexicographically ordered pairs (d, r) , where $d \geq 0$ is the degree and $r \geq 2$ is the rank of this cut. The basis of this induction, i.e. the case when $(d, r) = (0, 2)$, holds: in this case the last inference rule is not preceded by \bowtie -introduction or \leftrightarrow -introduction, since both premisses must be axiomatic sequents and the derivation is already normal.

If $r > 2$, then it is possible that the end of our derivation is of the form

$$\frac{\frac{\frac{\vdash \Gamma', \varphi, \psi \quad \vdash \Gamma', \varphi, \chi}{\vdash \Gamma', \psi \bowtie \chi, \varphi}}{\vdash \Gamma', \psi \bowtie \chi, \Delta}, \quad \vdash \Delta, \varphi}{\vdash \Gamma', \psi \bowtie \chi, \Delta},$$

and we transform it into the derivation ending as

$$\frac{\frac{\frac{\vdash \Gamma', \varphi, \psi \quad \vdash \Delta, \varphi}{\vdash \Gamma', \psi, \Delta}}{\vdash \Gamma', \psi \bowtie \chi, \Delta}, \quad \frac{\frac{\vdash \Gamma', \varphi, \chi \quad \vdash \Delta, \varphi}{\vdash \Gamma', \chi, \Delta}}{\vdash \Gamma', \psi \bowtie \chi, \Delta},}{\vdash \Gamma', \psi \bowtie \chi, \Delta},$$

where both applications of cut have the same degree but lower ranks. If the end of our derivation is of the form

$$\frac{\frac{\frac{\vdash \Gamma', \varphi, \psi \quad \vdash \Gamma'', \chi}{\vdash \Gamma', \Gamma'', \psi \leftrightarrow \chi, \varphi}}{\vdash \Gamma', \Gamma'', \psi \leftrightarrow \chi, \Delta}, \quad \vdash \Delta, \varphi}{\vdash \Gamma', \Gamma'', \psi \leftrightarrow \chi, \Delta},$$

then we transform it into the derivation ending as

$$\frac{\frac{\frac{\vdash \Gamma', \varphi, \psi \quad \vdash \Delta, \varphi}{\vdash \Gamma', \Delta, \psi}}{\vdash \Gamma', \Gamma'', \psi \leftrightarrow \chi, \Delta}, \quad \vdash \Gamma'', \chi}{\vdash \Gamma', \Gamma'', \psi \leftrightarrow \chi, \Delta},$$

where the new cut has the same degree but a lower rank.

On the other hand, if we assume that the end of our derivation is of the form

$$\frac{\frac{\frac{\vdash \Gamma_1, \varphi, \psi \quad \vdash \Gamma_2, \psi}{\vdash \Gamma_1, \Gamma_2, \varphi}}{\vdash \Gamma_1, \Gamma_2, \Delta}, \quad \vdash \Delta, \varphi}{\vdash \Gamma_1, \Gamma_2, \Delta}, \quad \text{or} \quad \frac{\frac{\frac{\vdash \Gamma_1, \varphi \quad \vdash \Gamma_2}{\vdash \Gamma_1, \Gamma_2, \varphi}}{\vdash \Gamma_1, \Gamma_2, \Delta}, \quad \vdash \Delta, \varphi}{\vdash \Gamma_1, \Gamma_2, \Delta},$$

then, since \bowtie -introduction and \leftrightarrow -introduction are not applied in the subderivations ending with $\vdash \Gamma_1, \Gamma_2, \varphi$, the formula φ must be atomic. If our derivation is not normal, then the subderivation ending with $\vdash \Delta, \varphi$ must have \bowtie -introduction or \leftrightarrow -introduction as the last rule and we are again in the former situation.

If $r = 2$ and $d > 0$, then we have two possibilities. If the end of our derivation is of the form

$$\frac{\frac{\frac{\vdash \Gamma, \varphi_1 \quad \vdash \Gamma, \varphi_2}{\vdash \Gamma, \varphi_1 \bowtie \varphi_2}}{\vdash \Gamma, \Delta}, \quad \frac{\frac{\vdash \Delta, \varphi_1 \quad \vdash \Delta, \varphi_2}{\vdash \Delta, \varphi_1 \bowtie \varphi_2}}{\vdash \Gamma, \Delta},}{\vdash \Gamma, \Delta},$$

then we transform it into the derivation whose end is the cut

$$\frac{\Gamma, \varphi_1 \quad \vdash \Delta, \varphi_1}{\vdash \Gamma, \Delta}$$

with a lower degree. If the end of our derivation is of the form

$$\frac{\frac{\frac{\vdash \Gamma', \varphi_1 \quad \vdash \Gamma'', \varphi_2}{\vdash \Gamma', \Gamma'', \varphi_1 \leftrightarrow \varphi_2} \quad \frac{\frac{\vdash \Delta', \varphi_1 \quad \vdash \Delta'', \varphi_2}{\vdash \Delta', \Delta'', \varphi_1 \leftrightarrow \varphi_2}}{\vdash \Gamma', \Gamma'', \Delta', \Delta''}}$$

then we transform it into the derivation whose end is of the form

$$\frac{\frac{\frac{\vdash \Gamma', \varphi_1 \quad \vdash \Delta', \varphi_1}{\vdash \Gamma', \Delta'} \quad \frac{\frac{\vdash \Gamma'', \varphi_2 \quad \vdash \Delta'', \varphi_2}{\vdash \Gamma'', \Delta''}}{\vdash \Gamma', \Gamma'', \Delta', \Delta''}}$$

where both upper cuts are of lower degree. By the induction hypothesis, there are two normal derivations ending with $\vdash \Gamma', \Delta'$ and $\vdash \Gamma'', \Delta''$, and, by Lemma 9.2, there is a normal derivation of $\vdash \Gamma', \Gamma'', \Delta', \Delta''$. \square

Corollary 9.4. *For every derivation of a sequent, there is a normal derivation of the same sequent.*

For a multiset Γ of formulae, let $\lambda(\Gamma)$ be the set of elements of W occurring in Γ and let $\kappa(\Gamma)$ be the number of elements (possibly with repetition) of Γ .

Lemma 9.5. *If $\vdash \Delta, \varphi$ is derivable, then $\lambda(\{\varphi\}) \subseteq \lambda(\Delta)$.*

Proof. Note that this property holds for the axiomatic sequents and the only interesting case in an inductive proof of this fact is when a derivation of $\vdash \Delta, \varphi$ ends as

$$\frac{\frac{\vdash \Delta', \psi \quad \vdash \Delta'', \varphi, \psi}{\vdash \Delta', \Delta'', \varphi}}$$

If $A \in \lambda(\{\varphi\})$ and $A \notin \lambda(\Delta'')$, then $A \in \lambda(\{\psi\})$, and by the induction hypothesis applied to the subderivation ending with $\vdash \Delta', \psi$, we have that $A \in \lambda(\Delta')$. \square

Lemma 9.6. *For every sequent $\vdash \Delta$ that occurs in a derivation of $\vdash \Gamma$, we have that $\lambda(\Delta) \subseteq \lambda(\Gamma)$ and $2 \leq \kappa(\Delta) \leq \kappa(\Gamma)$.*

Proof. By Lemma 9.5, every application of cut preserves the letters from the premisses, and it is obvious that the other rules satisfy this property. Also, all the rules are such that $2 \leq \kappa(\Delta) \leq \kappa(\Gamma)$ holds for $\vdash \Delta$ being a premise of the rule and $\vdash \Gamma$ being its conclusion. \square

The following is usually presupposed for a formal system.

Lemma 9.7. *The set of axiomatic sequents is decidable.*

Proof. It is straightforward to check that the properties (0)-(5) of Δ -complexes that define the \mathcal{M} -complexes are decidable. Hence the set of axiomatic sequents coming from \mathcal{M} -complexes is decidable. Also, it is straightforward to check whether a sequent is an instance of the two axiom schemata corresponding to permutation of vertices and switching of triangles. Therefore, the set of all axiomatic sequents is recursive. \square

Let the *atomic* Menelaus system be defined as the original system, save that \mathbb{X} and \leftrightarrow are omitted from the language, and \mathbb{X} -introduction and \leftrightarrow -introduction are omitted from the set of rules. The sequents of this system are called *atomic*.

Lemma 9.8. *The atomic Menelaus system is decidable.*

Proof. Let $\vdash \Gamma$ be an atomic sequent. By Lemma 9.6, if $\vdash \Delta$ occurs in a derivation of $\vdash \Gamma$, then $\lambda(\Delta) \subseteq \lambda(\Gamma)$ and $\kappa(\Delta) \leq \kappa(\Gamma)$. Let S be the set of atomic sequents

$$\{\vdash \Delta \mid \lambda(\Delta) \subseteq \lambda(\Gamma) \text{ and } \kappa(\Delta) \leq \kappa(\Gamma)\}.$$

Note that S is finite. Then the decision procedure may be carried out in the following way (cf. [10, Section IV.1.2]).

Let $S_0 \subseteq S$ be the set of axiomatic sequents in S . (Recall that the set of all axiomatic sequents is decidable by Lemma 9.7.) If $(\vdash \Gamma) \in S_0$, then we are done, and $\vdash \Gamma$ is derivable. If not, then let S_1 contain all the elements of S_0 and all the sequents from S obtained from two S_0 sequents by a single application of cut. If $S_1 = S_0$, then $\vdash \Gamma$ is not derivable, otherwise we proceed in this manner until either $\vdash \Gamma$ appears as a member of some S_i , in which case it is derivable, or the procedure yields no more derivable sequents. In the later case the sequent $\vdash \Gamma$ is not derivable in the atomic Menelaus system. \square

We say that the formula A is a *subformula* of Γ if A is a subformula of some formula in Γ .

Proof of Proposition 9.1. Let $\vdash \Gamma$ be a sequent. By Corollary 9.4, for every derivation of $\vdash \Gamma$ there exists a normal derivation of the same sequent, i.e. a derivation that can be divided into *atomic* and *non-atomic* part. Note that if $\vdash \Delta$ occurs in a non-atomic part of such a derivation, then all the formulae in Δ are subformulae of Γ . Moreover, by Lemma 9.6, we know that $\kappa(\Delta) \leq \kappa(\Gamma)$.

Let S be the following set of sequents

$$\{\vdash \Delta \mid \text{every formula in } \Delta \text{ is a subformula of } \Gamma \text{ and } \kappa(\Delta) \leq \kappa(\Gamma)\}.$$

Note that S is again finite and we proceed as in the proof of Lemma 9.8, save that for S_0 we take not just the axiomatic sequents but all the sequents from S derivable in the atomic Menelaus system. (By Lemma 9.8, this set is decidable.) If $\vdash \Gamma$ is not in S_i , then S_{i+1} contains all the elements of S_i and all the sequents from S obtained from two S_i sequents by a single application of \mathbb{X} -introduction or \leftrightarrow -introduction. \square

10. THE MENELAUS CYCLIC OPERAD

Operads are mathematical objects introduced by Boardman-Vogt [3] and May [16] in the early seventies in algebraic topology with the goal of encoding the algebraic structure of iterated loop spaces. That an operad encodes an algebra means that its data consists of all operations made by composing the structure operations of that algebra, whereas the underlying vector space of the algebra is abstracted away. In this way, operads provide a compact meta-algebraic setting, convenient for a uniform treatment of various algebraic problems. Operads govern algebras whose operations have multiple inputs and one output and, therefore, the underlying combinatorics of operads is given by rooted trees.

Since the nineties, operads have found applications in many other areas of modern mathematics, such as combinatorics, homotopy theory, higher category theory, proof theory and mathematical physics. Also, for the purpose of encoding more general algebraic structures, various generalizations of operads came to existence. One such structure is that of a cyclic operad, introduced by Getzler and Kapranov in [12]. If one accepts the intuitive description of operads as structures that formalize laws of grafting of rooted trees, then cyclic operads can be described as structures that formalize laws of grafting of unrooted trees.

The purpose of this section is to show that \mathcal{M} -complexes admit a natural cyclic operad structure, thereby answering positively the question of whether cyclic operads appear in the context of proof-theory, alongside with ordinary operads.

10.1. Cyclic operads. We start by recalling the definition of cyclic operads. In the definition that follows, \mathbf{Bij} denotes the category of finite sets and bijections. We call a functor of the form $\mathcal{C} : \mathbf{Bij}^{op} \rightarrow \mathbf{Set}$ a *collection* and, for a finite set X , we refer to the elements of $\mathcal{C}(X)$ as *operations*. Moreover, for a collection $\mathcal{C} : \mathbf{Bij}^{op} \rightarrow \mathbf{Set}$, $f \in \mathcal{C}(X)$ and a bijection $\sigma : X' \rightarrow X$, we write f^σ instead of $\mathcal{C}(\sigma)(f)$.

A *cyclic operad* \mathcal{C} is a collection $\mathcal{C} : \mathbf{Bij}^{op} \rightarrow \mathbf{Set}$, together with a family of functions

$$x \circ_y : \mathcal{C}(X) \times \mathcal{C}(Y) \rightarrow \mathcal{C}((X - \{x\}) \cup (Y - \{y\})),$$

called *insertions*, indexed by arbitrary non-empty finite sets X and Y and elements $x \in X$ and $y \in Y$, such that $(X - \{x\}) \cap (Y - \{y\}) = \emptyset$. These data must satisfy the axioms given below.

Associativity. For $f \in \mathcal{C}(X)$, $g \in \mathcal{C}(Y)$ and $h \in \mathcal{C}(Z)$, the following equality holds:

$$(A) (f \circ_x g) \circ_u h = f \circ_x (g \circ_u h), \text{ where } x \in X, y, u \in Y, z \in Z.$$

Commutativity. For $f \in \mathcal{C}(X)$, $g \in \mathcal{C}(Y)$, $x \in X$ and $y \in Y$, the following equality holds:

$$(C) f \circ_x g = g \circ_y f.$$

Equivariance. For bijections $\sigma_1 : X' \rightarrow X$ and $\sigma_2 : Y' \rightarrow Y$, and $f \in \mathcal{C}(X)$ and $g \in \mathcal{C}(Y)$, the following equality holds:

$$(E) f^{\sigma_1} \circ_{\sigma_1^{-1}(x)} g^{\sigma_2} = (f \circ_x g)^\sigma, \text{ where } \sigma = \sigma_1|^{X-\{x\}} \cup \sigma_2|^{Y-\{y\}}.$$

Additionally, cyclic operads may have *units*. We say that a cyclic operad is *unital* if, for each two-element set $\{x, y\}$, there exists a distinguished operation $id_{x,y} \in \mathcal{C}(\{x, y\})^1$, called the *identity* or *unit indexed by $\{x, y\}$* . The units are required to satisfy the following two axioms.

Unitality. For $f \in \mathcal{C}(X)$, $x \in X$ and a bijection σ that renames x to z , the following equality holds:

$$(U1) f \circ_x id_{y,z} = f^\sigma.$$

Moreover, the unit elements are preserved under the action of $\mathcal{C}(\sigma)$, i.e.

$$(U2) id_{x,y}^\sigma = id_{u,v},$$

for any two two-element sets $\{x, y\}$ and $\{u, v\}$, and a bijection $\sigma : \{u, v\} \rightarrow \{x, y\}$.

This completes the definition of (unital) cyclic operads. For an operation $f \in \mathcal{C}(X)$, we say that the elements of X are the *entries* of f .

Given a collection $\underline{\mathcal{C}} : \mathbf{Bij}^{op} \rightarrow \mathbf{Set}$, one can build the *free cyclic operad* $\mathcal{F}(\underline{\mathcal{C}})$ over $\underline{\mathcal{C}}$. The elements of $\mathcal{F}(\underline{\mathcal{C}})(X)$ are the unrooted trees with the external edges (i.e. leaves) indexed by X and with the nodes decorated by the operations of $\underline{\mathcal{C}}$, where the internal edges witness the insertions between the operations decorating the nodes.

¹It is understood that $id_{y,x} = id_{x,y}$.

10.2. The cyclic operad of homogeneous n -dimensional Δ -complexes. The cyclic operad structure of homogeneous n -dimensional Δ -complexes is given by means of (a combinatorial version of) the connected sum operation, which we define next.

Let X be an n -dimensional homogeneous Δ -complex and let $x \in X_n$. The Δ -complex $X \ominus x$ is obtained from X by deleting x from X_n and restricting the faces d_i^n to $X_n - \{x\}$.

The *boundary complex* $\partial\Delta^n$ is the Δ -complex such that for $0 \leq k \leq n-1$, we have

$$(\partial\Delta^n)_k = \{U \subseteq \{0, \dots, n\} \mid |U| = k+1\},$$

and for $U = \{j_0, \dots, j_k\}$, such that $0 \leq j_0 < \dots < j_k \leq n$,

$$d_i^k U = \{j_0, \dots, j_{i-1}, j_{i+1}, \dots, j_k\}.$$

For an n -dimensional homogeneous Δ -complex X and $x \in X_n$, let $f^x: \partial\Delta^n \rightarrow X \ominus x$ be a family of functions

$$\{f_k^x: (\partial\Delta^n)_k \rightarrow (X \ominus x)_k \mid 0 \leq k \leq n-1\}$$

defined so that for $i_1 < \dots < i_{n-k}$,

$$f_k^x(\{0, \dots, n\} - \{i_1, \dots, i_{n-k}\}) = d_{i_1} \dots d_{i_{n-k}} x.$$

It is straightforward to check that f^x is a morphism of Δ -complexes.

Let X and Y be two disjoint n -dimensional homogeneous Δ -complexes and let $x \in X_n$ and $y \in Y_n$. For $f^x: \partial\Delta^n \rightarrow X \ominus x$ and $f^y: \partial\Delta^n \rightarrow Y \ominus y$ defined as above, let K be the n -dimensional Δ -complex defined as

$$K_n = (X_n - \{x\}) \cup (Y_n - \{y\}), \quad K_k = (X_k \cup Y_k)_{\sim_k}, \quad 0 \leq k \leq n-1,$$

where the equivalence relations \sim_k are generated by $f_k^x(U) \sim_k f_k^y(U)$ for every $U \in (\partial\Delta^n)_k$. Moreover, the faces of K are defined so that for $z_{\sim_k} \in K_k$ we have $d_i z_{\sim_k} = (d_i z)_{\sim_k}$.

The complex K , which is the *connected sum* of X and Y with respect to x and y , is the pushout $(X \ominus x) \bigsqcup_{\partial\Delta^n} (Y \ominus y)$ in the category of Δ -complexes. It is homogeneous, and, as we shall see, in the case when X and Y are \mathcal{M} -complexes, K is an \mathcal{M} -complex too.

The isomorphisms in the category of n -dimensional homogeneous Δ -complexes are morphisms all of whose components are bijections. We call such an isomorphism an *$<n$ -isomorphism* if its n -th component is the identity function.

Let $\mathcal{D}: \mathbf{Bij}^{op} \rightarrow \mathbf{Set}$ be the functor that associates to a finite set X_n , the set of $<n$ -isomorphism classes of Δ -complexes that have X_n as set of n -cells. The action of the symmetric group is defined as follows. Let $\sigma: Z_n \rightarrow X_n$ be a bijection and K be an n -dimensional homogeneous Δ -complex such that $K_n = X_n$. We define a representative K' of $[K]^\sigma$ as follows: $K'_n = Z_n$, $K'_i = K_i$ for all $i < n$, and the face maps of K' coincide with those of K except for the maps d_i^n which for K' are defined by $d_i^n u = d_i^n(\sigma(u))$. We equip the functor \mathcal{D} with the family of connected sum functions

$$x \circ_y : \mathcal{D}(X_n) \times \mathcal{D}(Y_n) \rightarrow \mathcal{D}((X_n - \{x\}) \cup (Y_n - \{y\})),$$

where $x \in X_n$ and $y \in Y_n$, defined on $[X] \in \mathcal{D}(X_n)$ and $[Y] \in \mathcal{D}(Y_n)$ as the $<n$ -isomorphism class of the connected sum of X and Y with respect to x and y (note that we can always choose the representatives X and Y in such a way that their collections of cells in all dimensions $<n$ are disjoint).

Observe that the above definition does not depend on the choice of representatives of the classes from $\mathcal{D}(X_n)$ and $\mathcal{D}(Y_n)$, making the functions $x \circ_y$ well-defined. We shall, from now on, write simply $X \in \mathcal{D}(X_n)$.

Lemma 10.1. *The functor \mathcal{D} , equipped with the family of functions $x \circ_y$, is a unital cyclic operad.*

Proof. The unit is the Δ -complex given by 2 n -cells having the same boundaries. The associativity and commutativity laws of cyclic operads are immediate consequences of our ability to choose always disjoint sets of cells of all dimensions, and of the observation that union of disjoint sets is commutative and associative “on the nose”. \square

The Menelaus cyclic operad is the suboperad of the cyclic operad of homogeneous 2-dimensional Δ -complexes determined by the \mathcal{M} -complexes.

We finish this section by proving that the Menelaus cyclic operad is well-defined.

Lemma 10.2. *The \mathcal{M} -complexes are closed under the connected sum operation.*

Proof. The connected sum operation clearly preserves the properties (0)-(4) from the definition of \mathcal{M} -complexes.

In order to prove that the orientability of \mathcal{M} -complexes X and Y implies the orientability of the \mathcal{M} -complex $X_{x \circ_y} Y$, let c_1 (resp. c_2) be the fundamental cycle of X (resp. of Y), and let α (resp. β) be the coefficient of x in c_1 (resp. y in c_2). Then, by the orientability of X and Y , the boundary of the linear combination $(c_1 - \alpha x) \pm (c_2 - \beta y)$, where \pm is the Kronecker symbol of (α, β) , is zero. This proves that $H_2(X_{x \circ_y} Y; \mathbf{Z}) = \mathbf{Z}$, and, in particular, determines an orientation of $X_{x \circ_y} Y$. \square

10.3. Irreducible \mathcal{M} -complexes. With the goal of introducing a presentation of the Menelaus cyclic operad by generators and relations in Section 10.4, in this section we characterize the class of \mathcal{M} -complexes that generate the Menelaus cyclic operad: we call them irreducible \mathcal{M} -complexes.

Let K be an \mathcal{M} -complex and let $T = \{e_0, e_1, e_2\} \subseteq K_1$ be such that $\partial_1(e_0 - e_1 + e_2) = 0$, i.e. $e_0 - e_1 + e_2$ is a 1-cycle. Consider the binary relation on K_2 of sharing an edge from $K_1 - T$. Let τ be the transitive closure of this relation. We say that T is a *cut-triangle*, when τ is an equivalence relation with exactly two classes. If K contains a cut-triangle, then we say that it is *reducible*, otherwise it is *irreducible*. Note that if T is the set of edges of a 2-cell of K , then τ is not reflexive.

Let K be a reducible \mathcal{M} -complex and let $T = \{e_0, e_1, e_2\} \subseteq K_1$ be a cut triangle of K . We shall write K_T^l and K_T^r for the two equivalence classes of 2-cells with respect to the relation of sharing an edge from $K_1 - T$. We shall denote this relation with \sim_T .

We set up a bit of terminology. A *path* between two 2-cells u, v is a sequence of 2-cells starting from u and ending with v such that any two consecutive 2-cells share a 1-cell, and we say that the path *crosses* these 1-cells.

A key tool in this section is the following Lemma, which makes explicit the argument used in the proof of Proposition 2.9 (stating that the realisation of a \mathcal{M} -complex is a manifold).

Lemma 10.3 (Camembert lemma). *The axiom (4) in the definition of \mathcal{M} -complex can be reinforced as follows. For every 0-cell w , the faces of L_w can be displayed without repetition around w in a circle. Formally, we can arrange a cyclic order on $L_w = \{u_1, \dots, u_n\}$ in such a way that, for all i , u_i and u_{i+1} are w -neighbours, modulo n . We refer to L_w equipped with this cyclic order as the camembert of the 0-cell w .*

Proof. We first exhibit a circle like in the statement, except that we do not prove yet that all elements of L_w occur in it. By axiom (1), we know that L_w is not empty.

Let us pick $u_1 \in L_w$. We define a u_1 -sequence to be a path v_1, \dots, v_m of elements of L_w such that $v_1 = u_1$, v_i and v_{i+1} are w -neighbours, for all $1 \leq i < m$, and the path v_{i-1}, v_i, v_{i+1} crosses two different edges of v_i , for all $2 \leq i \leq m-1$. By axiom (3), we can construct an infinite u_1 -sequence v_1, \dots, v_m, \dots , in the following way: choose v_2 to be a w -neighbour of u_1 in L_w (notice that v_2 might be the unique such 2-cell, in which case $L_w = \{v_1, v_2\}$, or there might be a choice of exactly two such 2-cells), and then at each v_i continue with the w -neighbour of v_i along the edge of v_i incident to w that is not crossed from v_{i-1} to v_i . If L_w contains more than two 2-cells, then, by axiom (0), there must exist i and j such that $i < j$, $v_i = v_j$, and all 2-cells $v_i, v_{i+1}, \dots, v_{j-1}$ are distinct. We next show the following property (P): every two v_i -sequences that share their first two elements have to be prefixes of one another. Indeed, if the two sequences diverge at some 2-cell v , this would display a 1-cell shared by at least three 2-cells of K , contradicting axiom (3). We now show that all elements of L_w appear among $v_i, v_{i+1}, \dots, v_{j-1}$. Let $u \in L_w$. By axiom (4), there exists a v_i -sequence leading to u . By property (P) this sequence must be a prefix of a sufficiently long prefix of $v_i, v_{i+1}, \dots, v_{j-1}, v_j, v_{i+1}, \dots, v_{j-1}, v_j, \dots$ or of $v_i, v_{j-1}, \dots, v_{i+1}, v_j, v_{j-1}, \dots, v_{j+1}, v_i, \dots$. Therefore $u = v_k$, for some $i \leq k \leq j-1$. \square

Remark 10.4. The Camembert lemma provides us with the following reasoning concerning the paths between 2-cells: if the path crosses at least 2 edges of a single cut triangle T , using the Camembert lemma, we shall be able to transform that path into one that crosses strictly less edges of T . We shall tacitly use an induction on this number of crossings.

The following lemma provides a method of showing that three 1-cells of an \mathcal{M} -complex do *not* make a cut-triangle.

Lemma 10.5. *If $T = \{e_0, e_1, e_2\}$ is a cut-triangle of K , then for each e_i , $0 \leq i \leq 2$, each of the classes K_T^l and K_T^r contains exactly one 2-cell having e_i as a 1-cell.*

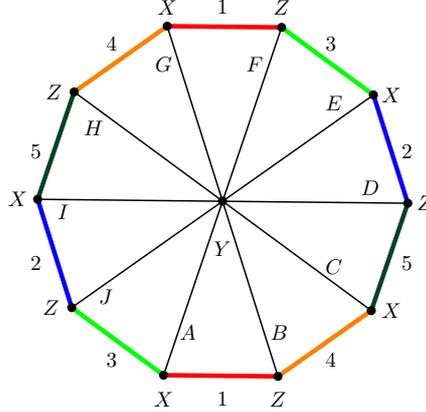
Proof. Fix an $i \in \{0, 1, 2\}$. By the property (3) for K , there exist exactly two 2-cells of K having e_i as an edge; denote them with u^l and u^r . We show that, if u^l and u^r are both contained in K_T^l , then all the 2-cells of K must also belong to K_T^l , thereby contradicting the fact that T is a cut triangle of K .

Let u and v be two arbitrary distinct 2-cells of K . As a consequence of the connectedness property and axiom (4) of K , there exists a path of 2-cells starting from u and ending in v . If the 1-cells crossed on this path all belong to $K_1 \setminus T$, then u and v are equivalent by definition. If the path crosses e_i , then, by the assumption that u^l and u^r are equivalent, this crossing can be replaced by a path between u^l and u^r that does not cross e_i . If the path crosses e_j for $e_j \in T \setminus \{e_i\}$, then let u_1 and u_2 be the faces sharing e_j on the path. By the Camembert lemma, we can display all of u^l, u^r, u_1, u_2 in a circle of 2-cells associated with the link of the 0-cell w common to e_i and e_j , in such a way that u_1, u_2 (resp. u^l, u^r) are neighbours. Then following the cells in clockwise or anticlockwise way, we see that there is a path from u_1 to u^l (or u^r) and a path from u_2 to u^r (or u^l) that do not cross e_i nor e_j (nor the third 1-edge of T , which is not incident to w). Therefore, again, we can replace the length 1 path from u_1 to u_2 by the concatenation of three paths from, say, u_1 to u^l to u^r to u_2 , witnessing that u_1 and u_2 are equivalent. This concludes the proof by contradiction. \square

EXAMPLE 5. The unique cut-triangle in the torus with two holes of Example 3, p. 15. is the triangle with edges D , 1 and I . Indeed, the two equivalence classes of 2-cells with respect to the relation of sharing an edge other than D , 1 and I are given by

- (1) $B1A, B2C, D3C, J2I, J3A$, and
- (2) $D4E, F1G, F5E, HAG, H5I$.

EXAMPLE 6. Consider the triangulation of the torus with two holes into ten 2-cells and three zero-cells in total, obtained by identifying the opposite sides of a decagon:



In order to see that this triangulation is irreducible, notice first that the symmetric nature of the triangulation allows us to look for a cut triangle by requiring that it contains a fixed side of the decagon, say 1, without loss of generality. Let us first examine the potential cut-triangles that contain edges 1 and A :

$$1AD, 1AF, 1AH, \text{ and } 1AJ.$$

(Notice that, since $1AB$ is a 2-cell of the triangulation, it is not a candidate for a cut-triangle.) By Lemma 10.5, in order for $1AD$ to be a cut triangle, the 2-cells $1AB$ and $1GF$ must belong to different equivalence classes induced by $1AD$. However, the sequence of 2-cells $1AB, 4BC, 4GH, 1GF$, in which each two successive members share an edge outside of $1AD$, witnesses that this is not possible. By the same argument, the sequence $1AB, 4BC, 5CD, 2DE, 3EF, 3AJ$ shows that $1AF$ cannot be a cut-triangle, the sequence $4HG, 1GF, 3FE, 2ED, 5DC, 5IH$ shows that $1AH$ cannot be a cut-triangle, and the sequence $3AJ, 3EF, 1FG, 4GH, 5IH, 2IJ$ shows that $1AJ$ cannot be a cut-triangle. By symmetry again, we can conclude that a cut-triangle cannot contain edges 1 and B . The remaining candidates for a cut triangle are $1ID, 1EJ$ and $1EH$, whereby, by the symmetry of the triangulation again, we can reduce the analysis to $1ID$ and $1EJ$. By Lemma 10.5, in order for $1ID$ to be a cut triangle, the 2-cells $1AB$ and $1GF$ must belong to different equivalence classes induced by $1ID$. However, the sequence of 2-cells $1AB, 3AJ, 3FE, 1GF$, in which each two successive members share an edge outside of $1ID$, witnesses that this is not possible. Similarly, the sequence of 2-cells $1AB, 4BC, 5CD, 5HI, 4HG, 1GF$ witnesses that $1EJ$ cannot be a cut triangle.

We next prove additional properties of cut-triangles that will ultimately lead to the proof that the Menelaus cyclic operad is generated by the irreducible \mathcal{M} -complexes.

Lemma 10.6. *If T is a cut triangle for K , then the two equivalence classes K_T^l and K_T^r of 2-cells do not share lower dimensional faces other than those of T .*

Proof. We just have to check that if u, v are two 2-cells of K sharing a 1-cell in $K_1 \setminus T$ or a cell in K_0 which is not a face of any element of T , then they are equivalent. This is obvious for 1-cells, by the very definition of the equivalence. Suppose now that u, v share a 0-cell w outside of T . These cells belong to L_w ,

hence by condition (4) applied to K and w , there is a path from u to v in L_w . Such a path does not cross any of the e_i 's, since none of these 1-cells is incident to w . Hence u and v are equivalent. \square

Proposition 10.7. *An \mathcal{M} -complex K is reducible if and only if it can be obtained as a connected sum of two simpler (with respect to the cardinality of the set of 2-cells) \mathcal{M} -complexes.*

Proof. Suppose that K is reducible and let $T = \{e_0, e_1, e_2\} \subseteq K_1$ be a cut triangle of K . Define, for $o \in \{l, r\}$ and $0 \leq j \leq 2$, the sets $(\hat{K}_T^o)_j$ as follows:

$$\begin{aligned} (\hat{K}_T^o)_2 &= K_T^o \cup \{t\}, & (\hat{K}_T^o)_1 &= \bigcup_{x \in (\hat{K}_T^o)_2 - \{t\}} \{d_0^2 x, d_1^2 x, d_2^2 x\} \quad \text{and} \\ (\hat{K}_T^o)_0 &= \bigcup_{y \in (K_T^o)_1} \{d_0^1 y, d_1^1 y\}, \end{aligned}$$

for $t \notin K_T^o$. Therefore, for $j < 2$, $(\hat{K}_T^o)_j$ is the set of all j -cells contained in the equivalence class K_T^o , whereby, for $j = 2$, $(\hat{K}_T^o)_2$ additionally contains t as a 2-cell. Define, moreover, for $1 \leq i \leq 2$, $1 \leq j \leq 2$ and $0 \leq k \leq j$, functions $(d_T^o)_k^j : (\hat{K}_T^o)_j \rightarrow (\hat{K}_T^o)_{j-1}$ as the appropriate restrictions of the corresponding faces $d_k^j : K_j \rightarrow K_{j-1}$ of the starting \mathcal{M} -complex K , except on the triangle t , for which the action of $(d_T^o)_k^j$ is defined by $(d_T^o)_k^j(t) = e_k$. We prove that, by taking the elements of $(\hat{K}_T^o)_j$ as j -cells and functions $(d_T^o)_k^j$ as the appropriate faces, the equivalence classes K_T^l and K_T^r get turned into \mathcal{M} -complexes \hat{K}_T^l and \hat{K}_T^r , respectively. Note that, by the construction, \hat{K}_T^l and \hat{K}_T^r satisfy the first two properties of an \mathcal{M} -complex. We verify below the remaining four for \hat{K}_T^l .

- (2) The property (2) follows by the fact that, in the cut-triangle T , all 0-cells and 1-cells are mutually different. Indeed, by the regularity of K , the two 0-cells of each e_i , $i \in \{0, 1, 2\}$, are mutually different, and, therefore, the three 0-cells of the 2-cell t must be mutually different. Consequently, the 1-cells of t must also be mutually different.
- (3) Note that, for all the 1-cells e of $(\hat{K}_T^l)_1 - \{e_0, e_1, e_2\}$, this property holds by the same property of K . (Observe that Lemma 10.6 disallows to apply the same argument twice on e , which would produce four 2-cells adjacent with e .) As for the 1-cells $\{e_0, e_1, e_2\}$, each one of them belongs to t and, by Lemma 10.5, to exactly one 2-cell of \hat{K}_T^l , which proves the claim.
- (4) Note that, for all the 0-cells of $(\hat{K}_T^l)_0 - \{d_0^1 e_0, d_1^1 e_0, d_1^1 e_1\}$, this property holds by the same property of K (and again Lemma 10.6). Suppose therefore that $w \in \{d_0^1 e_0, d_1^1 e_0, d_1^1 e_1\}$, say $w = d_0^1 e_0$, and let $u, v \in L_w$. Suppose, moreover, that both u and v are different than t . By the Camembert lemma for K , there exists a path u, u_1, \dots, u_k, v of 2-cells of K starting at u and ending at v , such that every two consecutive 2-cells are w -neighbours. Axiom (4) follows immediately if this path is entirely contained in \hat{K}_T^l . If the path contains 2-cells of \hat{K}_T^r , and, hence, crosses the two edges of T adjacent with w , then, by the Camembert lemma, we can transform it into a path that does not cross an edge of T , by the same reasoning as in the proof of Lemma 10.5. Suppose now that $u = t$. By Lemma 10.5, for the 1-cell e_0 of t , we can pick the 2-cell u' of \hat{K}_T^l that has e_0 as a 1-cell; note that $u' \in L_w$. By the Camembert lemma for K , there exists a sequence u, u_1, \dots, u_k, u' of 2-cells starting at u and ending at u' , such that every two consecutive 2-cells are w -neighbours, and which is, moreover, entirely contained in \hat{K}_T^l .

- The sequence of 2-cells starting at u and ending at t , such that every two consecutive 2-cells are w -neighbours, is then the sequence $u, u_1, \dots, u_k, u', t$.
- (5) Let $c = \sum_{i=1}^n \varepsilon_i u_i$ be the fundamental cycle of K , and let $c_l = \sum_{i \in I} \varepsilon_i u_i$ and $c_r = \sum_{j \in J} \varepsilon_j u_j$ be such that $c = c_l + c_r$, $K_T^l = \{u_i \mid i \in I\}$ and $K_T^r = \{u_j \mid j \in J\}$. Consider the boundary ∂c_l . By the property (3) for \hat{K}_T^l , each 1-cell e that appears in $\partial c_l - \{e_0, e_1, e_2\}$ will have exactly one more occurrence in this boundary. Therefore, by the orientability of K , we can conclude that $\partial c_l = \sum_{m=0}^2 \tau_m e_m$, for $\tau_m \in \{+1, -1\}$. Moreover, since $\partial^2 = 0$, we know that $\sum_{m=0}^2 \tau_m e_m$ is a 1-cycle. But since $e_0 - e_1 + e_2$ is also a 1-cycle, it must be the case that either

$$\partial c_l = e_0 - e_1 + e_2 \quad \text{or} \quad \partial c_l = -(e_0 - e_1 + e_2).$$

In the first case, the orientation of \hat{K}_T^l is obtained by taking for coefficient ε_l of T to be -1 and, in the second case, by taking it to be $+1$.

Note that, by Lemma 10.6, \hat{K}_T^l and \hat{K}_T^r have no common faces other than the ones of t . Also, both \hat{K}_T^l and \hat{K}_T^r have strictly less 2-cells than K . Indeed, if, say, \hat{K}_T^l would contain the same number of 2-cells as K , then \hat{K}_T^r would have to be the unit \mathcal{M} -complex (see the proof of Lemma 10.1), meaning that \sim_T would not be an equivalence relation. It follows by the definition of the connected sum operation that the \mathcal{M} -complex K is indeed a connected sum of \mathcal{M} -complexes \hat{K}_T^l and \hat{K}_T^r along the 2-cell t .

For the other direction, suppose that $M = K_x \circ_y L$, with K, L disjoint and different from the unit \mathcal{M} -complexes. We show that the triangle $T = \{e_0, e_1, e_2\}$ resulting from the identification of the 1-cells of x and y is a cut triangle. We show this in three stages:

- All 2-cells in $K_2 \setminus \{x\}$ are equivalent. Let $u, v \in K_2 \setminus \{x\}$, and consider a path of 2-cells between them in K . If this path does not cross any 1-cell from T , then u and v are equivalent by definition. If the path does cross $e_i \in T$, then it contains x and the other 2-cell u'_i of K sharing e_i with x . After x , the path has to cross another 1-cell e_j , and contains also the other 2-cell u'_j of K_2 sharing e_j with x . By the Camembert lemma applied to the common 0-cell of e_i and e_j , we can replace the portion u'_i, x, u'_j of the path by a path from u'_i to u'_j that does not go through x , and hence does not cross T . Therefore u and v are equivalent.
- All 2-cells in $L_2 \setminus \{y\}$ are equivalent. This is entirely symmetric to what we have just shown.
- A 2-cell in $K_2 \setminus \{x\}$ cannot be equivalent to a 2-cell in $L_2 \setminus \{y\}$. Suppose that $u \in K_2 \setminus \{x\}$ and $v \in L_2 \setminus \{y\}$ and consider an arbitrary path of 2-cells from u to v . Any such path must have the form $u, \dots, u', v', \dots, v$, where $u' \in K_2 \setminus \{x\}$ and $v' \in L_2 \setminus \{y\}$. But the only possibility for u', v' to share a 1-cell is that u' shares a 1-cell with x in K and v' shares a 1-cell with y in L , these 1-cells being identified in T . In other words, the path crosses T . Since the starting path from u to v was arbitrary, we conclude that u and v belong to different equivalence classes.

It follows obviously from these three properties that the equivalence relation induced by T has exactly two equivalence classes. \square

Remark 10.8. Observe that the two constructions from Proposition 10.7 are reciprocal to each other, in the sense that the action of “composing” the constructions results (up to isomorphism) in the original cut-triangle (resp. decomposition of an \mathcal{M} -complex as a connected sum).

10.4. A presentation of the Menelaus cyclic operad. Proposition 10.7 gives us a way to represent each \mathcal{M} -complex as an element of the free cyclic operad over the irreducible \mathcal{M} -complexes. In other words, the Menelaus cyclic operad is generated by the irreducible \mathcal{M} -complexes. However, as we show in the following example, the Menelaus cyclic operad is not free over this collection of generators.

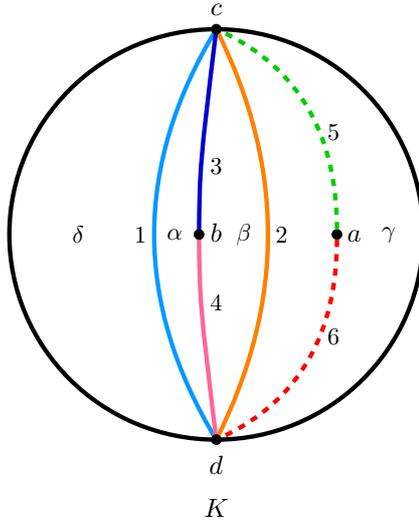
EXAMPLE 7. Here is an example of an \mathcal{M} -complex M which is <2 -isomorphic to two different connected sums of two irreducible \mathcal{M} -complexes. We define the irreducible \mathcal{M} -complexes K , U , L and V as follows:

- the \mathcal{M} -complex K is defined by

$$K_2 = \{\alpha, \beta, \gamma, \delta\} \quad \begin{cases} d_0^2 \alpha = 1 & d_1^2 \alpha = 4 & d_2^2 \alpha = 3 \\ d_0^2 \beta = 2 & d_1^2 \beta = 4 & d_2^2 \beta = 3 \\ d_0^2 \gamma = 2 & d_1^2 \gamma = 6 & d_2^2 \gamma = 5 \\ d_0^2 \delta = 1 & d_1^2 \delta = 6 & d_2^2 \delta = 5 \end{cases}$$

$$K_1 = \{1, 2, 3, 4, 5, 6\} \quad \begin{cases} d_0^1 1 = c & d_1^1 1 = d & d_0^1 2 = c & d_1^1 2 = d \\ d_0^1 3 = b & d_1^1 3 = c & d_0^1 4 = b & d_1^1 4 = d \\ d_0^1 5 = a & d_1^1 5 = c & d_0^1 6 = a & d_1^1 6 = d \end{cases}$$

$$K_0 = \{a, b, c, d\}$$

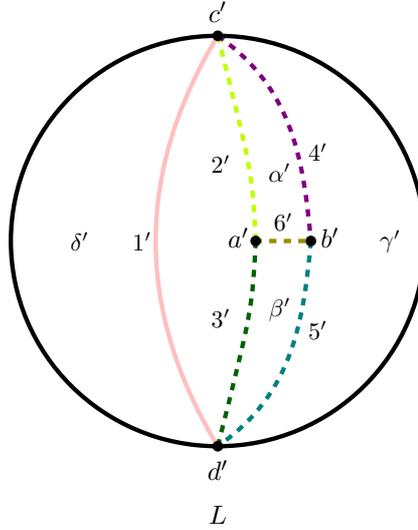


- the \mathcal{M} -complex U is defined by $U = K^\sigma$, where σ renames the 2-cells α , β , γ and δ of K into β , α , γ' and φ , respectively,
- the \mathcal{M} -complex L is defined by

$$L_2 = \{\alpha', \beta', \gamma', \delta'\} \quad \begin{cases} d_0^2 \alpha' = 4' & d_1^2 \alpha' = 2' & d_2^2 \alpha' = 6' \\ d_0^2 \beta' = 5' & d_1^2 \beta' = 3' & d_2^2 \beta' = 6' \\ d_0^2 \gamma' = 1' & d_1^2 \gamma' = 5' & d_2^2 \gamma' = 4' \\ d_0^2 \delta' = 1' & d_1^2 \delta' = 3' & d_2^2 \delta' = 2' \end{cases}$$

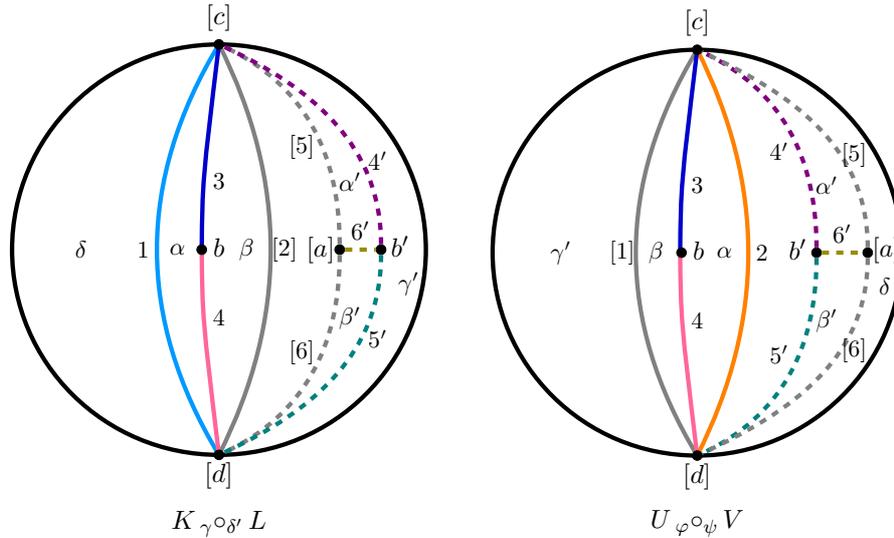
$$L_1 = \{1', 2', 3', 4', 5', 6'\} \quad \begin{cases} d_0^1 1' = c' & d_1^1 1' = d' & d_0^1 2' = a' & d_1^1 2' = c' \\ d_0^1 3' = a' & d_1^1 3' = d' & d_0^1 4' = b' & d_1^1 4' = c' \\ d_0^1 5' = b' & d_1^1 5' = d' & d_0^1 6' = a' & d_1^1 6' = b' \end{cases}$$

$$L_0 = \{a', b', c', d'\}$$



- the \mathcal{M} -complex V is defined by $V = L^\tau$, where τ renames the 2-cells α' , β' , γ' and δ' of L into α' , β' , δ and ψ , respectively.

The compositions $K_{\gamma \circ \delta'} L$ and $U_{\varphi \circ \psi} V$ are then the same operations of the Menelaus cyclic operad:



Indeed, taking into account the identifications

$$2 \sim 1' \quad 6 \sim 3' \quad 5 \sim 2' \quad a \sim a' \quad c \sim c' \quad d \sim d',$$

induced by the connected sum $K_{\gamma \circ \delta'} L$, and the identifications

$$1 \sim 1' \quad 6 \sim 3' \quad 5 \sim 2' \quad a \sim a' \quad c \sim c' \quad d \sim d',$$

induced by the connected sum $U_{\varphi \circ \psi} V$, the witnessing $\langle 2$ -isomorphism between $K_{\gamma \circ \delta'} L$ and $U_{\varphi \circ \psi} V$ is defined as follows:

$$\begin{aligned} 1 &\mapsto 2 & [2] &\mapsto [1] & 3 &\mapsto 3 & 4 &\mapsto 4 & [5] &\mapsto 4' & [6] &\mapsto 5' & 4' &\mapsto [5] & 5' &\mapsto [6] & 6' &\mapsto 6' \\ [a] &\mapsto b' & b &\mapsto b & [c] &\mapsto [c] & [d] &\mapsto [d] & b' &\mapsto [a] \end{aligned}$$

It follows that $K_{\gamma \circ \delta'} L = U_{\varphi \circ \psi} L$ is a relation, preventing the Menelaus cyclic operad to be free.

We shall now examine which of the \mathcal{M} -complexes induce and which do not induce relations.

Definition 1. Let K be an \mathcal{M} -complex and let $T_1 = \{e_0^1, e_1^1, e_2^1\} \subseteq K_1$ and $T_2 = \{e_0^2, e_1^2, e_2^2\} \subseteq K_1$ be two cut-triangles of K . We say that T_1 is *disjoint* from T_2 if all the edges of T_1 are 1-cells of one of the two \mathcal{M} -complexes induced by T_2 .

Lemma 10.9. *If K is an \mathcal{M} -complex and if $T_1 = \{e_0^1, e_1^1, e_2^1\} \subseteq K_1$ and $T_2 = \{e_0^2, e_1^2, e_2^2\} \subseteq K_1$ are two cut-triangles of K , then T_1 is disjoint from T_2 if and only if one of the equivalence classes of 2-cells induced by T_2 is entirely contained in one of the equivalence classes of 2-cells induced by T_1 .*

Proof. Suppose that T_1 is disjoint from T_2 , i.e., that, without loss of generality, all the edges of T_1 belong to $\hat{K}_{T_2}^l$. We show that any two 2-cells u and v of $\hat{K}_{T_2}^r$ belong to the same class with respect to T_1 . Pick a path p from u to v in $\hat{K}_{T_2}^r$ and suppose that it crosses T_1 . Observe that this is only possible if T_1 and T_2 share one edge, say e_1 . Moreover, the path contains the 2-cells w_1 (from $\hat{K}_{T_2}^r$) and w_2 (from $\hat{K}_{T_2}^l$) adjacent with e_1 . Since e_1 is an edge of T_2 , and u and v are both in $\hat{K}_{T_2}^r$, the path must cross again T_2 before reaching v , say, at the edge e_2 . Let w_1' and w_2' be the 2-cells from $\hat{K}_{T_2}^l$ and $\hat{K}_{T_2}^r$, respectively, adjacent with e_2 . Then, thanks to the Camembert lemma, the part of the path p from w_1 to w_1' can be replaced by another path that lies completely in $\hat{K}_{T_2}^r$. We conclude by induction.

For the opposite direction, suppose, without loss of generality, that $\hat{K}_{T_2}^l \subsetneq \hat{K}_{T_1}^l$. Suppose, again without loss of generality, that e_0^1 is a 1-cell of $\hat{K}_{T_2}^l$ and that e_1^1 is a 1-cell of $\hat{K}_{T_1}^r$. Observe that this is only possible if e_0^1 shares its two vertices, say A and B , with one of the edges of T_2 , say e_0^2 . Suppose that e_1^2 is the other edge of T_2 adjacent with A . Let u and v be the 2-cells of $\hat{K}_{T_2}^l$ adjacent with e_0^2 and e_1^2 , respectively. Since $\hat{K}_{T_2}^l \subsetneq \hat{K}_{T_1}^l$, the path on the camembert of A between u and v that lies in $\hat{K}_{T_2}^l$ does not cross T_1 . This means that e_0^1 is on the opposite path between u and v on the camembert of A , and, hence in $\hat{K}_{T_2}^r$. Contradiction. \square

The following lemma, which is a direct consequence of Lemma 10.9, shows that being disjoint is a symmetric relation on cut-triangles of an \mathcal{M} -complex.

Lemma 10.10. *If $T_1 = \{e_0^1, e_1^1, e_2^1\} \subseteq K_1$ and $T_2 = \{e_0^2, e_1^2, e_2^2\} \subseteq K_1$ are cut-triangles of an \mathcal{M} -complex K , and if T_1 is disjoint from T_2 , then T_2 is disjoint from T_1 .*

Lemma 10.11. *For two disjoint cut-triangles $T_1 = \{e_0^1, e_1^1, e_2^1\} \subseteq K_1$ and $T_2 = \{e_0^2, e_1^2, e_2^2\} \subseteq K_1$ of an \mathcal{M} -complex K , the following two claims hold.*

- (1) *If $\hat{K}_{T_1}^l \subsetneq \hat{K}_{T_2}^r$, then T_1 is a cut-triangle of the \mathcal{M} -complex $\hat{K}_{T_2}^r$.*
- (2) *There exist \mathcal{M} -complexes K_1 , K_2 and K_3 such that*

$$K = (K_1 \circ_{t_1} K_2) \circ_{t_2} K_3 = K_1 \circ_{t_1} (K_2 \circ_{t_2} K_3),$$

where t_1 and t_2 are the 2-cells induced by T_1 and T_2 as in the proof of Proposition 10.7.

Proof. (1) We prove that the two equivalence classes of triangles with respect to the relation of sharing an edge from $(\hat{K}_{T_2}^r)_1 - T_1$ are given by $\hat{K}_{T_1}^l$ and $\hat{K}_{T_1}^r \cap \hat{K}_{T_2}^r$. Note that, by the fact that T_1 is a cut triangle of K , we already know that these two sets of 2-cells make *at least* 2 equivalence classes. Therefore, it remains to show that all the 2-cells from $\hat{K}_{T_1}^r \cap \hat{K}_{T_2}^r$ are equivalent, i.e. that for $u, v \in \hat{K}_{T_1}^r \cap \hat{K}_{T_2}^r$ we can find a path from u to v that does not leave $\hat{K}_{T_1}^r \cap \hat{K}_{T_2}^r$. Note that, since $\hat{K}_{T_1}^r \cap \hat{K}_{T_2}^r \subseteq \hat{K}_{T_1}^r$, and $\hat{K}_{T_1}^r$ is a well-defined equivalence class with respect to

the cut-triangle T_1 , there exists a path u, u_1, \dots, u_k, v from u to v in $K_{T_1}^r$, which, however, might leave $K_{T_2}^r$. But if the 2-cells u_i and u_{i+1} are such that $u_i \in K_{T_2}^r$ and $u_{i+1} \in K_{T_1}^r - K_{T_2}^r$, by the Camembert lemma, we can replace u_{i+1} by the 2-cell $u'_{i+1} \in K_{T_2}^r$ adjacent to u_i , and we conclude by the induction hypothesis.

(2) This is a consequence of the claim (1). Indeed, supposing that $K_{T_1}^l \subsetneq K_{T_2}^r$ and $K_{T_2}^l \subsetneq K_{T_1}^r$, by (1) we know that T_1 (resp. T_2) is a cut triangle in $\hat{K}_{T_2}^r$ (resp. $\hat{K}_{T_1}^r$), and we define $K_1 = \hat{K}_{T_1}^l$, $K_3 = \hat{K}_{T_2}^l$ and K_2 as the \mathcal{M} -complex that, together with $\hat{K}_{T_1}^l$, makes the decomposition of the \mathcal{M} -complex $\hat{K}_{T_2}^r$ induced by T_1 . \square

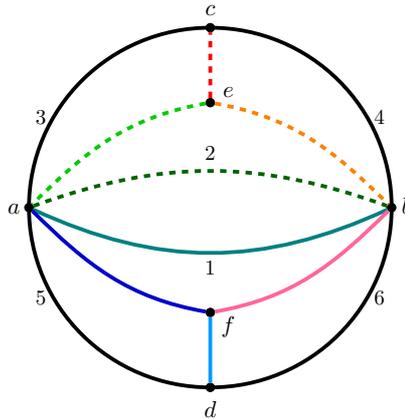
Definition 2. Two cut-triangles $T_1 = \{e_0^1, e_1^1, e_2^1\} \subseteq K_1$ and $T_2 = \{e_0^2, e_1^2, e_2^2\} \subseteq K_1$ of an \mathcal{M} -complex K are *imbricated* if they are not disjoint.

EXAMPLE 8. We give some examples of disjoint cut-triangles.

- (1) Let T_2 be a cut-triangle. If T_1 is a cut-triangle distinct from T_2 such that whenever two vertices A, B of T_1 are vertices of T_2 , then also the edge of T_1 connecting A, B is an edge of T_2 , then T_1 is disjoint from T_2 .

Indeed, suppose first that T_1 does not share two vertices with T_2 (and hence vacuously satisfies the above implication), and pick two vertices A, B of T_1 that are not vertices of T_2 . It then follows that none of the edges incident to A (resp. to B) are edges of T_2 . By using axiom (4) of \mathcal{M} -complexes, we conclude that all the faces of L_A (resp. L_B) are in the same component with respect to \sim_{T_2} . We note that $L_A \cap L_B$ is not empty, since it contains the two faces sharing the edge of T_1 between A and B . It follows that $L_A \cup L_B$, and hence the whole triangle T_1 , is contained in the same component of \sim_{T_2} . Suppose now that T_1 satisfies the above implication and does share the vertices A and B with T_2 , and that there is a 1-cell e whose faces are A, B lying in the two triangles. Then the third vertex C of T_1 cannot lie in T_2 as otherwise, by the implication applied twice, T_1 would coincide with T_2 . Then, applying again axiom (4), we obtain that all edges incident to C lie in the same component of \sim_{T_2} , and hence so does the whole of T_1 .

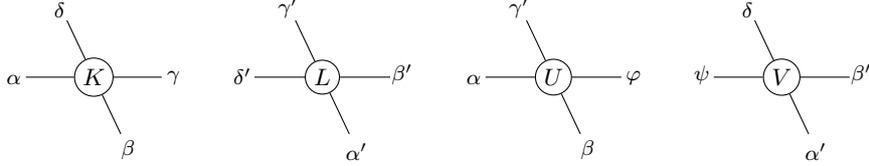
- (2) The following \mathcal{M} -complex contains disjoint cut-triangles, $T_1 = \{2, 3, 4\}$ and $T_2 = \{1, 5, 6\}$, that share two vertices (and no edge):



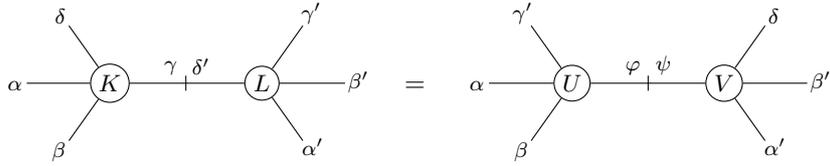
This is to be contrasted with Example 7, where we also have an \mathcal{M} -complex that contains two cut-triangles sharing two vertices and no edge, which are, however, not disjoint. In other words, some pairs of cut-triangles sharing two vertices and not an edge between them may be disjoint, and some may not be disjoint.

As a warm-up for the proof of our main theorem, we translate Example 7 to the formalism of unrooted trees (recall that, as a consequence of Proposition 10.7, each \mathcal{M} -complex can be represented as an element of the free cyclic operad generated by the irreducible \mathcal{M} -complexes).

EXAMPLE 9. In the free cyclic operad generated by the irreducible \mathcal{M} -complexes, the irreducible \mathcal{M} -complexes K , L , U and V from Example 7 are represented by the single-node unrooted trees



respectively. The tree-wise representation of the compositions $K \xrightarrow{\gamma \circ \delta'} L$ and $U \xrightarrow{\varphi \circ \psi} V$ is then obtained by grafting the corresponding single-node trees along the leaves indicated in the two insertions, and their identification is reflected by imposing the equality



of the resulting unrooted trees.

Theorem 10.12. *The Menelaus cyclic operad is the quotient of the free cyclic operad generated by the irreducible \mathcal{M} -complexes, under the equivalence relation generated by the equalities of the form*

$$\mathcal{T}_1 \circ_u \mathcal{T}_2 = \mathcal{T}'_1 \circ_{u'} \mathcal{T}'_2,$$

for each quadruple $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}'_1, \mathcal{T}'_2)$ of unrooted trees and a quadruple (u, v, u', v') of leaves, such that both hand sides are well formed and evaluate, up to < 2 -isomorphism, to the same \mathcal{M} -complex K , in which the cut-triangles T and T' , associated with the pairs (u, v) and (u', v') by Proposition 10.7, are imbricated.

Proof. We define an isomorphism α from the Menelaus cyclic operad to the cyclic operad $\mathcal{F}(I)/(R)$, where I denotes the collection of the isomorphism classes of the irreducible \mathcal{M} -complexes and R the set of relations from the statement of the theorem, by induction on the cardinality of the set of 2-cells of a given operation K of the Menelaus cyclic operad.

If K is irreducible, then $\alpha(K)$ is the unrooted tree with a single node decorated with K that has $|K_2|$ leaves, each labeled with a 2-cell of K . If K is reducible and if $T = \{e_0, e_1, e_2\}$ is a cut-triangle of K , we define $\alpha(K)$ to be the unrooted tree $\alpha(\hat{K}_T^l) \circ_t \alpha(\hat{K}_T^r)$, obtained by grafting the unrooted trees $\alpha(\hat{K}_T^l)$ and $\alpha(\hat{K}_T^r)$ along the corresponding leaves t , where t is the 2-cell of \hat{K}_T^l and \hat{K}_T^r induced from the cut-triangle T as in the proof of Proposition 10.7.

In order to show that α is well-defined, suppose that T_1 and T_2 are two different cut-triangles of K . We distinguish two cases.

- If T_1 and T_2 are imbricated, then the equality

$$\alpha(K_{T_1}^l) \circ_{t_1} \alpha(K_{T_1}^r) = \alpha(K_{T_2}^l) \circ_{t_2} \alpha(K_{T_2}^r)$$

belongs to R by definition.

- Suppose that T_1 and T_2 are disjoint. By Lemma 10.11(1), we can assume, without loss of generality, that $K_{T_1}^l \subsetneq K_{T_2}^r$ and $K_{T_2}^l \subsetneq K_{T_1}^r$. Let, moreover, K_1 , K_2 and K_3 be the \mathcal{M} -complexes induced by $K_{T_1}^l$, $K_{T_1}^r \cap K_{T_2}^r$ and $K_{T_2}^l$, respectively, as in the proof of Lemma 10.11(2). Note that, by the induction hypothesis, α is well-defined on all three \mathcal{M} -complexes K_1 , K_2 and K_3 . Then, by Lemma 10.11(3) and the associativity axiom of the operad of unrooted trees, we have

$$\begin{aligned} \alpha(K_{T_1}^l)_{t_1 \circ t_1} \alpha(K_{T_1}^r) &= \alpha(K_1)_{t_1 \circ t_1} (\alpha(K_2)_{t_2 \circ t_2} \alpha(K_3)) \\ &= (\alpha(K_1)_{t_1 \circ t_1} \alpha(K_2))_{t_2 \circ t_2} \alpha(K_3) \\ &= \alpha(K_{T_2}^l)_{t_2 \circ t_2} \alpha(K_{T_2}^r). \end{aligned}$$

Therefore, the definition of α does not depend on the choice of a cut-triangle. That α is indeed a morphism of cyclic operads follows by Proposition 10.7 and Remark 10.8: if $K = K_{1x \circ y} K_2$, then K is reducible with respect to the cut-triangle t resulting from the identification of the boundaries of x and y , and, hence

$$\alpha(K_{1x \circ y} K_2) = \alpha(\hat{K}_T^l)_{t \circ t} \alpha(\hat{K}_T^r) = \alpha(K_1)_{x \circ y} \alpha(K_2),$$

where the second equality additionally requires the equivariance axiom of the operad of unrooted trees. Finally, the inverse of α is defined in a straightforward manner by induction on the size of a given unrooted tree from the free cyclic operad generated by the irreducible \mathcal{M} -complexes. \square

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