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Abstract A bounded divergence measure field is a bounded measurable function $q = (q_1, \dots, q_n)$ on $\mathbb{R}^n$ whose weak divergence is a finite signed measure. The Gauss-Green theorem for this class of fields on sets of finite perimeter was established independently by Chen & Torres and the present author in 2005. To emphasize the essentially simple nature of this result, I outline my original proof with some amendments. In addition, future developments are briefly recapitulated together with some remarks on the later proof by Chen, Torres & Ziemer.

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Contents

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1 Introduction

Bounded divergence measure fields are vectorfields $q = (q_1, \dots, q_n) \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ whose weak divergence $\operatorname{div} q$ is a finite signed measure on $\mathbb{R}^n$ [see Equation (6), below]. They are much more general than, say, bounded vectorfields with components from $BV(\mathbb{R}^n)$. In the latter case, each partial derivative $\partial q_i/\partial x_j$, $1 \le i, j \le n$, is a measure, while in the former case, only the sum $\sum_{i=1}^n \partial q_i/\partial x_i$ is known to be a measure, while the individual terms $\partial q_i/\partial x_i$ can be much wilder (not to speak about the mixed derivatives $\partial q_i/\partial x_j$, $i \neq j$).

The Gauss-Green theorem for bounded divergence measure fields and open sets with lipschitzian boundary has been proved by Anzellotti in the classical paper [2]. The theorem was generalized to sets of finite perimeter simultaneously and independently in 2005 by Chen & Torres [7; Theorem 2] and Šilhavý [15; Theorem 4.4(i)]. These papers present identical, and final, forms of the Gauss-Green theorem for this class; the proofs are different, though. The theorem was later re-derived by Chen, Torres & Ziemer [8] (as corrected in [9]). The approach in [8] is based on approximations of the set of finite perimeter by smooth sets and the use of the classical Gauss-Green theorem for these smooth sets. As declared by the authors, the main theorem of [8] is Theorem 5.2, a detailed account of the approximation. The Gauss-Green theorem, identical to that in [7] and [15], is Theorem 5.3.

I do not believe that the complicated apparatus of approximations by "nicer" sets as in [8] increases the "credibility" of the Gauss-Green theorem for the present class. No analogous extra support is given to the Federer-De Giorgi version of the Gauss-Green theorem for lipschitzian functions and sets of finite perimeter. What is needed in applications is the Gauss-Green theorem *per se.* In line with these ideas, Comi & Payne, in their recent paper [10], attempted do simplify the proofs of [8] in some way.

Here I present my original proof of 2005, which seems to be straightforward: mollify the vectorfield q, then apply the Federer-De Giorgi version of the Gauss-Green theorem, and then let the mollification parameter tend to 0. Some amendments and future developments are included also.

2 Sets of finite perimeter

Sets of finite perimeter are treated in detail in many sources, such as [1, 11–12]. This section gives a brief recapitulation.

We denote by $\mathscr{L}^n$ the Lebesgue measure and by $\mathscr{H}^{n-1}$ and the $n-1$-dimensional Hausdorff measure in $\mathbb{R}^n$. Let M be a $\mathscr{L}^n$ measurable subset of $\mathbb{R}^n$. If $x \in \mathbb{R}^n$, the *density* $\Theta^n(x, M)$ of M at x is defined by

$$\Theta^n(x, M) = \lim_{r \to 0} \mathscr{L}^n(M \cap B(x, r))/\alpha_n r^n \tag{1}$$

provided the limit exists. Here $B(x, r)$ is the open ball of center x and radius r and α_n is the volume of the unit ball in $\mathbb{R}^n$. If $0 \le t \le 1$, we introduce the set

$$M^t = \{x \in \mathbb{R}^n : \Theta^n(x, M) = t\}$$

of of points of density t. The *essential boundary* $\partial^* M$ of M is defined by

$$\partial^* M = \mathbb{R}^n \sim (M^0 \cup M^1).$$

Thus $\partial^* M$ is the set of all points $x \in \mathbb{R}^n$ such that either the limit in (1) does not exist or, if it exists, then $0 < \Theta^n(x, M) < 1$. We say that a unit vector ν is an *exterior normal* to M at $x \in \mathbb{R}^n$ if the half-space

$$H(x,\nu) := \{y \in \mathbb{R}^n : \nu \cdot (x - y) < 0\} \tag{2}$$

locally approximates the set M at x in the density sense. This means that the symmetric difference $M \triangle H(x,\nu) := [M \sim H(x,\nu)] \cup [H(x,\nu) \sim M]$ has vanishing density at x, i.e.,

$$\Theta^n(x, M \triangle H(x,\nu)) = 0.$$

It is easy to show that the exterior normal, if it exists, is uniquely determined. Furthermore, since $\Theta^n(x, H(x,\nu)) = 1/2$, one easily deduces that if the exterior normal at x exists, then $\Theta^n(x, M) = 1/2$ and hence $x \in \partial^* M$.

The following theorem introduces sets of finite perimeter in a symmetric way by formulating four equivalent conditions characterizing them. Any of them can serve as a definition.

Theorem 1 *The following four conditions are equivalent*:
(i) $\mathscr{H}^{n-1}(\partial^* M) < \infty$;
(ii) *there exists a finite* $\mathbb{R}^n$*-valued measure* μ *on* $\mathbb{R}^n$ *such that*

$$\int_M \nabla\varphi \, d\mathscr{L}^n = \int_{\mathbb{R}^n} \varphi \, d\mu$$

for every lipschitzian function φ *on* $\mathbb{R}^n$ *with compact support*;
(iii) *there exists an* $\mathscr{H}^{n-1}$ *integrable function* $\nu^M : \partial^* M \to \mathbb{S}^{n-1}$ *such that*

$$\int_M \nabla\varphi \, d\mathscr{L}^n = \int_{\partial^* M} \varphi \nu^M \, d\mathscr{H}^{n-1} \tag{3}$$

for every lipschitzian function φ *on* $\mathbb{R}^n$ *with compact support;*
(iv) *there exists an* $\mathscr{H}^{n-1}$ *integrable function* $\nu : \partial^* M \to \mathbb{S}^{n-1}$ *such that*

$$\int_M \operatorname{div} v \, d\mathscr{L}^n = \int_{\partial^* M} v \cdot \nu^M \, d\mathscr{H}^{n-1} \tag{4}$$

for every lipschitzian vectorfield $v : \mathbb{R}^n \to \mathbb{R}^n$ *with compact support.*

If these conditions are satisfied, M is said to be a set of finite perimeter. In that case the exterior normal is defined for $\mathscr{H}^{n-1}$ almost every $x \in \partial^* M$ and hence

$$\Theta^n(x, M) = \frac{1}{2} \qquad \text{for } \mathscr{H}^{n-1} \text{ almost every } x \in \partial^* M. \tag{5}$$

The function ν^M of Items (iii) and (iv) coincides with the exterior normal at $\mathscr{H}^{n-1}$ almost every point of $\partial^* M$. Equations (3) and (4) are Federer-De Giorgi Gauss-Green theorems for lipschitzian functions and vectorfields on sets of finite perimeter.

3 The Gauss-Green theorem

A bounded $\mathscr{L}^n$ measurable function $q : \mathbb{R}^n \to \mathbb{R}^n$ is called a *bounded divergence measure field* if there exists a finite signed measure $\operatorname{div} q$ on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} \nabla\varphi \cdot q \, d\mathscr{L}^n = -\int_{\mathbb{R}^n} \varphi \, d \operatorname{div} q \tag{6}$$

for every lipschitzian function φ on $\mathbb{R}^n$ with compact support. We denote by $|q|_\infty$ the norm of q in $L^\infty(\mathbb{R}^n, \mathbb{R}^n)$, and by $|\operatorname{div} q|$ the total variation measure for $\operatorname{div} q$. If $A \subset \mathbb{R}^n$ is a Borel set then $(\operatorname{div} q) \llcorner A$ and $\mathscr{H}^{n-1} \llcorner A$ denote the restrictions of $\operatorname{div} q$ and $\mathscr{H}^{n-1}$ to A.

We are going to prove the Gauss-Green theorem for q on a bounded set $M \subset \mathbb{R}^n$ of finite perimeter. It will be clear that it suffices to make a seemingly weaker hypothesis that q is defined on some open neighborhood N of the closure of M. However, one can always extend this q from N to $\tilde{q}$ on $\mathbb{R}^n$ by putting $\tilde{q} = \psi q$ where $\psi : \mathbb{R}^n \to \mathbb{R}$ is any smooth function with $\psi = 1$ on the closure of M and $\psi = 0$ outside N. We therefore assume that the divergence measure fields are defined on the whole of $\mathbb{R}^n$.

The following auxiliary result will be needed.

Proposition *If q is a bounded divergence measure then for every Borel set $A \subset \mathbb{R}^n$ with $\mathscr{H}^{n-1}(A) < \infty$ there exists a $s^A \in L^\infty(A, \mathscr{H}^{n-1})$ such that*

$$(\operatorname{div} q) \llcorner A = s^A \mathscr{H}^{n-1} \llcorner A; \tag{7}$$

moreover, the norm of s^A in $L^\infty(A, \mathscr{H}^{n-1})$ satisfies

$$|s^A|_\infty \le c_n |q|_\infty \tag{8}$$

where $c_n = n[2n/(n+1)]^{(n-1)/2} \alpha_n/\alpha_{n-1}$.

Chen & Frid [4; Proposition 3.1] proved that $\operatorname{div} q$ is absolutely continuous with respect to $\mathscr{H}^{n-1}$. The Radon-Nikodym theorem then gives an integrable density $s^A \in L^1(A, \mathscr{H}^{n-1})$ satisfying (7). The boundedness of s^A stated above appears to be new.

Proof Let $\mathscr{S}^{n-1}$ be the $n-1$ dimensional spherical measure, see [11; §2.10.2]. We shall first prove that if K is a compact subset of $\mathbb{R}^n$ with $\mathscr{S}^{n-1}(K) < \infty$ then

$$|\operatorname{div} q(K)| \le n\alpha_n/\alpha_{n-1} \mathscr{S}^{n-1}(K). \tag{9}$$

If $0 < h < 1$, we define $\sigma_h : \mathbb{R}^n \to \mathbb{R}$ by

$$\sigma_h(x) = \begin{cases} 1 & \text{if } |x| < 1, \\ 1 - (1 - |x|)/h & \text{if } 1 \le |x| \le 1 + h, \\ 0 & \text{if } |x| > 1 + h, \end{cases}$$

$x \in \mathbb{R}^n$. The support of σ_h is $B(0, 1+h)$ and σ_h is a lipschitzian function with $|\nabla \sigma_h(x)| = 1/h$ if $1 \le |x| + h$ and $\nabla \sigma_h = 0$ elsewhere.

Let K be a compact set with $\mathscr{S}^{n-1}(K) < \infty$ and let $\varepsilon > 0$ be arbitrary. Invoking a general property of Radon measures, we find that there exists a bounded open set $U \subset \mathbb{R}^n$ with $K \subset U$ such that

$$|\operatorname{div} q|(U \sim K) < \varepsilon. \tag{10}$$

The definition of $\mathscr{S}^{n-1}$ gives that for each sufficiently small $\delta > 0$ we find a covering of the compact K by open balls $B(x_i, r_i)$, $i = 1, \dots, N$, with $r_i < \delta$, such that

$$K \cap B(x_i, r_i) \neq \emptyset, \quad \text{and} \quad \alpha_{n-1} \sum_{i=1}^{N} r_i^{n-1} < \mathscr{S}^{n-1}(K) + \varepsilon. \tag{11}$$

Moreover, we can consider all δ small enough to satisfy

$$\{x \in \mathbb{R}^n : \operatorname{dist}(x, K) \leq 4\delta\} \subset U. \tag{12}$$

Let h be fixed, $0 < h < 1$, and put

$$\varphi(x) = \sup\{\varphi_i(x) : i = 1, 2, \ldots, N\}, \quad \text{where} \quad \varphi_i(x) = \sigma_h\big((x - x_i)/r_i\big),$$

$x \in \mathbb{R}^n$. Then $0 \leq \varphi \leq 1$ on $\mathbb{R}^n$, $\varphi = 1$ on K, φ is lipschitzian, and, by (12), $\operatorname{spt}\varphi \subset U$. We now apply (6) to the present function φ. We can thus replace the original integration limits, $\mathbb{R}^n$, by U. A rearrangement of the result provides

$$\int_K \varphi \, d \operatorname{div} q = -\int_U \nabla\varphi \cdot q \, d\mathscr{L}^n - \int_{U \sim K} \varphi \, d \operatorname{div} q.$$

Since $\operatorname{spt}\varphi_i = B\big(x_i, (1+h) r_i\big)$, we have

$$\Big|\int_U \nabla\varphi \cdot q \, d\mathscr{L}^n\Big| \leq \sum_{i=1}^{N} \Big|\int_{B(x_i, (1+h) r_i)} \nabla\varphi_i \cdot q \, d\mathscr{L}^n\Big|. \tag{13}$$

We note that $\nabla\varphi_i(x)$ is different from 0 only if x belongs to the shell $S_i = \{r_i \leq |x| \leq (1+h) r_i\}$, in which case $|\nabla\varphi_i(x)| = 1/h$. Since the volume of the shell S_i is $\alpha_n\big((r_i + h)^n - r_i^n\big)$, the magnitude of each of the integrals on the right-hand side of (13) is estimated by $\alpha_n h^{-1}\big((r_i + h)^n - r_i^n\big)|q|_\infty$ and so

$$\Big|\int_U \nabla\varphi \cdot q \, d\mathscr{L}^n\Big| \leq \alpha_n |q|_\infty h^{-1} \sum_{i=1}^{N} \big((r_i + h)^n - r_i^n\big)$$

for any h with $0 < h < 1$. Letting $h \to 0$ in the right-hand side and the subsequent use of $(11)_2$ provides

$$\Big|\int_U \nabla\varphi \cdot q \, d\mathscr{L}^n\Big| \leq n\alpha_n |q|_\infty \sum_{i=1}^{N} r_i^{n-1} \leq n\alpha_n/\alpha_{n-1} |q|_\infty \big(\mathscr{S}^{n-1}(K) + \varepsilon\big).$$

Furthermore, since $|\varphi| \leq 1$ on $\mathbb{R}^n$, Equation (10) provides $|\int_{U \sim K} \varphi \, d \operatorname{div} q| < \varepsilon$. Consequently, (13) yields

$$|\operatorname{div} q(K)| \leq n\alpha_n/\alpha_{n-1} \big(\mathscr{S}^{n-1}(K) + \varepsilon\big)|q|_\infty + \varepsilon.$$

Thus the arbitrariness of $\varepsilon > 0$ yields (9).

If E is a Borel set with $\mathscr{S}^{n-1}(E) < \infty$, the values of the measures $\operatorname{div} q$ and $\mathscr{S}^{n-1}$ on E can be approximated by the respective values on compact subsets of E; thus (9) extends to all Borel sets $K = E$ with $\mathscr{S}^{n-1}(E) < \infty$. If E_i, $i = 1, \ldots$, is any Borel partition of E then

$$|\operatorname{div} q|(E) \leq \sum_{i=1}^{\infty} |\operatorname{div} q(E_i)| \leq n\alpha_n/\alpha_{n-1} \sum_{i=1}^{\infty} \mathscr{S}^{n-1}(E_i) = n\alpha_n/\alpha_{n-1} \mathscr{S}^{n-1}(E),$$

i.e.,

$$|\operatorname{div} q|(E) \leq n\alpha_n/\alpha_{n-1} \mathscr{S}^{n-1}(E).$$

A combination with the inequality

$$\mathscr{S}^{n-1}(E) \le \left[2n/(n+1)\right]^{(n-1)/2} \mathscr{H}^{n-1}(E),$$

see [11; §2.10.42], then gives

$$|\operatorname{div} q|(E) \le c_n \mathscr{H}^{n-1}(E) \tag{14}$$

where c_n is as in the statement of the proposition. In particular, $|\operatorname{div} q|$ is absolutely continuous with respect to $\mathscr{H}^{n-1}$. The Radon-Nikodym theorem then gives an integrable density $s^A \in L^1(A, \mathscr{H}^{n-1})$ satisfying (7), and (14) then implies (8). □

The next theorem is the Gauss-Green theorem for bounded divergence measure fields. Since the essential boundary of any two sets which differ by a $\mathscr{L}^n$ null set coincide, without any loss in generality we formulate it for normalized sets. These are defined as bounded $\mathscr{L}^n$ measurable sets $M \subset \mathbb{R}^n$ which coincide with their own points of density, i.e., $M = M^1$.

Theorem 2 (Chen & Torres [7; Theorem 2]; Šilhavý [15; Theorem 4.4(i)]) *If M is a normalized set of finite perimeter and if q is a bounded divergence measure field then there exists a bounded $\mathscr{H}^{n-1}$ measurable function $t^M : \partial^* M \to \mathbb{R}$ such that*

$$\int_{\partial^* M} \varphi\, t^M \, d\mathscr{H}^{n-1} = \int_M \nabla\varphi \cdot q \, d\mathscr{L}^n + \int_M \varphi \, d \operatorname{div} q \tag{15}$$

for every lipschitzian function φ on $\mathbb{R}^n$ with compact support. The norm of t^M satisfies

$$|t^M|_{L^\infty} \le \frac{\alpha_n}{2\alpha_{n-1}} |q|_{L^\infty}. \tag{16}$$

The function t^M is called the normal trace of q on $\partial^* M$. Inequality (16) corrects the inequality $|t^M|_{L^\infty} \le |q|_{L^\infty}$ occurring in [15; Theorem 4.4(i)].

Proof Let q_ρ, $\rho > 0$, be the ρ-mollification of q, given by

$$q_\rho(x) = \int_{\mathbb{R}^n} \omega_\rho(x-y) q(y) \, d\mathscr{L}^n, \quad x \in \mathbb{R}^n,$$

where $\omega_\rho(z) = \rho^{-n}\omega(z/\rho)$ for each $z \in \mathbb{R}^n$, where ω is any mollifier. The Gauss-Green theorem for sets of finite perimeter (4) with $v = \varphi q_\rho$ gives

$$\int_M \nabla\varphi \cdot q_\rho \, d\mathscr{L}^n + \int_M \varphi \operatorname{div} q_\rho \, d\mathscr{L}^n = \int_{\partial^* M} \varphi\, q_\rho \cdot \nu^M \, d\mathscr{H}^{n-1}. \tag{17}$$

We now let $\rho \to 0$ in (17) and evaluate the limits of the three terms in (17) separately. First, we have

$$\int_M \nabla\varphi \cdot q_\rho \, d\mathscr{L}^n \to \int_M \nabla\varphi \cdot q \, d\mathscr{L}^n \tag{18}$$

since $q_\rho \to q$ uniformly. To evaluate the limit of the second term in (17), we first note that

$$(\operatorname{div} q_\rho)(x) = (\operatorname{div} q)_\rho(x) := \int_{\mathbb{R}^n} \omega_\rho(x-y) \, d \operatorname{div} q(y); \tag{19}$$

thus

$$\int_M \varphi \operatorname{div} q_\rho \, d\mathscr{L}^n = \int_M \varphi(x) \int_{\mathbb{R}^n} \omega_\rho(x-y) \, d \operatorname{div} q(y) \, dx = \int_{\mathbb{R}^n} \varphi^*_\rho \, d \operatorname{div} q$$

where

$$\varphi_\rho^*(y) = \int_M \varphi(x)\omega_\rho(x-y)\,dx, \quad y \in \mathbb{R}^n.$$

Since M is a normalized set, we have $\Theta^n(y,M) = 1$ for all $y \in M \equiv M^1$, $\Theta^n(y,M) = 0$ for all $y \in M^0$ and $\Theta^n(y,M) = 1/2$ for $\mathscr{H}^{n-1}$ almost every $y \in \partial^* M$ by (5). Consequently,

$$\lim_{\rho\to 0} \varphi_\rho^*(y) = \begin{cases} \varphi(y) & \text{for all} & y \in M, \\ \dfrac{1}{2}\varphi(y) & \text{for } \mathscr{H}^{n-1} \text{ almost every} & y \in \partial^* M, \\ 0 & \text{for all} & y \in N. \end{cases}$$

Splitting the integral $\int_{\mathbb{R}^n} \varphi_\rho^*(y)\,dy$ into the sum of integrals over M, $\partial^* M$ and M^0, and using the last equation in combination with the dominated convergence theorem, we obtain

$$\int_M \varphi_\rho^*\,d\operatorname{div} q \to \int_M \varphi\,d\operatorname{div} q, \quad \int_N \varphi_\rho^*\,d\operatorname{div} q \to 0,$$

and since $\operatorname{div} q$ is absolutely continuous with respect to $\mathscr{H}^{n-1}$,

$$\int_{\partial^* M} \varphi_\rho^*\,d\operatorname{div} q \to \frac{1}{2}\int_{\partial^* M} \varphi\,d\operatorname{div} q = -\int_{\partial^* M} \varphi t_1\,d\mathscr{H}^{n-1}$$

where $-t_1 \in L^\infty(\partial^* M, \mathscr{H}^{n-1})$ is the density of the measure $\frac{1}{2}(\operatorname{div} q) \llcorner \partial^* M$ which exists by Proposition above. Hence (19) gives

$$\int_M \varphi \operatorname{div} q_\rho\,d\mathscr{L}^n \to \int_M \varphi\,d\operatorname{div} q - \int_{\partial^* M} \varphi t_1\,d\mathscr{H}^{n-1}. \tag{20}$$

Finally, to take the limit of the third term in (17), we note that $|q_\rho|_\infty \le |q|_\infty$ for every $\rho > 0$. Hence $|q_\rho \cdot \nu^M|_\infty \le |q|_\infty$ for every $\rho > 0$. A standard property of the weak* convergence (see, e.g., [14; Theorem 3.15]) then implies that there is a sequence $\rho_k \downarrow 0$ and a function $t_2 \in L^\infty(\partial^* M, \mathscr{H}^{n-1})$ such that $q_{\rho_k} \cdot \nu^M \overset{*}{\rightharpoonup} t_2$ in $L^\infty(\partial^* M, \mathscr{H}^{n-1})$; in particular

$$\int_{\partial^* M} \varphi\, q_{\rho_k} \cdot \nu^M\,d\mathscr{H}^{n-1} \to \int_{\partial^* M} \varphi t_2\,d\mathscr{H}^{n-1}. \tag{21}$$

Using (18), (20) and (21), we see that the limit $\rho \to 0$ in (17) gives (15) with $t^M = t_1 + t_2$. Inequality (8) gives $|t^M|_{L^\infty} \le (c_n/2+1)|q|_{L^\infty}$; the stronger Inequality (16) is a consequence of Equation (22) (below). □

4 Complements and outlook

The Gauss-Green theorem 2 asserts the existence of the normal trace t^M but does not say how to determine it. We now give a formula for t^M. For a given bounded divergence measure field q we define a function $q^0 : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$ by

$$q^0(x,\nu) = \begin{cases} \lim\limits_{r\to 0} \dfrac{n}{\alpha_{n-1} r^n} \displaystyle\int\limits_{B(x,r)\cap H(x,\nu)} q(y)\cdot\frac{y-x}{|y-x|}\,d\mathscr{L}^n(y) & \\ \qquad\qquad\text{if the limit exists and is finite,} & \\ 0 \qquad\qquad \text{if the limit either does not exist or is infinite,} & \end{cases}$$

for any $x \in \mathbb{R}^n$ and $\nu \in \mathbb{S}^{n-1}$, where $H(x,\nu)$ is the half-space from (2).

Theorem 3 ([15; Equation (4.7)]) *If M is a normalized set of finite perimeter and if q is a bounded divergence measure field then the normal trace t^M on $\partial^* M$ is given by*

$$t^M(x) = q^0(x, \nu^M(x)) \tag{22}$$

for $\mathscr{H}^{n-1}$ almost every $x \in \partial^* M$. If q is continuous then $q^0(x,\nu) = q(x)\cdot\nu$, i.e.,

$$t^M(x) = q(x)\cdot\nu^M(x),$$

and the Gauss-Green theorem (15) *takes a more classical form*

$$\int\limits_{\partial^* M} \varphi\, q\cdot\nu^M\, d\mathscr{H}^{n-1} = \int\limits_M \nabla\varphi\cdot q\, d\mathscr{L}^n + \int\limits_M \varphi\, d\operatorname{div} q.$$

By (22) the normal trace $t^M(x)$ at $x \in \partial^* M$ is completely determined by x and the exterior normal $\nu^M(x)$; the higher-order characteristics such as the curvature etc. are irrelevant. A continuum mechanics analogue of this result says that the surface traction on the boundary of a body depends only on the position and normal. (*Cauchy's postulate.*)

The paper [15] treats also the more general case of vectorfields $q \in L^p(\mathbb{R}^n, \mathbb{R}^n)$, $1 \le p \le \infty$, with divergence measure. The set of such vectorfields is denoted by $\mathscr{DM}^p(\mathbb{R}^n)$, or, if they are defined only on an open subset $\Omega \subset \mathbb{R}^n$, by $\mathscr{DM}^p(\Omega)$. In the case $p \neq \infty$, the analogue of Theorem 2 is no longer simple: the normal trace function t^M must be replaced by a (nonlocal) functional on the space of Lipschitz continuous functions on the essential boundary $\partial^* M$ [15; Proposition 4.4]. This nonlocality is related to the singularities of q, not to the complications of the boundary: [18; Example 2.5] shows that it occurs even on a flat part of the boundary.

An extended divergence measure field on $\Omega \subset \mathbb{R}^n$ is a finite $\mathbb{R}^n$-valued measure q on Ω whose divergence is a finite signed measure. These were introduced by Chen and Frid in [5] and the corresponding space denoted by $\mathscr{DM}^{\mathrm{ext}}(\Omega)$. Clearly, this is a generalization of $\mathscr{DM}^p(\Omega)$. The paper [16] discusses extended divergence measure fields in from the point of view of the geometric measure theory, specifically the theory of flat chains of Whitney [19], as presented in Federer [11]. In Federer's terminology, extended divergence measure fields are normal one-dimensional currents in $\mathbb{R}^n$, which form a dense subset of the set of one-dimensional flat chains. Among other things, the paper [16; Section 8] establishes a decomposition of an extended divergence measure field into the absolutely continuous part, diffuse (= Cantor) part, and a part represented by a one-dimensional rectifiable $\mathbb{R}^n$-valued measure. This is an analogue of the well known decomposition of the derivative of a BV function into the absolutely continuous, Cantor, and the jump parts.

Closely related to divergence measure fields is the Cauchy stress theorem, a fundamental discovery of Cauchy [3] that the internal force system in a continuous body is described by a stress tensor field. The treatment of the Cauchy stress theorem on the level of extended divergence measure fields is given in [17], where also further literature is to be found.

Finally, to the best knowledge of the author, the most general form of the Gauss-Green theorem is presented in [18]: the class of regions are general open sets, even those for which the normal cannot be defined ("fractal" or "rough" sets). The vector fields are the extended divergence measure fields from $\mathcal{DM}^{\mathrm{ext}}(\Omega)$. Again, the normal trace is a functional on the class of lipschitzian functions on the topological boundary.

The fields from $\mathcal{DM}^{p}(\Omega)$, $1 \le p \le \infty$, on general open sets are treated at length in a recent paper of Chen, Comi and Torres [6] by a method different from [18]. The results of [18], of course, apply to this special case. This provides results close to those of [6]. The only difference is in the class of testfunctions, i.e., functions φ occurring in the Gauss-Green theorem (like φ in (15)). Related to the smaller space $\mathcal{DM}^{p}(\Omega) \subset \mathcal{DM}^{\mathrm{ext}}(\Omega)$, the paper [6] admits a larger class of testfunctions. Specifically, continuous functions from $W^{1,p'}(\Omega)$ (p'= the Hölder conjugate of p), whereas [18] admits only lipschitzian testfunctions. However, the density of $C^{\infty}(\Omega) \cap W^{1,p}(\Omega)$ in $W^{1,p}(\Omega)$ for general open sets (see [13]) allows, not unexpectedly, to extend the Gauss-Green theorem from [18] to continuous testfunctions from $W^{1,p'}(\Omega)$ by a simple limiting procedure. This possibility, overlooked in [6], will be discussed in a future paper, together with a number of amendments. (Of course, such an extension is not possible for general extended divergence measure fields.)

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