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An Exponential Lower Bound for Proofs in Focused Calculi ⋆

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Abstract. In [7], Iemhoff introduced a special form of sequent-style rules and axioms, which she called focused, and studied the relationship between the focused proof systems, the systems only consisting of this kind of rules and axioms, and the uniform interpolation of the logic that the system captures. Subsequently, as a negative consequence of this relationship, she excludes almost all super-intuitionistic logics from having these focused proof systems. In this paper, we will provide a complexity theoretic analogue of her negative result to show that even in the cases that these systems exist, their proof-length would computationally explode. More precisely, we will first introduce two natural subclasses of focused rules, called PPF and MPF rules. Then, we will introduce some **CPC**-valid (**IPC**-valid) sequents with polynomially short tree-like proofs in the usual Hilbert-style proof system for classical logic, or equivalently **LK** + **Cut**, that have exponentially long proofs in the systems only consisting of PPF (MPF) rules.

Keywords: Focused calculi · Propositional proof complexity · Feasible interpolation · Super-intuitionistic logics.

1 Introduction

In the field of proof theory, proof systems, as the main players of the game, deserve to be considered as the objects of the study themselves. Regarding this matter, there are various problems to attack. One of them is investigating whether some special kinds of proof systems exist and if they do, what properties they or their corresponding logics posses, including the Craig or uniform interpolation of the corresponding logic, or the complexity of proofs in the given proof system.

These problems have been studied by many researchers (for instance [3], [6] and [7]). In [6] and [7], Iemhoff inspected the relationship between a specific kind of proof system and the uniform interpolation property of the logic that

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the proof system captures. She introduced the so-called focused rules and axioms, and studied the sequent calculi only consisting of these rules and axioms, which she named focused calculi. Roughly speaking, a focused axiom is just a modest generalization of the axioms of the classical sequent-style proof system, **LK**. A focused rule is a rule where only one side of its sequents, either left or right, is active in all the premises and in the conclusion and also all the variables in its premises occur in its conclusion. For instance, the usual conjunction and disjunction rules in **LK**, are focused, while the cut rule is not. After her formalization of the focused rules and focused axioms, she provided a method to prove that a super-intuitionistic logic enjoys the uniform interpolation property if it has a terminating focused proof system. Since there are only seven super-intuitionistic logics with the uniform interpolation property, she finally excluded almost all the super-intuitionistic logics (except at most seven of them) from having a focused proof system.

Inspired by Iemhoff's work, [1] proposed a generalization of focused rules, called semi-analytic rules, to cover a wider range of proof systems for a wider range of logics. Stated informally, in a semi-analytic rule, the side condition is relaxed and the formulas can appear freely in any side of the sequents in the premises and the conclusion. Iemhoff's results in [6] and [7] are then strengthened to also hold for these rules. It implies that many substructural logics and almost all super-intuitionistic logics (except at most seven of them) do not have a sequent style proof system only consisting of semi-analytic rules and focused axioms.

This paper is a sequel of [1] in its extension of the negative results of [6] and [7] to the remaining cases in which the interpolation property exists. For this purpose, we change our focus from the existence of a proof system of some kind to its efficiency to show an exponential lower bound for the focused proof systems of a certain sort. Beside the clear impacts in the study of focused rules, these lower bounds can also be considered as the basic steps in a universal approach to the proof complexity of the propositional proof systems. In such an approach, we are interested in investigating the proof lengths of a given sequence of tautologies in a generically given proof system with a certain form of axioms and rules. The method we use here is the well-known technique in proof complexity called the feasible interpolation. It reduces a problem in proof complexity to a problem in circuit complexity by extracting a Boolean circuit for an interpolant from a given proof for an implication, where the size of the circuit is polynomially bounded by the size of the proof. The feasible interpolation property for various classical calculi has been studied by Krajíček[9], Pudlák[10], and Pudlák and Sgall[13]. For the intuitionistic calculus, the feasible interpolation theorem was proved by Pudlák in [12] based on the feasible witnessing of the disjunction property developed in [14]. Buss and Pudlák in [15] and Buss and Mints in [14] studied the connection between intuitionistic propositional proof lengths and Boolean circuits. In [5], Hrubeš showed the connection is tighter in the sense that the circuit in question in [15] and [14] is monotone. Here we will use the technique of [5] as we will explain in a moment. For more information on feasible interpolation and its role in proof complexity, the reader is refered to [11].

In this paper, we will prove two lower bounds, one for the classical logic and the other for super-intuitionistic logics. For the first one, we will define a natural subclass of the focused rules, which we will call *polarity preserving focused*, PPF, rules. Then, we show that there are **CPC**-tautologies with exponential proof lengths in any proof system only consisting of PPF rules and focused axioms, which we call PPF calculi, while they have polynomial proof lengths in **LK**. This shows an exponential speed-up of the Frege-style proof system for classical logic with respect to any PPF calculus. To prove the similar exponential lower bound for intuitionistically valid formulas, we first define *monotonicity preserving focused*, MPF, rules and subsequently MPF calculi. Then, we will use the mentioned lower bound technique developed by Hrubeš in [4] and [5] to obtain an exponential lower bound for the lengths of proofs of particular **IPC**-tautologies in MPF calculi, while they have polynomial length proofs in **LK**.

2 Preliminaries

In this section, we will present some definitions and notions that will be needed in the rest of the paper.

Note that any finite object O that we use here, such as a formula or a proof, can be represented by a fixed suitable binary string and by $|O|$ we mean the length of the string representing the object.

In this paper, we work with the usual propositional language $\{\wedge, \vee, \neg, \rightarrow, \bot, \top\}$. By **IPC** and **CPC** we mean intuitionistic and classical propositional logics, respectively. By meta-language, we mean the language in which we define the sequent calculi. A meta-formula is defined inductively; an atom and a formula symbol are meta-formulas and we can construct new meta formulas using the existing ones and the connectives of the language. A meta-multiset is a set of meta-formulas and meta-multiset variables. By $V(A)$, we mean the atoms and meta-formula variables of the meta-formula A.

By a sequent $\Gamma \Rightarrow \Delta$, we mean an expression where Γ and Δ are multisets and it is interpreted as $\bigwedge \Gamma \rightarrow \bigvee \Delta$. A meta-sequent is essentially a sequent defined by meta-multisets. A rule is an expression of the form:

$$\frac{T_1, \cdots, T_n}{T}$$

where T_i's and T are meta-sequents. A sequent calculus is a set of rules.

By monotone **LK**, **mLK**, we mean the sequent calculus consisting of the axioms of **LK**, the structural rules (exchange, weakening, contraction), and its usual conjunction and disjunction rules.

A calculus G is *sound* for logic L, if $G \vdash \Gamma \Rightarrow \Delta$ implies $L \vdash \bigwedge \Gamma \to \bigvee \Delta$. It is called *complete* if $L \vdash \bigwedge \Gamma \to \bigvee \Delta$ implies $G \vdash \Gamma \Rightarrow \Delta$ and *feasibly complete* if the length of the tree-like proof is polynomially bounded by the sequent, i.e., there exists a tree-like proof π of $\Gamma \Rightarrow \Delta$ in G such that $|\pi| \leq |\Gamma \Rightarrow \Delta|^{O(1)}$. We say that logic M is an *extension of logic* L, if $L \vdash A$ implies $M \vdash A$. We say a calculus H is an *extension of a calculus* G, if for any rule of G, if all the premises are provable in H, then the consequence is also provable in H. Moreover, H is called an *axiomatic extension* of G, when all the provable sequents of G are considered as axioms of H, and H can add some rules to them.

A logic L is called *sub-classical* if **CPC** extends L. In the same way, a calculus G is called sub-classical if **LK** extends G.

A logic L (calculus G) has the *Craig interpolation* property when for any formula $\phi \to \psi$ (sequent $\Gamma \Rightarrow \Delta$), if $L \vdash \phi \to \psi$ ($G \vdash \Gamma \Rightarrow \Delta$) then there exists a formula θ such that $V(\theta) \subseteq V(\phi) \cap V(\psi)$ ($V(\theta) \subseteq V(\Gamma) \cap V(\Delta)$) and $L \vdash \phi \to \theta$ and $L \vdash \theta \to \psi$ ($G \vdash \Gamma \Rightarrow \theta$ and $G \vdash \theta \Rightarrow \Delta$). The calculus G has *feasible interpolation* if for any tree-like proof π of $\Gamma \Rightarrow \Delta$, there exists an interpolant θ such that $|\theta| \leq |\pi|^{O(1)}$.

3 Focused Calculi

In this section we will give the definitions of the focused axioms, rules and calculi, which are the building blocks of the rest of the paper.

Definition 3.1. A rule is called *focused* (a left focused rule, L, or a right focused rule, R) if it has one of the following forms:

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_{r=1}^{m_i} \rangle_{i=1}^{n}}{\Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n} L \qquad \frac{\langle\langle \Gamma_i \Rightarrow \bar{\phi}_{ir}, \Delta_i \rangle_{r=1}^{m_i} \rangle_{i=1}^{n}}{\Gamma_1, \cdots, \Gamma_n \Rightarrow \phi, \Delta_1, \cdots, \Delta_n} R$$

where Γ_i's and Δ_i's are meta-multiset variables, $\bar{\phi}_{ir}$ is a multi-set of formulas, and $\bigcup_{i,r} V(\phi_{ir}) \subseteq V(\phi)$. By the notation $\langle\langle \cdot \rangle_r \rangle_i$, we mean the sequents first range over $1 \leq r \leq m_i$ and then over $1 \leq i \leq n$.

Example 3.2. The usual conjunction and disjunction rules in **LK** are focused. On the other hand, the implication rules:

$$\frac{\Gamma \Rightarrow \phi, \Delta \qquad \Sigma, \psi \Rightarrow \Lambda}{\Gamma, \Sigma, \phi \to \psi \Rightarrow \Delta, \Lambda} \qquad \frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \to \psi, \Delta}$$

are not focused, simply because both sides of the sequents are active.

Definition 3.3. A sequent is called a *focused axiom* if it is of the following form:

(1) Identity axiom: $(\phi \Rightarrow \phi)$

(2) Context-free right axiom: $(\Rightarrow \bar{\alpha})$
(3) Context-free left axiom: $(\bar{\beta} \Rightarrow)$
(4) Contextual left axiom: $(\Gamma, \bar{\phi} \Rightarrow \Delta)$
(5) Contextual right axiom: $(\Gamma \Rightarrow \bar{\phi}, \Delta)$

where Γ and Δ are meta-multiset variables and in $2-5$, the set of the variables of any two elements of $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\phi}$ must be the same.

Example 3.4. It is easy to see that the axioms of **LK**, $(\phi \Rightarrow \phi)$, $(\Gamma \Rightarrow \top, \Delta)$ and $(\Gamma, \bot \Rightarrow \Delta)$ are focused. Here are some more examples which are not in **LK**:

$$\phi, \neg\phi \Rightarrow \quad , \quad \Rightarrow \phi, \neg\phi$$

$$\Gamma, \neg\top \Rightarrow \Delta \quad , \quad \Gamma \Rightarrow \Delta, \neg\bot$$

First let us investigate the power of focused rules and focused axioms. The natural question to ask is whether it is possible to have a calculus consisting only of these rules and axioms, that is complete for some given logic. For **CPC** the answer is yes, and the following theorem can be considered as a witness of the power and naturalness of focused axioms and rules.

Theorem 3.5. **CPC** *has a sequent calculus consisting only of focused rules and focused axioms.*

Proof. Consider a sequent calculus containing the usual axioms of **CPC** and the following axioms:

Axioms:

$$\frac{}{\phi \Rightarrow \phi} \qquad \frac{}{\phi, \neg\phi \Rightarrow} \qquad \frac{}{\Rightarrow \phi, \neg\phi}$$

$$\frac{}{\Gamma \Rightarrow \neg\bot, \Delta} \qquad \frac{}{\Gamma, \neg\top \Rightarrow \Delta}$$

The usual left and right rules for disjunction and conjunction and the following rules for implication:

$$\frac{\Gamma \Rightarrow \neg\phi, \psi, \Delta}{\Gamma \Rightarrow \phi \to \psi, \Delta} \qquad \frac{\Gamma_1, \neg\phi \Rightarrow \Delta_1 \qquad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \phi \to \psi \Rightarrow \Delta_1, \Delta_2}$$

And finally, for any combination $\neg\vee$, $\neg\wedge$, and $\neg\neg$ we have the corresponding right and left rules, using De Morgan's laws. For instance, we have

$$\frac{\Gamma \Rightarrow \neg\phi, \Delta}{\Gamma \Rightarrow \neg(\phi \wedge \psi), \Delta} R\neg\wedge$$

It is easy to check that all the rules of this sequent calculus are focused and the system is sound and complete for **CPC**. The proof of the completeness part is based on the observation that if $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ is provable in the usual calculus for classical logic, then $\Gamma, \neg\Delta \Rightarrow \neg\Gamma', \Delta'$ is provable in the new calculus. The proof is an easy application of induction on the length of the usual **LK** proof of $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$. □

So far, we have seen some definitions and a sequent calculus consisting only of focused axioms and rules. Now, it is time to examine how effective such a characterization can be. For this purpose, from now on we will restrict our investigations to two natural sub-classes of focused rules, polarity preserving focused, PPF rules, and monotonicity preserving focused, MPF rules.

Definition 3.6. Let $\mathcal{P}$ be a set of meta-formula variables or atomic constants. A meta-formula ψ is called *$\mathcal{P}$-monotone* if for any $\phi \in \mathcal{P}$, all occurrences of ϕ in ψ are positive, i.e., ϕ does not occur in the scope of negations or in the antecedents of implications. A multiset Γ of meta-formulas is called $\mathcal{P}$-monotone if all of its elements are $\mathcal{P}$-monotone.
A meta-formula is called *monotone* if it is constructed by conjunctions and disjunctions on meta-formula variables, atomic constants and variable-free formulas.

Remark 3.7. Note that since any variable-free formula is classically equivalent to $\top$ or $\bot$, then any monotone formula in our sense is classically equivalent to the usual monotone formulas i.e., the formulas constructed from atomic formulas by applying conjunctions and disjunctions. Therefore, from now on, in the classical settings, we always assume that a monotone formula has the mentioned simpler form.

Definition 3.8. A focused rule is called *polarity preserving*, PPF, if it preserves $\mathcal{P}$-monotonicity backwardly for any $\mathcal{P}$, i.e., if the antecedent of the consequence is $\mathcal{P}$-monotone, then the antecedents of all the premises are also $\mathcal{P}$-monotone. It is *monotonicity preserving*, MPF, if it is focused and preserves monotonicity backwardly, in the same way.

Example 3.9. All analytic focused rules in the language of **CPC**, the focused rules in which any formula in the premises is a subformula of a formula in the consequence, are both PPF and MPF.

3.1 The Classical Case

Let us first see a relationship between focused calculi and the Craig interpolation property.

Theorem 3.10. *Let G be a sequent calculus extending* **mLK** *and only consisting of focused rules and focused axioms. Then, G has feasible interpolation property. Moreover, if the rules are also* PPF *and Γ is $\mathcal{P}$-monotone, then $\Gamma \Rightarrow \Delta$ has a feasible $\mathcal{P}$-monotone interpolant.*

Proof. We need to prove that to any provable sequent $\Gamma \Rightarrow \Delta$, we can assign a formula C such that $G \vdash \Gamma \Rightarrow C$ and $G \vdash C \Rightarrow \Delta$ and $V(C) \subseteq V(\Gamma) \cap V(\Delta)$. Use induction on the length of the proof π of the sequent $\Gamma \Rightarrow \Delta$ in G. If $\Gamma \Rightarrow \Delta$ is a focused axiom, it is easy to see that in different cases of the focused axioms, the interpolant C is either ϕ or $\bot$ or $\top$. We check the case 4 of the focused axioms. The rest are similar. In this case, we have to find C such that $\Gamma, \bar{\phi} \Rightarrow C$

and $C \Rightarrow \Delta$. We claim that $C = \bot$ works here. Note that in the focused axioms, since Γ and Δ are meta-multiset variables, we can substitute anything for them. Hence, we have $\Gamma, \bar{\phi} \Rightarrow \bot$, since it is an instance of the axiom 4 when Δ is substituted by $\bot$. And $\bot \Rightarrow \Delta$ is an instance of the axiom $\bot$ in **mLK** which is weaker than the system G by assumption.

For the rules, suppose the last rule used in the proof π is the following left focused rule:

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_r\rangle_i}{\Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n}$$

Then, by induction, there are formulas C_{ir} such that $\Gamma_i, \bar{\phi}_{ir} \Rightarrow C_{ir}$ and $C_{ir} \Rightarrow \Delta_i$. Using the right and left disjunction rules we have $\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bigvee_r C_{ir}$ and $\bigvee_r C_{ir} \Rightarrow \Delta_i$. By the left disjunction rule we have $\bigvee_i \bigvee_r C_{ir} \Rightarrow \Delta_1, \cdots, \Delta_n$. And if we substitute the sequents $\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bigvee_r C_{ir}$ in the original left focused rule we get $\Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \bigvee_r C_{1r}, \cdots, \bigvee_r C_{nr}$ and then using the right disjunction rule we get $\Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \bigvee_i \bigvee_r C_{ir}$.

Note that for any i and r, by induction we have $V(C_{ir}) \subseteq V(\Gamma_i \cup \{\phi_{ir}\}) \cap V(\Delta_i)$. Using this and the fact that for focused rules $\bigcup_{ir} V(\phi_{ir}) \subseteq V(\phi)$, we can easily show that $V(\bigvee_i \bigvee_r C_{ir}) \subseteq V(\Gamma \cup \{\phi\}) \cap V(\Delta)$, where $\Gamma = \Gamma_1, \cdots, \Gamma_n$ and $\Delta = \Delta_1, \cdots, \Delta_n$. Therefore, we have shown that $\bigvee_i \bigvee_r C_{ir}$ is the interpolant.

The case for a right focused rule is dual to the previous case.

The proof for the upper bound for the length of the interpolant goes as follows. We claim that our previously constructed interpolant C has the property $|C| \leq |\pi|^2$ and we will prove it by induction on π.

For the axioms, as we have seen, the interpolant is either ϕ (in the case that the sequent is of the form of the first axiom ($\phi \Rightarrow \phi$)) or $\bot$ or $\top$ (in other cases). In these cases, we have $|C| \leq |\pi|$.

For the left focused rules, we have shown that $C = \bigvee_i \bigvee_r C_{ir}$. Let $N_{\mathcal{R}}$ be the number of the premises of the rule $\mathcal{R}$, which is the last rule used in the proof. We have that the number of the formulas which appear in C , i.e. C_{ir}, is equal to $N_{\mathcal{R}}$. The rest of the symbols appeared in C are connectives, and the number of them is again equal to $N_{\mathcal{R}}$. Since the sequent $\Gamma \Rightarrow \Delta$ is the conclusion of a rule in G, the lengths of the proofs of its premises are less than the length of π and we can use the induction hypothesis for them. Then $|C| \leq \Sigma_{i,r}|C_{ir}| + N_{\mathcal{R}}$. By induction hypothesis we have $|C_{ir}| \leq |\pi_{ir}|^2$, where π_{ir} is the proof of the sequent whose interpolant is C_{ir}. But since the proof is tree-like, we have $\Sigma_{ir}|\pi_{ir}| \leq |\pi|$. It is easy to see that $|C| \leq \Sigma_{i,r}|\pi_{i,r}|^2 + N_{\mathcal{R}} \leq \Sigma_{i,r}|\pi_{i,r}|^2 + \Sigma_{i,r}|\pi_{i,r}| \leq (\Sigma_{i,r}|\pi_{i,r}|)^2 \leq |\pi|^2$, and the claim follows. We have usesd the fact that $N_{\mathcal{R}} \leq \Sigma_{i,r}|\pi_{i,r}|$. The latter is an easy consequence of the fact that the number of $\pi_{i,r}$ in total is $N_{\mathcal{R}}$.

Finally, for $\mathcal{P}$-monotonicity note that since Γ is $\mathcal{P}$-monotone and all the rules are PPF, all the antecedents in the proof must be $\mathcal{P}$-monotone, as well. Therefore, the interpolants of the axioms are $\mathcal{P}$-monotone. Because, for the axioms, except for the axiom $\phi \Rightarrow \phi$, the interpolants are variable-free and hence $\mathcal{P}$-monotone. And for the identity axiom $\phi \Rightarrow \phi$, the interpolant is ϕ itself which is also $\mathcal{P}$-monotone. Finally, since the interpolants are constructed by the interpolants of the axioms via disjunctions and conjunctions, the interpolant for $\Gamma \Rightarrow \Delta$ is also $\mathcal{P}$-monotone. □

The following theorem is our first example of the mentioned ineffectiveness of the combination of focused axioms and PPF rules. It shows that none of the combinations of focused axioms and PPF rules can simulate the cut rule in a feasible way.

Corollary 3.11. *There is no calculus G consisting of only focused axioms and* PPF *rules, sound and feasibly complete for* **CPC**. *More precisely, if G is a complete calculus for* **CPC***, then there exists a sequence of* **CPC***-valid sequents $\phi_n \Rightarrow \psi_n$, with polynomially short tree-like proofs in the Hilbert-style system or equivalently in* **LK**+**Cut** *such that $||\phi_n \Rightarrow \psi_n||_G$, the length of the shortest tree-like G-proof of $\phi_n \Rightarrow \psi_n$, is exponential in n. Therefore, the* PPF *rules together with focused axioms are either incomplete or feasibly incomplete for* **CPC**.

Proof. Assume that G is a calculus for **CPC** consisting of PPF rules and focused axioms. In the following, we bring the definitions for clique and coloring formulas from [8]. Note that we use $[n]$ to denote $\{1, 2, \cdots, n\}$. Let $Clique^k_n(\bar{p}, \bar{q})$ be the proposition asserting that $\bar{q}$ is a clique of size at least k on a graph with vertices $[n]$. There are $\binom{n}{2}$ atoms p_{ij} where $p_{ij} = 1$ if and only if there is an edge between nodes $\{i, j\} \in \binom{n}{2}$. There are also $k.n$ atoms q_{ui} where their intended meaning is to describe a mapping from $[k]$ to $[n]$. $Clique^k_n(\bar{p}, \bar{q})$ is the conjunction of the following clauses:

- $\bigvee_{i \in [n]} q_{ui}$, all $u \leq k$,
- $\neg q_{ui} \vee \neg q_{uj}$, all $u \in [k]$ and $i \neq j \in [n]$,
- $\neg q_{ui} \vee \neg q_{vi}$, all $u \neq v \in [k]$ and $i \in [n]$,
- $\neg q_{ui} \vee \neg q_{vj} \vee p_{ij}$, all $u \neq v \in [k]$ and $\{i, j\} \in \binom{n}{2}$.

The proposition $Color^m_n(\bar{p}, \bar{r})$ asserts that $\bar{r}$ is an m-coloring of the same graph represented by $\bar{p}$ and also uses $n.m$ atoms r_{ia} where $i \in [n]$ and $a \in [m]$. $Color^m_n(\bar{p}, \bar{r})$ is the conjunction of the following clauses:

- $\bigvee_{a \in [m]} r_{ia}$, all $i \in [n]$,
- $\neg r_{ia} \vee \neg r_{ib}$, all $a \neq b \in [m]$ and $i \in [n]$,
- $\neg r_{ia} \vee \neg r_{ja} \vee \neg p_{ij}$, all $a \in [m]$ and $\{i, j\} \in \binom{n}{2}$.

Note that by the formalization of the Clique formula, every occurrence of $\bar{p}$ in $Clique^k_n(\bar{p}, \bar{q})$ is positive (which means it is monotone in $\bar{p}$). We know that for $m < k$, the formula $\neg Clique^k_n(\bar{p}, \bar{q}) \vee \neg Color^m_n(\bar{p}, \bar{r})$ is a tautology in classical logic which implies that

$$Clique_n^k(\bar{p}, \bar{q}) \Rightarrow \neg Color_n^m(\bar{p}, \bar{r})$$

is **CPC**-valid.

First observe that by the Craig interpolation theorem for **CPC** and the fact that the antecedent is monotone in $\bar{p}$, there exists a monotone interpolant $I(\bar{p})$ such that

$$Clique_n^k(\bar{p}, \bar{q}) \Rightarrow I(\bar{p}) \Rightarrow \neg Color_n^m(\bar{p}, \bar{r})$$

which means that if the graph H represented by $\bar{p}$ has a k-clique then $I(\bar{p}) = 1$ and if H has an m-coloring then $I(\bar{p}) = 0$. In other words, if $I(\bar{p}) \neq 0$ then H does not have an m-coloring and if $I(\bar{p}) \neq 1$ then H does not have a k-clique. By the result in [2], every such monotone interpolant I must have exponential length in n for suitable polynomially bounded choices for k and m.

Secondly, define $\phi_n(\bar{p}, \bar{q}) = Clique_n^k(\bar{p}, \bar{q})$ and $\psi_n(\bar{p}, \bar{r}) = \neg Color_n^m(\bar{p}, \bar{r})$. We will show that this family of sequents, $\phi_n(\bar{p}, \bar{q}) \Rightarrow \psi_n(\bar{p}, \bar{r})$, serve as the **CPC**-valid sequents mentioned in the theorem. The idea is simple. First note that the fact that the sequent

$$Clique_n^k(\bar{p}, \bar{q}) \Rightarrow \neg Color_n^m(\bar{p}, \bar{r})$$

has a tree-like proof of the size $n^{O(1)}$ in the classical Hilbert-style proof system or equivalently **LK** + **Cut** is a folklore well-known fact in the proof complexity community. Now pick π_n as the shortest tree-like proof of the sequent in G. Note that the antecedent of our sequent, $Clique_n^k(\bar{p}, \bar{q})$, is $\bar{p}$-monotone. Hence, by Lemma 3.10, the interpolant for the sequent $\phi_n(\bar{p}, \bar{q}) \Rightarrow \psi_n(\bar{p}, \bar{r})$ will be $\bar{p}$-monotone, as well. And since $\bar{p}$ are the only common variables and hence the only variables in the interpolant, the interpolant is monotone. However, G captures **CPC**. Therefore, the whole process provides a classical monotone interpolant for the sequent

$$Clique_n^k(\bar{p}, \bar{q}) \Rightarrow \neg Color_n^m(\bar{p}, \bar{r})$$

which we will call C_n. By Lemma 3.10, we have $|C_n| \leq |\pi_n|^2$. However, any such C_n should be exponentially long in n as we explained before. Therefore, the shortest proof π_n for our sequent is exponentially long. □

3.2 The Intuitionistic Case

It is also possible to lower down the previous exponential lower bound to the level of the **IPC**-valid sequents. For that purpose we need a new form of interpolation and its preservation theorem.

Definition 3.12. A sequent is called a *strongly focused* axiom if it has one of the following forms:

(1) $\phi \Rightarrow \phi$
(2) $\Rightarrow \bar{\alpha}$

(3) $\bar{\beta} \Rightarrow$
(4) $\Gamma, \bar{\phi} \Rightarrow \Delta$
(5) $\Gamma \Rightarrow \bar{\phi}, \Delta$

where in (2) and (5), $\bar{\alpha}$ and $\bar{\phi}$ have no variable and Γ and Δ are meta-multiset variables.

Example 3.13. For the strongly focused axioms, note that all the axioms of **LK** are strongly focused. An example of a focused axiom which is not strongly focused is $(\Rightarrow \phi, \neg\phi)$. Since otherwise it would have been an instance of either 2 or 5, which is not possible. The reason is that ϕ can have a variable which must not appear in the right side of the sequent.

Definition 3.14. Let G and H be two sequent calculi. G has *H-monotone feasible interpolation with the exponent* $m \geq 1$ if for any k and any sequent $S = (\Sigma \Rightarrow \Lambda_1, \cdots, \Lambda_k)$ if S is provable in G by a tree-like proof π and for any $1 \leq j \leq k$, $\Lambda_j \neq \emptyset$, then there exist formulas $|C_j| \leq |\pi|^m$ for $1 \leq j \leq k$ such that $(\Sigma \Rightarrow C_1, \cdots, C_k)$ and $(C_j \Rightarrow \Lambda_j)$ are provable in H and $V(C_j) \subseteq V(\Sigma) \cap V(\Lambda_j)$, where $V(A)$ is the set of the atoms of A. Moreover, if Σ is monotone, then C_j is also monotone for all $1 \leq j \leq k$. We call C_j's, the interpolants of the partition $\Lambda_1, \cdots, \Lambda_k$ of the succedent of the sequent S. The system G has *H-monotone feasible interpolation* if it has H-monotone feasible interpolation with some exponent $m \geq 1$.

Theorem 3.15. *Let G and H be two sequent calculi such that G is a set of strongly focused axioms, H extends* **mLK** *and any sequent in G is provable in H. Then G has H-monotone feasible interpolation with the exponent one.*

Proof. We will consider the strongly focused axioms one by one:

(1) In this case the sequent S is of the form $(\phi \Rightarrow \phi)$. Therefore, $\Lambda_1 = \phi$. Pick $C_1 = \phi$. It is easy to see that this C_1 works and since ϕ is monotone, C_1 is also monotone.
(2) For the case $(\Rightarrow \bar{\alpha})$, consider C_j to be $\bigvee \Lambda_j$. We can easily see that these C_j's work, using the left and right disjunction rules. For the variables, since $V(\bar{\alpha}) = \emptyset$, we have $V(C_j) \subseteq V(\emptyset) \cap V(\Lambda_j)$. And for the monotonicity, since $V(C_j) = \emptyset$, then C_j is monotone.
(3) The case $(\bar{\beta} \Rightarrow)$ does not happen.
(4) If S is of the form $\Gamma, \bar{\phi} \Rightarrow \Delta$ define $C_j = \bot$. First note that we have $\Gamma, \bar{\phi} \Rightarrow \bot, \bot, \cdots, \bot$ where in the right hand-side we have k many $\bot$'s. The reason is that this sequent is an instance of the axiom (4) itself. Moreover, for every j we have $\bot \Rightarrow \Lambda_j$ since it is an instance of the axiom $\bot$. And again $V(C_j) = \emptyset$.
(5) If S is of the form $(\Gamma \Rightarrow \bar{\phi}, \Delta)$ define $C_j = \bigvee(\Lambda_j \cap \bar{\phi})$. It is easy to see that this C_j works. Because, $C_j \Rightarrow \Lambda_j$ is an instance of an axiom. We also have $\Gamma \Rightarrow C_1, \cdots, C_k$, since in the right hand-side we will have the formula $\bar{\phi}$ (together with some other formulas which we will treat as the context) and it will become an instance of the same axiom. Note that since $V(\bar{\phi}) = \emptyset$, there is nothing to check for the variables. For the monotonicity, note that $V(C_j) = \emptyset$, therefore C_j is monotone.

Note that in all cases and for all $1 \leq j \leq k$, $|C_j| \leq |\pi|$. □

The next theorem shows that MPF rules preserve the monotone feasible interpolation property. We will use this theorem later in the lower bound result that we have promised before.

Theorem 3.16. *(monotone feasible interpolation) Let G and H be two sequent calculi such that H extends* **mLK** *and axiomatically extends G by* MPF *rules. Then if G has H-monotone feasible interpolation property, so does H.*

Proof. To prove the theorem, we will prove the following claim:

Claim. Let G and H be two sequent calculi such that H extends **mLK** and axiomatically extends G by MPF rules and G has H-monotone feasible interpolation with the exponent m. Then for any H-provable sequent $\Gamma \Rightarrow \Delta$ and any non-trivial partition of Δ as $\Lambda_1, \cdots, \Lambda_k$ (non-trivial means that none of the Λ_j's are empty), there exist the required interpolants C_j as in the Definition 3.14 such that $\Sigma_j |C_j| \leq |\pi|^M$ where $M = m + 1$.

The proof uses induction on the H-length of π (the number of the rules of H in the proof π). First we will explain how to construct C_j's. Then we will prove the bound for the given construction.

If the H-length of π is zero, it means that the proof is in G. Hence the claim is clear by the assumption. There are two cases to consider based on the last rule of the proof.

- If the last rule used in the proof is a right focused one, then it is of the following form:

$$\frac{\langle\langle \Gamma_i \Rightarrow \bar{\phi}_{ir}, \Delta_i \rangle_r \rangle_i}{\Gamma \Rightarrow \phi, \Delta}$$

  where $\Gamma = \Gamma_1, \cdots, \Gamma_n$ and $\Delta = \Delta_1, \cdots, \Delta_n$. And, again $\Lambda_1, \cdots, \Lambda_k$ are given such that they are non-empty and $\bigcup_{j=1}^{k} \Lambda_j = \Delta \cup \{\phi\}$. W.l.o.g. suppose $\phi \in \Lambda_1$ and we denote $\Lambda_1 - \{\phi\}$ by Λ'_1. Consider the case that all of the $\Lambda_{ij} = \Delta_i \cap \Lambda_j$ and $\bar{\phi}_{ir} \cup \Lambda'_{i1}$ are non-empty where $\Lambda'_{i1} = \Delta_i \cap \Lambda'_1$. By induction hypothesis for the premises, there exist formulas $D_{ir1}, \cdots, D_{irk}$ such that for every i, r and $j \neq 1$

$$D_{ir1} \Rightarrow \bar{\phi}_{ir}, \Lambda'_{i1} \quad , \quad D_{irj} \Rightarrow \Lambda_{ij} \quad , \quad \Gamma_i \Rightarrow D_{ir1}, \cdots, D_{irk}$$

  Again, note that if some of Λ_{ij}'s or $\bar{\phi}_{ir}, \Lambda'_{i1}$ are empty, we can eliminate them from the partition to have a non-trivial partition and hence to apply the IH. Then in these cases, we can simply pick D_{irj} as $\bot$. Now using the rules $(R\vee)$, $(L\vee)$, $(R\wedge)$ and $(L\wedge)$, we get for every i and $j \neq 1$

$$\bigwedge_r D_{ir1} \Rightarrow \bar{\phi}_{ir}, \Lambda'_{i1} \ , \ \bigvee_r D_{irj} \Rightarrow \Lambda_{ij} \ , \ \Gamma_i \Rightarrow \bigwedge_r D_{ir1}, \bigvee_r D_{ir2}, \cdots, \bigvee_r D_{irk}$$

Note that in the right sequent, we first use $(R\vee)$ to get $\Gamma_i \Rightarrow D_{ir1}, \bigvee_r D_{ir2}, \cdots, \bigvee_r D_{irk}$, and then we can use the rule $(R\wedge)$. Now, we can substitute the left sequents in the original rule to get

$$\bigwedge_r D_{ir1} \Rightarrow \phi, \Lambda'_1$$

and using the rule $(L\wedge)$ we have

$$\bigwedge_i \bigwedge_r D_{ir1} \Rightarrow \phi, \Lambda'_1$$

We denote $\bigwedge_i \bigwedge_r D_{ir1}$ by C_1. Using the rule $(L\vee)$ for the sequents $\bigvee_r D_{irj} \Rightarrow \Lambda_{ij}$ we get

$$\bigvee_i \bigvee_r D_{irj} \Rightarrow \Lambda_j$$

and we denote $\bigvee_i \bigvee_r D_{irj}$ by C_j for $j \neq 1$. We can see that first using the rule $(R\vee)$ and after that using the rule $(R\wedge)$ we get

$$\Gamma \Rightarrow \bigwedge_i \bigwedge_r D_{ir1}, \bigvee_i \bigvee_r D_{ir2}, \cdots, \bigvee_i \bigvee_r D_{irk}$$

which is

$$\Gamma \Rightarrow C_1, \cdots, C_k$$

It only remains to check the variables. If a variable is in C_j, then it is in one of D_{irj}'s. By induction hypothesis we have $V(D_{ir1}) \subseteq V(\Gamma_1) \cap V(\{\{\bar{\phi}_{ir}\} \cup \Lambda'_{i1}\}) \subseteq V(\Gamma) \cap V(\{\{\phi\} \cup \Lambda'_1\})$ and $V(D_{irj}) \subseteq V(\{\Gamma_i\}) \cap V(\Lambda_{ij}) \subseteq V(\Gamma) \cap V(\Lambda_j)$, since the rule is occurrence preserving, and this is what we wanted.

- The case of the left focused rule is similar to the case for right.

For the monotonicity part, since the extending rules are MPF, it is easy to prove that if the antecedent of the consequence is monotone, then all the antecedents, everywhere in the proof up to the sequents in G, are also monotone. Since G has H-monotone feasible interpolation property, the interpolants in the base case are monotone. Finally, since the conjunctions and disjunctions do not change monotonicity, our constructed interpolants are also monotone.

For the upper bound part, use a similar proof to the corresponding part in Lemma 3.10, this time using the induction on π to show that $\Sigma_j |C_j| \leq |\pi|^M$. For the axioms note $|C_j| \leq |\pi|^m$ for $1 \leq j \leq k$ by the assumption that G has H-monotone feasible interpolation with the exponent m. Since the partition is non-trivial $k \leq |S| \leq |\pi|$, hence $\Sigma_{j=1}^{k} |C_j| \leq k|\pi|^m \leq |\pi|^{m+1} = |\pi|^M$.
For the rules, define X as the set of all (i, r, j)'s where D_{irj} is $\bot$ coming from handling the empty cases. It is clear that X has at most $N_{\mathcal{R}}$ elements, the number of the premises of the rule $\mathcal{R}$. We have $\Sigma_j |C_j| \leq \Sigma_{(i,r,j) \notin X} |D_{irj}| + |X| + N_{\mathcal{R}} \leq$

$\Sigma_{ir}|\pi_{ir}|^M + 2N_{\mathcal{R}} \leq (\Sigma_{ir}|\pi_{ir}| + 1)^M \leq |\pi|^M$. The second inequality holds using the induction hypothesis and the third inequality holds because $N_{\mathcal{R}} \leq \Sigma_{ir}|\pi_{ir}|$ and $M \geq 2$.

Finally, the theorem is a clear consequence of the Claim. It is enough to apply the Claim to provide the formulas C_j such that $\Sigma_j|C_j| \leq |\pi|^M$ which implies $|C_j| \leq |\pi|^M$. □

Lemma 3.17. *[5] Let $A(\bar{p}, \bar{r_1})$ and $B(\bar{q}, \bar{r_2})$ be propositional formulas and $\bar{p}$, $\bar{q}$, $\bar{r_1}$ and $\bar{r_2}$ be mutually disjoint. Let $\bar{p} = p_1, \cdots, p_n$ and $\bar{q} = q_1, \cdots, q_n$. Assume that A is monotone in $\bar{p}$ or B is monotone in $\bar{q}$ and $A(\bar{p}, \bar{r_1}) \vee B(\neg\bar{p}, \bar{r_2})$ is a classical tautology. Then*

$$\bigwedge_{i=1}^{n}(p_i \vee q_i) \Rightarrow \neg\neg A(\bar{p}, \bar{r_1}), \neg\neg B(\bar{q}, \bar{r_2})$$

is **IPC***-valid.*

Proof. For the details, the reader is referred to [5]. □

Theorem 3.18. *Let G and H be two sequent calculi such that H is sub-classical, extends* **mLK***, axiomatically extends G by* MPF *rules and G has H-monotone feasible interpolation property. Then there exists a family of* **IPC***-valid sequents $\phi_n \Rightarrow \psi_n$ with the length of $\phi_n \Rightarrow \psi_n$ bounded by a polynomial in n such that either there exists some n such that $H \nvdash \phi_n \Rightarrow \psi_n$ or $||\phi_n \Rightarrow \psi_n||_H$, the shortest tree-like H-proof of $\phi_n \Rightarrow \psi_n$, is exponential in n. Therefore, the* MPF *rules together with strongly focused axioms are either incomplete or feasibly incomplete for* **IPC**.

Proof. The proof is similar and also inspired by the lower bound proof given in [5]. Similar to the proof of Theorem 3.11, consider the **CPC**-valid sequent

$$Clique_n^k(\bar{p}, \bar{r_2}) \Rightarrow \neg Color_n^m(\bar{p}, \bar{r_1})$$

which is equivalent to

$$\Rightarrow \neg Clique_n^k(\bar{p}, \bar{r_2}), \neg Color_n^m(\bar{p}, \bar{r_1})$$

Then, using the Lemma 3.17, if we rewrite $\neg Clique_n^k(\bar{p}, \bar{r_2})$ as $B(\neg\bar{p}, \bar{r_2})$ and $\neg Color_n^m(\bar{p}, \bar{r_1})$ as $A(\bar{p}, \bar{r_1})$, we can easily see that A is monotone in $\bar{p}$ and $A(\bar{p}, \bar{r_1}) \vee B(\neg\bar{p}, \bar{r_2})$ is a classical tautology. Hence, we can transfer the **CPC**-valid sequent

$$\Rightarrow \neg Clique_n^k(\bar{p}, \bar{r_2}), \neg Color_n^m(\bar{p}, \bar{r_1})$$

to a sequent of the form

$$\bigwedge_i(p_i \vee q_i) \Rightarrow \neg\neg A(\bar{p}, \bar{r_1}), \neg\neg B(\bar{q}, \bar{r_2})$$

valid in **IPC**. Now, let

$$\phi_n(\bar{p}, \bar{q}) \Rightarrow \psi_n(\bar{p}, \bar{r_1}), \theta_n(\bar{q}, \bar{r_2})$$

be this sequent. We will show that this family of sequents, $\phi_n(\bar{p},\bar{q}) \Rightarrow \psi_n(\bar{p},\bar{r_1}), \theta_n(\bar{q},\bar{r_2})$, serve as the **IPC**-valid sequents mentioned in the theorem.

If for some n we have $H \nvdash \phi_n \Rightarrow \psi_n, \theta_n$, then the claim follows. Therefore, suppose that for every n we have $H \vdash \phi_n \Rightarrow \psi_n, \theta_n$. Let π_n be the shortest tree-like proof of the sequent $\phi_n \Rightarrow \psi_n, \theta_n$ in H. By Theorem 3.16, for every n, there exist monotone formulas $C_n(\bar{p})$ and $D_n(\bar{q})$ such that $|C_n| \leq |\pi_n|^{O(1)}$ and $|D_n| \leq |\pi_n|^{O(1)}$ and the followings are provable in H: $(\phi_n \Rightarrow C_n, D_n)$, $(C_n \Rightarrow \psi_n)$, $(D_n \Rightarrow \theta_n)$. Since H captures a sub-classical logic we have $(\phi_n \Rightarrow C_n, D_n)$, $(C_n \Rightarrow \psi_n)$, $(D_n \Rightarrow \theta_n)$ in **CPC**. Since $(\phi_n \Rightarrow C_n, D_n)$ is valid in classical logic, we have $C_n(\bar{p}) \vee D_n(\neg\bar{p}) = 1$. On the other hand, since A_n is classically equivalent to ψ_n we know that $C_n(\bar{p}) = 1$ implies $A(\bar{p},\bar{r_1}) = 1$. Similarly, we have that $D_n(\bar{q}) = 1$ implies $B(\bar{q},\bar{r_2}) = 1$. We Claim that $C_n(\bar{p})$ interpolates $\neg B(\neg\bar{p},\bar{r_2}) \Rightarrow A(\bar{p},\bar{r_1})$. One direction is proved. For the other direction, note that if $B(\neg\bar{p},\bar{r_2}) = 0$ then $D_n(\neg\bar{p}) = 0$ and since $C_n(\bar{p}) \vee D_n(\neg\bar{p}) = 1$ we have $C_n(\bar{p}) = 1$. Hence the monotone formula C_n interpolates $\neg B(\neg\bar{p},\bar{r_2}) \Rightarrow A(\bar{p},\bar{r_1})$ or equivalently the sequent

$$Clique^k_n(\bar{p},\bar{r_2}) \Rightarrow \neg Color^m_n(\bar{p},\bar{r_1})$$

However, in the proof of the Theorem 3.11, we mentioned that any such monotone interpolant must have exponential length. Together with the fact that $|C_n(\bar{p})| \leq |\pi_n|^{O(1)}$, we can conclude that $||\phi_n \Rightarrow \psi_n, \theta_n||_H$ is exponential in n which implies the claim. □

Corollary 3.19. *There is no calculus consisting only of strongly focused axioms and* MPF *rules, sound and feasibly complete for super-intuitionistic logics.*

Proof. This is an obvious consequence of Theorem 3.16, Theorem 3.15 and Theorem 3.18 . The only point that we have to explain is that if a calculus G consisting only of strongly focused axioms and MPF rules is sound and complete for a super-intuitionistic logic, then G extends **mLK**. The reason is that G is complete for a super-intuitionistic logic and any calculus complete even for **IPC** extends **mLK**. □

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