

INSTITUTE OF MATHEMATICS

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Martin Doležal

Jan Hladký

Jan Kolář

Themis Mitsis

Christos Pelekis

Václav Vlasák

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A Turán-type theorem for large-distance graphs in Euclidean spaces, and related isodiametric problems

Martin Doležal · Jan Hladký · Jan Kolář · Themis Mitsis ·
Christos Pelekis · Václav Vlasák

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Abstract Given a measurable set $A \subset \mathbb{R}^d$ we consider the *large-distance graph* \mathcal{G}_A , on the ground set A , in which each pair of points from A whose distance is bigger than 2 forms an edge. We consider the problems of maximizing the $2d$ -dimensional Lebesgue measure of the edge set as well as the d -dimensional Lebesgue measure of the vertex set of a large-distance graph in the d -dimensional Euclidean space that contains no copies of a complete graph on k vertices. The former problem may be seen as a continuous analogue of Turán’s classical graph theorem, and the latter as a “graph-theoretic” analogue of the classical isodiametric problem. Our main result yields an analogue of Mantel’s theorem for large-distance graphs. Our approach employs an isodiametric inequality in an annulus, which might be of independent interest.

Keywords Turán’s theorem · isodiametric problem · distance graph

Mathematics Subject Classification (2010) 05C63 · 51K99 · 05C35 · 51M16

1 Prologue, related work and main results

Let us begin with a folklore result of Turán which pertains to graphs that contain no complete graph (also called a *clique*) on k vertices. Given a graph $G = (V, E)$, we denote by $|V|$ and $|E|$ the number of its vertices and edges, respectively.

Theorem 1.1 (Turán [21]) *Let $G = (V, E)$ be a graph which does not contain a complete graph on k vertices. Then $|E| \leq \frac{1}{2} \left(1 - \frac{1}{k-1}\right) \cdot |V|^2$.*

Martin Doležal

Institute of Mathematics, Czech Academy of Sciences, Žitná 25, 115 67, Praha 1, Czech Republic. Research supported by the GAČR project 17-27844S and RVO: 67985840. E-mail: dolezal@math.cas.cz

Jan Hladký

Institute of Mathematics, Czech Academy of Sciences, Žitná 25, Praha 1, Czech Republic. Research supported by GAČR project 18-01472Y and RVO: 67985840. E-mail: honzahladky@gmail.com

Jan Kolář

Institute of Mathematics, Czech Academy of Sciences, Žitná 25, Praha 1, Czech Republic. Research supported by the EPSRC grant EP/N027531/1 and by RVO: 67985840. E-mail: kolar@math.cas.cz

Themis Mitsis

Department of Mathematics and Applied Mathematics, University of Crete, 70013 Heraklion, Greece. E-mail: themis.mitsis@gmail.com

Christos Pelekis

Institute of Mathematics, Czech Academy of Sciences, Žitná 25, Praha 1, Czech Republic. Research supported by the Czech Science Foundation, grant number GJ16-07822Y, by GAČR project 18-01472Y and RVO: 67985840. E-mail: pelekis.chr@gmail.com

Václav Vlasák

Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 18675 Praha 8, Czech Republic. E-mail: vlasakvv@gmail.com

For $k = 3$, this result is due to Mantel (see [15]). The extremal graph is obtained by dividing the vertex set into $k - 1$ pairwise disjoint subsets whose sizes are as equal as possible, and by joining two vertices with an edge if and only if they belong to different subsets. In other words, the extremal graph is the complete balanced $(k - 1)$ -partite graph. Turán's theorem is a fundamental result in extremal graph theory that has been generalised in a plethora of ways (see [3, 20]).

In this paper, we consider similar extremal questions for measure graphs whose edge set definition is based on distances in an Euclidean space.

1.1 Large-distance graphs

Already in the 1970's it was realized (see [2, 9, 10, 11]) that several results from extremal graph theory have measure-theoretic counterparts. In this setting, one is interested in the maximum “number of edges” in *measure graphs*, i.e., graphs whose vertex set corresponds to some measure space X and whose edge set corresponds to a symmetric subset of the product space that does not intersect the diagonal, i.e., it contains no points of the form (x, x) , where $x \in X$. This line of research received a substantial impetus recently due to the emergence of the theory of limits of dense graphs (see [13]). In this work we look at a special case of measure graphs whose vertex-set is formed by the points of a measurable set in the Euclidean space, and whose edge-set is formed by pairs of vertices that are at sufficiently large distance. In order to be more precise we need to introduce some piece of notation.

Here and later, $\lambda_d(\cdot)$ denotes d -dimensional Lebesgue measure. Given a point $p \in \mathbb{R}^d$ and a positive real r , we denote by $D(p, r)$ the set of points whose distance from p is less than or equal to r , and by $D(p, r)^\circ$ its interior. The boundary of $D(p, r)$ will be denoted as $S(p, r)$. If $r = 1$ we will refer to $D(p, 1)$ as a *unit ball*, or a *unit d -ball* if we want to emphasize the dimension of the underlying Euclidean space. The d -dimensional Lebesgue measure of $D(p, 1)$ is denoted ω_d ; recall that $\omega_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$. The Euclidean distance between two points $p, q \in \mathbb{R}^d$ is denoted $\|p - q\|$ and the distance between two sets $A, B \subset \mathbb{R}^d$ is defined as

$$\text{dist}(A, B) = \inf\{\|a - b\| : a \in A, b \in B\}.$$

We shall be interested in the “maximum number” of edges in graphs that are associated to the distance set of measurable subsets of the Euclidean space, and are defined as follows.

Definition 1.2 (Large-distance graphs) Let A be a measurable subset of \mathbb{R}^d . The *large-distance graph* corresponding to A , denoted \mathcal{G}_A , is defined as follows: The vertex set of \mathcal{G}_A is the set A . The edge set of \mathcal{G}_A , denoted \mathcal{E}_A , is defined as

$$\mathcal{E}_A = \{(p, q) \in A \times A : \|p - q\| > 2\}.$$

The “number of edges” in \mathcal{G}_A is defined as $e(\mathcal{G}_A) := \frac{1}{2} \cdot \lambda_{2d}(\mathcal{E}_A)$.

Let us remark that the reason for setting the distance threshold to 2 is that our main results have a particularly nice formulation. On the other hand, only trivial rescaling would have to be introduced in order to change the distance threshold to any other positive number.

To the best of our knowledge, large-distance graphs corresponding to measurable subsets of the Euclidean space have not been systematically studied so far. There exists a vast amount of literature on the, so-called, *distance graphs*, where two points are joined with an edge if their distance is equal to 1 (see, for example, [19]), but most research is mainly driven by the well-known problem regarding the chromatic number of the plane. A rather similar graph, whose vertex set corresponds to points of the d -dimensional sphere and whose edge set is formed by pairs of points that are at sufficiently large Euclidean distance, is the so-called *Borsuk graph* (see [16, p. 30], or [8]). However, most of the related research so far appears to focus on the chromatic number of Borsuk graphs. In this article we focus on the problem of maximizing the “amount” of vertices as well as the “amount” of edges in large-distance graphs subject to the constraint that they do not contain a copy of a fixed finite graph. The main object of our study in large-distance graphs is introduced in the following.

Definition 1.3 (*H-free large-distance graphs*) Suppose that $A \subset \mathbb{R}^d$ is measurable and let $H = (V(H), E(H))$ be a finite, simple, graph on $k \geq 2$ labeled vertices, whose labels are represented by the set $\{1, \dots, k\}$. Let \mathcal{G}_A be the large-distance graph corresponding to A , and let $\mathcal{G}_A \langle H \rangle \subset A^k$ be the set consisting of all k -tuples, (p_1, \dots, p_k) , of points in A for which $(p_i, p_j) \in \mathcal{E}_A$ whenever $ij \in E(H)$. We say that \mathcal{G}_A is *H-free* if $\mathcal{G}_A \langle H \rangle = \emptyset$. We say that \mathcal{G}_A is *essentially H-free* if $\lambda_{kd}(\mathcal{G}_A \langle H \rangle) = 0$.

Of course, the most basic case to investigate is the case of K_k -free large-distance graphs, where K_k denotes a clique of order k . We shall be interested in the problem of maximizing the d -dimensional Lebesgue measure of the vertex set, as well as in the problem of maximizing the $2d$ -dimensional Lebesgue measure of the edge set of a K_k -free large-distance graph corresponding to a measurable subset of \mathbb{R}^d . The former problem has been considered in [17], where some bounds are obtained in the 2-dimensional case. The latter has been considered in [2], where a Turán-type theorem is obtained in the setting of measure graphs. In this article we further investigate both problems and we additionally demonstrate that they are interrelated.

Let us now state a counterpart of Turán's theorem for large-distance graphs.

Theorem 1.4 *Fix a positive integer $k \geq 2$. Suppose that $A \subset \mathbb{R}^d$ is a measurable set such that \mathcal{G}_A is essentially K_k -free.*

Then

$$e(\mathcal{G}_A) \leq \frac{1}{2} \cdot \left(1 - \frac{1}{k-1}\right) \cdot \lambda_d(A)^2. \quad (1.1)$$

Moreover, the inequality becomes an equality if and only if there exists $c \in [0, \omega_d]$ such that A is a disjoint union of a set of λ_d -measure zero and $k-1$ disjoint sets A_1, \dots, A_{k-1} whose pairwise distances are at least 2 and satisfy $\text{diam}(A_i) \leq 2$ and $\lambda_d(A_i) = c$, for all $i = 1, \dots, k-1$.

Theorem 1.4 is not really new. The bound (1.1) follows from an old result of Bollobás, [2, Theorem 1]. In Section 2, we deduce it, including the moreover part, using the formalism of graphons.

Remark 1.5 Note that the constant c in Theorem 1.4 cannot be arbitrary. Indeed the classical Isodiametric inequality (see e.g. [5, Theorem 11.2.1]) asserts that if $X \subset \mathbb{R}^d$ is a measurable set of diameter at most 2 then $\lambda_d(X) \leq \omega_d$. The equality is attained only when X is a unit ball (modulo a nullset).

At this point it may seem that the setting of large-distance graphs does not provide anything new: the bound in the above theorem follow in a rather straightforward way from a general theory of measure graphs (and from the recent progress on graph limits), and there are obvious constructions showing that these general bounds are tight even within the class of large-distance graphs. There is one important difference, though. Turán's theorem is scale-free; for example knowing the optimal construction of a 100-vertex triangle-free graph allows us also to construct the optimal 100000-vertex triangle-free graph. However, this is not the case for large-distance graphs as we saw in Remark 1.5. So, we ask for absolute bounds on the measure of vertices and edges an K_k -free large-distance graph may have. We call this the Clique-isodiametric problem.

Problem 1.6 (Clique-isodiametric problem) Suppose that we are given $k, d \in \mathbb{N}$. Find

$$\mathfrak{V}_{d,k} = \sup_A \lambda_d(A) \quad \text{and} \quad \mathfrak{E}_{d,k} = \sup_A e(\mathcal{G}_A),$$

where the suprema range over all measurable K_k -free sets $A \subset \mathbb{R}^d$.

It is not difficult to see that $\mathfrak{V}_{d,k} \leq (k-1)2^d \omega_d$ and $\mathfrak{E}_{d,k} \leq (k-1)^2 2^{2d-1} \omega_d^2$, and in particular are finite. Indeed, a K_k -free set A cannot contain more than $k-1$ points $\{p_i\}$ that satisfy $\|p_i - p_j\| > 2$, whenever $i \neq j$. If $\{p_1, \dots, p_m\} \subset A$, where $m \leq k-1$, is a maximal family of points with this property then $A \subset \bigcup_{i=1}^m D(p_i, 2)$. Hence, by the Isodiametric inequality, $\lambda_d(A) \leq (k-1)2^d \omega_d$. Consequently, $e(\mathcal{G}_A) \leq (k-1)^2 2^{2d-1} \omega_d^2$ (and an improvement by a factor of $1 - \frac{1}{k-1}$ follows from Theorem 1.4).

Notice that the first part of Problem 1.6 makes sense even for $k=2$, i.e., when \mathcal{G}_A has no edges. Note that in this case Problem 1.6 is equivalent to the isodiametric problem.

We are unable to solve Problem 1.6 in general, and focus on the case $d=2$ and $k=3$. For this particular choice of parameters, we obtain the following 2-dimensional analogue of Mantel's theorem.

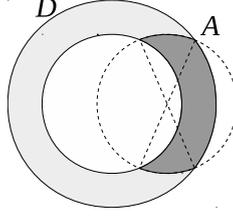


Fig. 1.1 A set A showing that the bound in Theorem 1.8 is optimal can be obtained by intersecting D with a unit disk placed in such a way that its center is in the intersection of diagonals of the “corners” of A . (The calculations necessary to verify this are implicit in our proof of Theorem 1.8.)

Theorem 1.7 *Let $A \subset \mathbb{R}^2$ be a measurable set for which \mathcal{G}_A is K_3 -free. Then $\lambda_2(A) \leq 2\pi$ as well as $e(\mathcal{G}_A) \leq \pi^2$.*

Moreover, each set attaining the first bound is (up to changes on a set of λ_2 -measure zero) a union of two non-overlapping unit balls. Each set attaining the second bound is (up to changes on a set of λ_2 -measure zero) a disjoint union of two unit balls whose distance is larger than or equal to 2.

In other words, the large-distance graph of an optimal K_3 -free set in \mathbb{R}^2 is a complete, λ_2 -balanced bipartite graph. The proof of Theorem 1.7 is based on the following result, which may be of independent interest.

Theorem 1.8 (Isodiametric inequality on the annulus) *Let $R \in [2\sqrt{2}, 4]$ and consider the set $D := D(0, R) \setminus D(0, 2)^o \subset \mathbb{R}^2$. Assume that A is a measurable subset of D such that $\text{diam}(A) \leq 2$. Then*

$$\lambda_2(A) \leq R^2 \arcsin\left(\frac{a}{R}\right) - 4 \arcsin\left(\frac{a}{2}\right) + 2 \arccos(a), \quad (1.2)$$

where $a = \sqrt{\frac{-R^4 + 16R^2}{8(R^2 + 2)}}$.

The bound (1.2) in Theorem 1.8 is optimal; see Figure 1.1.

1.2 Organisation of the paper

In Section 2 we provide a short proof of Theorem 1.4. In Section 3 we prove Theorem 1.7. The proof of Theorem 1.7 is based upon Theorem 1.8, whose proof is deferred to Section 4, and relies on Pólya’s circular symmetrization, which is employed along the perimeters of co-centric circles. Our paper ends with Section 5, in which we collect some remarks and an open problem.

2 Proof of Theorem 1.4

For our proof of Theorem 1.4, the formalism of graphons will be convenient. We actually need only very little, so we give a self-contained introduction. We refer the reader to [13] for a thorough treatise. Suppose that X is a probability measure space. A *graphon* is a symmetric measurable function $W : X^2 \rightarrow [0, 1]$. The density of a complete graph K_k in W is defined as

$$t(K_k, W) := \int_{x_1} \cdots \int_{x_k} \prod_{1 \leq i < j \leq k} W(x_i, x_j). \quad (2.1)$$

A graphon W is said to be K_k -free if $t(K_k, W) = 0$. The *edge density* of W is $t(K_2, W) = \int_x \int_y W(x, y)$.

Our proof of Theorem 1.4 is a rather straightforward application of the following folklore version of Turán’s theorem for graphons (see Corollary 16.11 in [13]).

Theorem 2.1 (Turán’s theorem for graphons) *Suppose that $k \in \mathbb{N}$, X is a probability measure space, and $W : X^2 \rightarrow [0, 1]$ is a K_k -free graphon. Then the edge density of W is at most $1 - \frac{1}{k-1}$. Moreover, if the edge density of W equals $1 - \frac{1}{k-1}$, then X can be partitioned into sets X_1, \dots, X_{k-1} of measure $\frac{1}{k-1}$ each, and such that W restricted to $X_i \times X_j$ equals to $\mathbf{1}_{i \neq j}$ (modulo a nullset) for each $i, j \in [k-1]$.*

Suppose now that A in Theorem 1.4 is given. There is nothing to prove if $\lambda_d(A) = 0$, so let us assume that $\lambda_d(A) > 0$. Also, recall that below Problem 1.6, we argued that $\lambda_d(A) < \infty$. So, let X be a measure space whose ground set is A and whose measure is the Lebesgue measure on A rescaled by $\frac{1}{\lambda_d(A)}$. Let $W(x, y)$ be 1 or 0, depending on whether xy forms an edge in \mathcal{G}_A or not. Since \mathcal{G}_A is essentially K_k -free, we have that W is K_k -free. Hence, Theorem 2.1 implies that the edge density of W is at most $1 - \frac{1}{k-1}$. Taking into account the rescaling, we get that $e(\mathcal{G}_A) \leq \frac{1}{2} \cdot (1 - \frac{1}{k-1}) \cdot \lambda_d(A)^2$, as was needed.

Let us now turn to the moreover part of the theorem. Suppose that we have an equality in (1.1). Hence, the edge density of W equals $1 - \frac{1}{k-1}$, and hence we can partition X into sets X_1, \dots, X_{k-1} as in Theorem 2.1. For each $i = 1, \dots, k-1$, let $A_i \subset X_i$ be the set of points of Lebesgue density 1 in A_i . We claim that for each pair $(c, d) \in A_i^2$ we have $\|c - d\| \leq 2$. Indeed, if there existed a single pair $(c, d) \in A_i^2$ with $\|c - d\| > 2$, then we could find a set $C \subset A_i$ (consisting only of points very close to c) of positive measure and a set $D \subset A_i$ (consisting only of points very close to d) of positive measure such that $\|c' - d'\| > 2$ for each $c' \in C$ and $d' \in D$. This would imply that W equals 1 on a subset of $X_i \times X_i$ of positive measure, which contradicts the assertion of Theorem 2.1. By similar reasoning, we get that if $i \neq j$ and $x \in A_i$ and $y \in A_j$, then $\|x - y\| \geq 2$. This concludes the proof of the moreover part of the theorem.

2.1 Discussion

Recall that some extremal questions for distance graphs were considered in [19]. However, there seems to be fundamental difference in extremal questions for distance graphs and for large-distance graphs. Large-distance graphs structurally behave similar to dense finite graphs, that is, graphs that have quadratically many edges in the number of vertices. This is exactly the class of graphs which is suitable for the graphon approach, and the above proof of Theorem 1.4 demonstrates this. The same type of reductions would yield optimal versions of the Erős–Stone Theorem, Razborov-Reiher Clique Density Theorem, and many other, for large-distance graphs.

Distance graphs, on the other hand, correspond more to sparse finite graphs. Let us point out that some constructions of sparse finite graphs in extremal graph theory (such as [12]) can be actually seen as finitarizations of certain distance graphs.

3 Proof of Theorem 1.7

In this section we prove Theorem 1.7. We begin by showing that $\lambda_2(A) \leq 2\pi$, where $A \subset \mathbb{R}^2$ is a measurable set for which \mathcal{G}_A is K_3 -free. The inner regularity of Lebesgue measure implies that it is enough to prove the result under the additional assumption that A is compact.

The proof is based upon Theorem 1.8, which we assume to be true throughout this section and whose proof is deferred to the next section. We distinguish three cases.

Suppose first that A satisfies $\text{diam}(A) < 2\sqrt{2}$. Then the Isodiametric inequality implies $\lambda_2(A) < 2\pi$, as desired.

Next assume that $\text{diam}(A) \geq 4$. Fix two points $p, q \in A$ such that $\|p - q\| = \text{diam}(A)$ and notice that $A \subset D(p, 2) \cup D(q, 2)$. Now observe that the set $A_p := A \cap D(p, 2)$ has diameter less than or equal to 2; indeed, if there existed two points $x, y \in A_p$ such that $\|x - y\| > 2$ then they would form together with the point q a triple of points all of whose pairwise distances are larger than 2, a contradiction to the fact that \mathcal{G}_A is K_3 -free. Similarly, we obtain that the set $A_q := A \cap D(q, 2)$ has diameter less than or equal to 2. The Isodiametric inequality yields

$$\lambda_2(A) \leq \lambda_2(A_p) + \lambda_2(A_q) \leq 2\pi, \quad (3.1)$$

as desired. Furthermore, in case of equality, $\lambda_2(A) = 2\pi$, we must have $\lambda_2(A_p) = \lambda_2(A_q) = \pi$. Hence, by the characterization of equality in the Isodiametric inequality, A_p and A_q are two unit balls, modulo a nullset. Since $\text{diam}(A) \geq 4$, their intersection is either empty or a singleton.

Hence we are left with the case $R := \text{diam}(A) \in [2\sqrt{2}, 4]$. Fix two points $x, y \in A$ such that $\|x - y\| = R$, and consider the sets

$$A_x = A \setminus D(x, 2) \text{ and } A_y = A \setminus D(y, 2).$$

For the same reasons as above, we have that $A \subset D(x, 2) \cup D(y, 2)$. Hence

$$A \subset A_x \cup A_y \cup (D(x, 2) \cap D(y, 2)). \quad (3.2)$$

Moreover, we have $\text{diam}(A_x), \text{diam}(A_y) \leq 2$ as well as

$$A_x \subset D(x, R) \setminus D(x, 2) \text{ and } A_y \subset D(y, R) \setminus D(y, 2).$$

Notation 3.1 Below, we define functions $g, a, h, H, f : [2\sqrt{2}, 4] \rightarrow \mathbb{R}$. We denote their derivatives as g', a', h', H', f' , respectively.

$$\begin{aligned} g(z) &= 4 \arccos\left(\frac{z}{4}\right) - \frac{z}{2} \sqrt{4 - \frac{z^2}{4}}, \\ a(z) &= \sqrt{\frac{-z^4 + 16z^2}{8(z^2 + 2)}}, \\ h(z) &= z^2 \arcsin\left(\frac{a(z)}{z}\right) - 4 \arcsin\left(\frac{a(z)}{2}\right) + 2 \arccos(a(z)), \\ H(z) &= z \frac{za'(z) - a(z)}{\sqrt{z^2 - (a(z))^2}} - \frac{4a'(z)}{\sqrt{4 - (a(z))^2}} - \frac{2a'(z)}{\sqrt{1 - (a(z))^2}}, \\ f(z) &= 2(g(z) + h(z)). \end{aligned}$$

Lemma 3.2 We have $\lambda_2(A) \leq f(R)$.

Proof It is not difficult to see that

$$\lambda_2(D(x, 2) \cap D(y, 2)) = 2g(R).$$

Moreover, by Theorem 1.8 we have

$$\lambda_2(A_x) \leq h(R) \text{ and } \lambda_2(A_y) \leq h(R).$$

Plugging this into (3.2), we get

$$\lambda_2(A) = \lambda_2(A_x) + \lambda_2(A_y) + \lambda_2(D(x, 2) \cap D(y, 2)) \leq f(R),$$

as required. \square

Clearly, we have $f(4) = 2\pi$ and Lemma 3.2 implies that it is enough to show that the function $f(\cdot)$ is increasing on the interval $[2\sqrt{2}, 4]$. Now, straightforward calculations provide the following expressions for the derivatives of the functions under consideration:

$$\begin{aligned} g'(z) &= -\frac{1}{2} \sqrt{16 - z^2}, \\ \frac{a'(z)}{a(z)} &= \frac{2z}{2a(z)^2} \frac{-z^4 - 4z^2 + 32}{8(z^2 + 2)^2} = \frac{(z^2 - 4)(z^2 + 8)}{z(z^2 + 2)(z^2 - 16)}, \\ a'(z) &= -\frac{(z^2 - 4)(z^2 + 8)}{(2(z^2 + 2))^{\frac{3}{2}} \sqrt{16 - z^2}}, \\ h'(z) &= 2 \arcsin\left(\frac{a(z)}{z}\right) z + H(z). \end{aligned} \quad (3.3)$$

Next, we want to express $H(z)$. To this end, write

$$\begin{aligned} Q &= 8(z^2 + 2), \\ S_1 &= z^2 - a(z)^2 = \frac{8z^4 + 16z^2 + z^4 - 16z^4}{8(z^2 + 2)} = \frac{9z^4}{Q}, \\ S_2 &= 4 - a(z)^2 = \frac{(z^2 + 8)^2}{Q}, \\ S_3 &= 1 - a(z)^2 = \frac{(z^2 - 4)^2}{Q}. \end{aligned}$$

Furthermore, let $R_i = \sqrt{S_i Q}$, for $i = 1, 2, 3$. Then

$$R_1 = 3z^2, \quad R_2 = z^2 + 8, \quad R_3 = z^2 - 4.$$

Now

$$\begin{aligned} \frac{H(z)}{\sqrt{Q}} &= \left(\frac{z^2}{R_1} - \frac{4}{R_2} - \frac{2}{R_3} \right) a'(z) - \frac{z}{R_1} \cdot a(z) \\ &= \frac{a(z)}{3z} \left(\frac{a'(z)}{a(z)} \cdot \frac{z(z^2 + 2)(z^2 - 16)}{(z^2 - 4)(z^2 + 8)} - 1 \right). \end{aligned}$$

Using (3.3), we obtain

$$H(z) = 0. \tag{3.4}$$

Observe that on $z \in [2\sqrt{2}, 4]$, we have $0 \leq \frac{a(z)}{z} \leq 1$. Using this together with $\arcsin(w) \geq w$ for $w \in [0, 1]$, we obtain

$$\arcsin\left(\frac{a(z)}{z}\right) z \geq a(z). \tag{3.5}$$

We are now ready to express $f'(z)$:

$$\begin{aligned} f'(z) &= 2(g'(z) + h'(z)) \stackrel{(3.4)}{=} 4 \arcsin\left(\frac{a(z)}{z}\right) z - \sqrt{16 - z^2} \\ &\stackrel{(3.5)}{\geq} 4a(z) - \sqrt{16 - z^2} \\ &= z \sqrt{\frac{2(16 - z^2)}{z^2 + 2}} - \sqrt{16 - z^2} \\ &= \sqrt{\frac{16 - z^2}{z^2 + 2}} (\sqrt{2}z - \sqrt{z^2 + 2}). \end{aligned}$$

It follows that $f'(z) > 0$ whenever $z < 4$ and $\sqrt{2}z - \sqrt{z^2 + 2} > 0$, which holds true for $z \in (2\sqrt{2}, 4)$. As f is clearly continuous on the interval $[2\sqrt{2}, 4]$, the first bound in the main part of Theorem 1.7 follows. The second bound follows from Theorem 1.4.

Let us quickly comment on the “moreover” part of Theorem 1.7. We showed that f is strictly increasing on the interval $[2\sqrt{2}, 4]$; hence $\lambda_2(A) < 2\pi$, when $\text{diam}(A) < 4$. If $\text{diam}(A) \geq 4$ then (3.1) implies that $\lambda_2(A) \leq 2\pi$. The isodiametric inequality yields the first statement of the “moreover” part. The second statement follows from the “moreover” part of Theorem 1.4.

4 Proof of Theorem 1.8

This section is devoted to the proof of Theorem 1.8, i.e., we prove the isodiametric inequality in the annulus. Recall that D denotes the annulus $D(0, R) \setminus D(0, 2)^o$, where $R \in [2\sqrt{2}, 4]$ is fixed. We write $(r, \alpha)_{\text{pol}}$ to denote the element of \mathbb{R}^2 represented by the polar coordinates (r, α) . That is, $(r, \alpha)_{\text{pol}} = (r \cos \alpha, r \sin \alpha)$.

The idea behind the proof is to find a “well behaved” and “maximal” subset of D whose diameter is less than or equal to 2, and then compute its Lebesgue measure. Let us begin by clarifying the meaning of the term “maximal”.

Definition 4.1 Set $\mathcal{M} = \{A \subset D : A \text{ is compact and } \text{diam}(A) \leq 2\}$. We say that $M \in \mathcal{M}$ is D -maximal if $\lambda_2(M) = \sup\{\lambda_2(A) : A \in \mathcal{M}\}$.

Our first result shows that D -maximal sets do exist. Throughout this section, $\text{conv}(\cdot)$ denotes convex hull.

Lemma 4.2 *There exists a D -maximal set.*

Proof For every positive integer n , there exists $K_n \in \mathcal{M}$ such that

$$\lambda_2(K_n) > \sup\{\lambda_2(A) : A \in \mathcal{M}\} - \frac{1}{n}.$$

Since \mathcal{M} endowed with the Hausdorff metric is a compact space, we can find a convergent subsequence $\{K_{n_k}\}_{k \in \mathbb{N}}$.

Let $K = \lim_{k \rightarrow \infty} K_{n_k}$. Clearly, $K \in \mathcal{M}$. For every $\varepsilon > 0$ there exists an open set $G \supset K$ such that $\lambda_2(G) \leq \lambda_2(K) + \varepsilon$. For all k sufficiently large, we have $K_{n_k} \subset G$, so that $\lambda_2(K_{n_k}) - \varepsilon \leq \lambda_2(K)$. Since $\varepsilon > 0$ was arbitrary, $\lambda_2(K) \geq \sup\{\lambda_2(A) : A \in \mathcal{M}\}$ and K is a D -maximal set. \square

To prove Theorem 1.8, we wish to show that the inequality (1.2) holds true for every measurable subset of D with diameter at most 2. Since the closure of a set preserves its diameter, it clearly suffices to show that (1.2) holds true for every $A \in \mathcal{M}$. By Lemma 4.2, it is enough to show that (1.2) holds true for every D -maximal set $A \in \mathcal{M}$. We will restrict our attention to such sets A .

Running assumption 4.3 $A \in \mathcal{M}$ is a D -maximal set.

Since $\text{diam}(A) \leq 2$ (and therefore the same is true for the “radial projection” of A on $S(0, 2)$) we may assume without loss of generality that

$$\alpha \in (-\pi/2, \pi/2) \quad \text{for every } (r, \alpha)_{\text{pol}} \in A. \quad (4.1)$$

Next we recall the notion of circular symmetrization, which “symmetrizes” a given set along the perimeters of balls of fixed radius. This notion is due to Pólya (see [18] or [4, p. 77]). Actually, the following definition is slightly adjusted to our needs as we will symmetrize compact subsets of an annulus.

Definition 4.4 (Circular symmetrization) Let $A \subset D$ be compact and $r \in [2, R]$. Set $A_{or} = A \cap S(0, r) = \{x \in A : \|x\| = r\}$. Define

$$A^{st} = \left\{ (r, \alpha)_{\text{pol}} \in \mathbb{R}^2 : r \in [2, R], A_{or} \neq \emptyset \text{ and } |\alpha| \leq \frac{\mathcal{H}^1(A_{or})}{2r} \right\},$$

where $\mathcal{H}^1(A_{or})$ denotes the 1-dimensional Hausdorff measure (or, the length) of A_{or} . Note that $\mathcal{H}^1(A_{or}) = r \lambda_1(\{\alpha \in [0, 2\pi) : (r, \alpha)_{\text{pol}} \in A\})$.

In other words, the circular symmetrization replaces the set A_{or} with an arc, having the same 1-dimensional Hausdorff measure as A_{or} , that is centered on the x -axis. Our aim is to show that we may assume that the set A is symmetric. Before doing so, we first show that the circular symmetrization of a compact set is compact.

Lemma 4.5 *If $K \subset D$ is compact, then so is K^{st} .*

Proof Suppose that $((r_n, \alpha_n)_{\text{pol}} \in K^{st})_n$ is a sequence converging to some $(r, \alpha)_{\text{pol}}$. We need to show that $(r, \alpha)_{\text{pol}} \in K^{st}$. For every $t > 0$, we set $K_{ot} = K \cap S(0, t)$ and $C_t = \{\alpha \in (-\pi, \pi] : (t, \alpha)_{\text{pol}} \in K_{ot}\} \subset \mathbb{R}$. As $K_{or_n} \neq \emptyset$ for every n , compactness of K implies that $K_{or} \neq \emptyset$ as well. So it clearly suffices to show that $\mathcal{H}^1(K_{or}) \geq \limsup_{n \rightarrow \infty} \mathcal{H}^1(K_{or_n})$. Fix $\varepsilon > 0$ and find an open set $G \subset \mathbb{R}$ such that $C_r \subset G$ and $\mathcal{H}^1(G) < \mathcal{H}^1(C_r) + \varepsilon$.

We claim that there is a natural number n_0 such that for every $n \geq n_0$ we have $C_{r_n} \subset G$. Indeed, otherwise, there exists an infinite sequence $n_1 < n_2 < \dots$ and for each $k \in \mathbb{N}$ we can find a $\beta_{n_k} \in C_{r_{n_k}} \setminus G$. By passing to a subsequence, we can assume that $(\beta_{n_k})_k$ is convergent. Let $\beta := \lim_k \beta_{n_k}$. Since G is open and $\beta_{n_k} \notin G$, we have $\beta \notin G$. But then the sequence $((r_{n_k}, \beta_{n_k})_{\text{pol}})_k$ converges to $(r, \beta)_{\text{pol}} \notin K$, which contradicts our assumption that K is compact. Hence the existence of n_0 as above follows.

For every $n \geq n_0$ we have $\mathcal{H}^1(C_{r_n}) \leq \mathcal{H}^1(G) < \mathcal{H}^1(C_r) + \varepsilon$. As $\varepsilon > 0$ was arbitrary, we conclude that $\mathcal{H}^1(C_r) \geq \limsup_{n \rightarrow \infty} \mathcal{H}^1(C_{r_n})$, and so

$$\mathcal{H}^1(K_{or}) = r \mathcal{H}^1(C_r) \geq r \limsup_{n \rightarrow \infty} \mathcal{H}^1(C_{r_n}) = \limsup_{n \rightarrow \infty} r_n \mathcal{H}^1(C_{r_n}) = \limsup_{n \rightarrow \infty} \mathcal{H}^1(K_{or_n}),$$

as was needed. \square

Lemma 4.6 *If A is D -maximal then A^{st} is also D -maximal.*

Proof Clearly, $A^{st} \subset D$, and it is compact by Lemma 4.5. Moreover, Fubini's theorem yields $\lambda_2(A) = \lambda_2(A^{st})$ and therefore it only remains to show that $\text{diam}(A^{st}) \leq 2$. To this end, we need to prove that for every $(r, \alpha), (t, \beta) \in [2, R] \times (-\pi, \pi]$ such that $A_{or} \neq \emptyset, A_{ot} \neq \emptyset, |\alpha| \leq \frac{\mathcal{H}^1(A_{or})}{2r}$ and $|\beta| \leq \frac{\mathcal{H}^1(A_{ot})}{2t}$, we have $\|(r, \alpha)_{\text{pol}} - (t, \beta)_{\text{pol}}\| \leq 2$. Note that there exist $\alpha_1 > \alpha_2$ and $\beta_1 > \beta_2$ such that $(r, \alpha_i)_{\text{pol}}, (t, \beta_i)_{\text{pol}} \in A$, for $i \in \{1, 2\}$, and

$$\alpha_1 - \alpha_2 \geq \frac{\mathcal{H}^1(A_{or})}{r} \geq 2|\alpha| \quad \text{and} \quad \beta_1 - \beta_2 \geq \frac{\mathcal{H}^1(A_{ot})}{t} \geq 2|\beta|. \quad (4.2)$$

We distinguish two cases. Assume first that $\alpha_1 + \alpha_2 \geq \beta_1 + \beta_2$. Then we have

$$|\alpha_1 - \beta_2| \geq \left| \alpha_1 - \frac{\alpha_1 + \alpha_2}{2} \right| + \left| \frac{\beta_1 + \beta_2}{2} - \beta_2 \right| \stackrel{(4.2)}{\geq} |\alpha| + |\beta| \geq |\alpha - \beta|. \quad (4.3)$$

Recall that $\alpha, \beta, \alpha_i, \beta_i \in (-\frac{\pi}{2}, \frac{\pi}{2})$ by (4.1). It is easy to see that $\|(r, \gamma)_{\text{pol}} - (t, 0)_{\text{pol}}\|$ is an increasing function of $\gamma \in [0, \pi)$. Therefore (4.3) implies that

$$\begin{aligned} \|(r, \alpha_1)_{\text{pol}} - (t, \beta_2)_{\text{pol}}\| &= \|(r, \alpha_1 - \beta_2)_{\text{pol}} - (t, 0)_{\text{pol}}\| \geq \|(r, \alpha - \beta)_{\text{pol}} - (t, 0)_{\text{pol}}\| \\ &= \|(r, \alpha)_{\text{pol}} - (t, \beta)_{\text{pol}}\|. \end{aligned}$$

Hence $\|(r, \alpha)_{\text{pol}} - (t, \beta)_{\text{pol}}\| \leq \text{diam} A \leq 2$, as required. If $\alpha_1 + \alpha_2 < \beta_1 + \beta_2$ then we proceed in the same way, the only difference being that we select α_2, β_1 , instead of α_1, β_2 , and obtain the same result. \square

Our next lemma says that we can find a D -maximal set which, in addition to being symmetric, also contains point $(R, 0)$.

Lemma 4.7 *There exists a D -maximal set A such that $A = A^{st}$ and $(R, 0) \in A$.*

Proof Let M be a D -maximal set and let $t = \min\{v \in \mathbb{R}; M \subset D(0, v)\}$. Notice that $(t, 0) \in M^{st}$. Define the set

$$A = (M^{st} + (R - t, 0))^{st},$$

where $M^{st} + x$ denotes the set M^{st} shifted by vector x . By construction, and using Lemmata 4.5 and 4.6, we deduce $A = A^{st}$ and $A \in \mathcal{M}$. Moreover, the D -maximality of M implies that A is D -maximal. Also, $(R, 0) \in A$ as desired. \square

Hence, for the remaining part of this section, we may assume that the set A satisfies the properties given by Lemma 4.7.

Running assumption 4.8 *The set A satisfies $A = A^{st}$ and $(R, 0) \in A$.*

Note that the assumption implies that $A \subset D((R, 0), 2)$.

Lemma 4.9 *The point $(2, 0)$ belongs to A .*

Proof Assume, towards a contradiction, that $(2, 0) \notin A$ and notice that the assumption $A = A^{st}$ implies $A \cap S(0, 2) = \emptyset$. We distinguish two cases.

Assume first that $A \subset D((R, 0), 2)^o$. Then there exists $\varepsilon > 0$ such that

$$\text{dist}(A, S(0, 2) \cup S((R, 0), 2)) > \varepsilon.$$

Consider the set $\tilde{A} = (A \cup D((R, 0), \varepsilon)) - (\varepsilon, 0)$. Clearly, $\lambda_2(\tilde{A}) > \lambda_2(A)$ and $\tilde{A} \in \mathcal{M}$. This implies that A is not D -maximal, which contradicts Running assumption 4.3.

Now, assume $A \cap S((R, 0), 2) \neq \emptyset$ and fix $(x_1, x_2) \in A \cap S((R, 0), 2)$. Recall that our Running assumption 4.8 implies that

$$(x_1, \pm x_2) \in A \cap S((R, 0), 2). \quad (4.4)$$

For a while, let us assume there exists a point $(y_1, y_2) \in A \setminus D((R - 2, 0), 2)$. Because of the symmetry in (4.4), we can assume $y_2 x_2 \geq 0$. Since $(y_1, y_2) \in A \subset D((R, 0), 2)$, it follows that there exists a real number $r \in [R - 2, R]$ such that $(r, 0) \in \text{conv}(\{(y_1, y_2), (x_1, -x_2)\})$. The triangle inequality, gives

$$\begin{aligned} \|(y_1, y_2) - (x_1, -x_2)\| &= \|(y_1, y_2) - (r, 0)\| + \|(r, 0) - (x_1, -x_2)\| \\ &\geq \|(y_1, y_2) - (R - 2, 0)\| - (r - (R - 2)) + \\ &\quad + \|(R, 0) - (x_1, -x_2)\| - (R - r). \end{aligned} \quad (4.5)$$

We have $\|(y_1, y_2) - (R - 2, 0)\| > 2$ since $(y_1, y_2) \notin D((R - 2, 0), 2)$. Further, we have $\|(R, 0) - (x_1, -x_2)\| = 2$ since $(x_1, x_2) \in S((R, 0), 2)$. Plugging this into (4.5), we get $\|(y_1, y_2) - (x_1, -x_2)\| > 2$ which contradicts the fact that $\text{diam}(A) \leq 2$. So there is no such point (y_1, y_2) , meaning that $A \subset D((R - 2, 0), 2)$. Since $\text{diam}(A) \leq 2$ we obtain $\text{diam}(A \cup \{(R - 2, 0)\}) \leq 2$. Now consider the set

$$\tilde{A} = \text{conv}(A \cup \{(R - 2, 0)\}) \cap D.$$

Clearly, $\tilde{A} \in \mathcal{M}$. We have also $\lambda_2(\tilde{A}) > \lambda_2(A)$ because $\tilde{A} \setminus A$ contains the point $(2, 0)$ and its small neighborhood (relative to D). Hence A is not D -maximal, contrary to Running assumption 4.3. The result follows. \square

Notation 4.10 *The following quantities will remain fixed for the remaining part of this section.*

$$\begin{aligned} \alpha &= \max\{\gamma \in (-\pi/2, \pi/2); (R, \gamma)_{\text{pol}} \in A\} \\ \beta &= \max\{\gamma \in (-\pi/2, \pi/2); (2, \gamma)_{\text{pol}} \in A\} \\ X^1 &= (x_1^1, x_2^1) = (R, \alpha)_{\text{pol}} \\ X^2 &= (x_1^2, x_2^2) = (R, -\alpha)_{\text{pol}} \\ Y^1 &= (y_1^1, y_2^1) = (2, \beta)_{\text{pol}} \\ Y^2 &= (y_1^2, y_2^2) = (2, -\beta)_{\text{pol}} \end{aligned}$$

Finally, let $S = (s, 0)$ be the point of intersection of the line segments $\text{conv}\{X^1, Y^2\}$ and $\text{conv}\{X^2, Y^1\}$.

Lemma 4.11 *There exists a convex set T such that $A = T \setminus D(0, 2)^o$.*

Proof Since the diameter does not increase upon taking convex hulls, this immediately follows from the fact that $\tilde{A} := \text{conv}(A) \setminus D(0, 2)^o \in \mathcal{M}$. Indeed, the assumption that A is D -maximal implies that $\lambda_2(\tilde{A}) = \lambda_2(A)$. It remains to observe that the set $\tilde{A} \setminus A$ is empty. Indeed, assuming that it is non empty, one can easily use Running assumption 4.8 to conclude that $\lambda_2(\tilde{A} \setminus A) > 0$, a contradiction. \square

The next two lemmata are concerned with the distances between the points defined in Notation 4.10.

Lemma 4.12 *We have $\|X^1 - Y^2\| = \|X^2 - Y^1\| = 2$.*

Proof As $\|X^1 - Y^2\| = \|X^2 - Y^1\|$ we only need to prove that $\|X^1 - Y^2\| = 2$. Assume, towards a contradiction, that

$$\|X^1 - Y^2\| < 2. \quad (4.6)$$

There are three cases to consider.

First, assume that $A \subset D(X^1, 2)^o$. Then there exists $\varepsilon > 0$ such that $A \subset D(X^1, 2 - \varepsilon)^o$. Set $\tilde{A} = (A \cup D(X^1, \varepsilon)) \cap D$. Obviously, $\tilde{A} \in \mathcal{M}$. Furthermore, we claim that $\lambda_2(\tilde{A}) > \lambda_2(A)$. Indeed, observe first that $S(0, R) \cap \tilde{A}$ has bigger length than $S(0, R) \cap A$ by the maximality of the angle α . Now, since both \tilde{A} and A are closed, we get that $\lambda_2(\tilde{A}) > \lambda_2(A)$. So, A is not D -maximal, which contradicts Running assumption 4.3.

Now, assume that $A \subset D(Y^2, 2)^o$. Then there exists $\varepsilon > 0$ such that $A \subset D(Y^2, 2 - \varepsilon)^o$. Set $\tilde{A} = (A \cup D(Y^2, \varepsilon)) \cap D$. By the same arguments as above, $\tilde{A} \in \mathcal{M}$ and $\lambda_2(\tilde{A}) > \lambda_2(A)$. This implies that A is not D -maximal, and contradicts Running assumption 4.3.

It remains to consider the case where there exist points $V^1 = (v_1^1, v_2^1) \in A \cap S(Y^2, 2)$ and $V^2 = (v_1^2, v_2^2) \in A \cap S(X^1, 2)$. We may assume further that $v_2^1 \geq 0$ and $v_2^2 \leq 0$; indeed, if $v_2^1 < 0$ then $\|(v_1^1, -v_2^1) - Y^2\| \geq \|(v_1^1, v_2^1) - Y^2\| = 2$ and $(v_1^1, -v_2^1) \in A$. Thus we can choose the point $V^1 = (v_1^1, -v_2^1)$ instead. Similarly for $v_2^2 > 0$. Using (4.6), we get $\|V^1 - Y^2\| = 2 > \|X^1 - Y^2\|$.

From Lemma 4.11 we easily see that $y_1^2 \leq x_1^1$ and that the real numbers v_1^1, v_1^2 belong to the interval $[y_1^2, x_1^1]$. Since $S(Y^2, 2) \cap ([y_1^2, x_1^1] \times (0, \infty))$ is the graph of a decreasing function, we have $v_2^1 > x_2^1$. In a similar way, we get $v_2^2 < y_2^2$. Therefore, the line segments $\text{conv}(\{V^1, V^2\})$ and $\text{conv}(\{X^1, Y^2\})$ intersect in a point, say, Z . The triangle inequality now yields

$$\begin{aligned} \|V^1 - V^2\| &= \|V^1 - Z\| + \|Z - V^2\| \\ &\geq (\|V^1 - Y^2\| - \|Y^2 - Z\|) + (\|X^1 - V^2\| - \|Z - X^1\|) \\ &= \|V^1 - Y^2\| + \|X^1 - V^2\| - \|Y^2 - X^1\| > 2 + 2 - 2 = 2, \end{aligned}$$

contrary to the fact that $\text{diam}(A) \leq 2$. The result follows. \square

Lemma 4.13 *We have $\|X^1 - X^2\| = \|Y^1 - Y^2\|$.*

Proof Consider any point $V = (v_1, v_2) \in (D(Y^1, 2) \cap D(Y^2, 2)) \setminus D(0, R)^o$. Note that $v_2 \in [x_2^2, x_2^1]$ and $v_1 \geq x_1^1$. We claim that $A \subset D(V, 2)$. To see the claim, suppose that there exists $T = (t_1, t_2) \in A \setminus D(V, 2)$. We can use Lemma 4.11 in the same way as before to deduce that $t_1 \in [y_1^2, x_1^1]$. Also note that $Y^1, Y^2 \in D(V, 2)$ while $T \notin D(V, 2)$. We have either $\text{conv}(\{X^1, T\}) \cap \text{conv}(\{Y^2, V\}) \neq \emptyset$ or $\text{conv}(\{X^2, T\}) \cap \text{conv}(\{Y^1, V\}) \neq \emptyset$ and, in a similar way as in Lemma 4.12, the triangle inequality implies that either $\|T - X^1\| > 2$ or $\|T - X^2\| > 2$, which is a contradiction.

Now consider the set

$$\hat{A} = \text{conv}(A) \cup ((D(Y^1, 2) \cap D(Y^2, 2)) \setminus D(0, R)^o)$$

and notice that, by the previous consideration, it holds $((\hat{A} + (\tau, 0)) \cap D) \in \mathcal{M}$ for any $\tau \in \mathbb{R}$. We distinguish two cases.

Suppose first that $\|Y^1 - Y^2\| - \|X^1 - X^2\| > 2\varepsilon$, for some $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$\lambda_2((\hat{A} + (\delta, 0)) \cap D) \geq \lambda_2(A) + (\|Y^1 - Y^2\| - \varepsilon)\delta - (\|X^1 - X^2\| + \varepsilon)\delta > \lambda_2(A).$$

In other words, the set $((\hat{A} + (\delta, 0)) \cap D) \in \mathcal{M}$ has larger measure than A and contradicts Running assumption 4.3.

Similarly, if $\|X^1 - X^2\| - \|Y^1 - Y^2\| > 2\varepsilon$, for some $\varepsilon > 0$, then there exists $\delta > 0$ such that

$$\lambda_2((\hat{A} - (\delta, 0)) \cap D) \geq \lambda_2(A) - (\|Y^1 - Y^2\| + \varepsilon)\delta + (\|X^1 - X^2\| - \varepsilon)\delta > \lambda_2(A),$$

which contradicts Running assumption 4.3. We conclude $\|X^1 - X^2\| = \|Y^1 - Y^2\|$, as desired. \square

By Lemma 4.13 and Running assumption 4.8, it follows that the four points X^1, X^2, Y^1, Y^2 form the vertices of a rectangle with center S , where S is defined in Notation 4.10. In the remaining part of this section, we use the following.

Notation 4.14 *The following quantities will remain fixed until the end of this section.*

$$\begin{aligned} a &= \frac{1}{2} \|X^1 - X^2\|, \\ K &= \{(v_1, v_2) \in D(X^1, 2) \cap D(X^2, 2) \cap D(Y^1, 2) \cap D(Y^2, 2); v_2 \notin [x_2^2, x_2^1]\}, \\ O &= \{(x, y) \in D; y \in [x_2^2, x_2^1] \text{ and } x > 0\}. \end{aligned}$$

Last, let $L = \{(s, \tau) \in \mathbb{R}^2; \tau \in \mathbb{R}\}$ be the line orthogonal to $\text{conv}(\{0, (R, 0)\})$ that contains S .

The proof of Theorem 1.8 is almost complete.

Proof of Theorem 1.8 Clearly we have $A \subset O \cup K$ and $K \subset D$. Moreover, notice that K is symmetric with respect to L . Now we apply the classical Steiner symmetrization (see [6, p. 87]) with respect to L to $K \cap A$ to obtain a new set \tilde{A} . This guarantees that $\lambda_2(\tilde{A}) = \lambda_2(A \cap K)$. Moreover, since $\text{diam}(K \cap A) \leq 2$, we have $\text{diam}(\tilde{A}) \leq 2$. Since K is symmetric with respect to L we obtain $\tilde{A} \subset K \subset D$. Now observe that for every $V \in K$ we have $\text{diam}(O \cup \{V\}) \leq 2$. Since $\tilde{A} \subset K$ and $\text{diam}(\tilde{A}) \leq 2$ we have $\text{diam}(\tilde{A} \cup O) \leq 2$, and it follows that $\tilde{A} \cup O$ is D -maximal. Since $A = A^{st}$, \tilde{A} is also symmetric with respect to the line $\{(\tau, 0); \tau \in \mathbb{R}\}$. Thus \tilde{A} is symmetric with respect to S . Since $\text{diam}(\tilde{A}) \leq 2$ we have $\tilde{A} \subset D(S, 1)$. Clearly, $O \subset D(S, 1)$. Thus $D(S, 1) \cap D$ is D -maximal. Now we compute the measure of the ‘‘well behaved’’ set $D(S, 1) \cap D$. Straightforward calculations give

$$a^2 = \frac{-R^4 + 16R^2}{8(R^2 + 2)}$$

as well as

$$\lambda_2(D(S, 1) \cap D) = R^2 \arcsin\left(\frac{a}{R}\right) - 4 \arcsin\left(\frac{a}{2}\right) + 2 \arccos(a).$$

This finishes the proof of Theorem 1.8. □

5 Concluding remarks

5.1 Problem 1.6 in higher dimension

When $d \leq 4$, we conjecture that a measurable subset $A \subset \mathbb{R}^d$ for which \mathcal{G}_A is K_k -free satisfies $\lambda_d(A) \leq (k-1) \cdot \omega_d$. Perhaps rather surprisingly, for $d \geq 5$ the answer is different. This can already be seen from the case of K_3 -free sets. Notice that the radius of a 2-ball that circumscribes an equilateral triangle whose sides are equal to 2 is equal to $2/\sqrt{3}$. Now it is easy to see that any d -ball of radius $2/\sqrt{3}$ (for arbitrary $d \geq 2$) is K_3 -free. The volume of such a d -ball is equal to $(2/\sqrt{3})^d \cdot \omega_d$. Now it is not difficult to see that $2\omega_d < (2/\sqrt{3})^d \cdot \omega_d$, when $d \geq 5$, and therefore the optimal K_3 -free set is not a disjoint union of two unit balls that are at sufficiently large distance. We do not have a conjecture for the optimal set in higher dimensions.

Notice that in Theorem 1.7 we found the best possible upper bound on the measure of the ‘‘edge set’’ of a 2-dimensional K_3 -free set by exploiting the fact that the optimal configuration maximizing the measure of the ‘‘vertex set’’ is a disjoint union of two unit balls that are at distance at least 2. From the discussion of the previous paragraph, we know that a disjoint union of two unit balls is not the optimal configuration in higher dimensions and therefore, and perhaps rather surprisingly, there might be instances for which the set A which maximizes $e(\mathcal{G}_A)$, in the setting of Problem 1.6, is different from the set A which maximizes $\lambda_d(A)$.

5.2 Isodiametric inequality for annuli in higher dimension

Central to our proof of Theorem 1.7 was the isodiametric inequality for annuli, Theorem 1.8. However, Theorem 1.8 was stated only in dimension 2. It would be of interest to extend the result to other dimensions.

Problem 5.1 Let $R \in (2, 4]$ and consider the set $D = D(0, R) \setminus D(0, 2)^o \subset \mathbb{R}^d$. Suppose that A is a compact subset of D such that $\text{diam}(A) \leq 2$. What is a sharp upper bound on the λ_d -measure of A ?

Problem 5.1 can in turn be employed to find the maximum λ_d -measure of a K_3 -free set. It appears that an approach that is similar to the approach of Section 4 can be employed in higher dimensions. The only subtlety is in the calculations involving the higher-dimensional analogue of the circular symmetrization and at the moment it is unclear how to proceed. We leave this as an open problem for the reader and we hope that we will be able to report on that matter in the future.

5.3 The structure of large-distance graphs

What is the structure of large-distance graphs? There are many ways to formalize this problem, but let us put forward a particular one.

Problem 5.2 Given a dimension $d \geq 1$, describe the set

$$\mathcal{S}_d := \left\{ (\lambda_{v(H),d}(\mathcal{G}_A(H)))_H \text{ graph} : A \subset \mathbb{R}^d \text{ of finite measure} \right\} \subset \mathbb{R}^{\text{graphs}}.$$

Recall that in Remark 1.5 and directly beneath it we pointed out that Problem 5.2 is not scale-free. Even for $d = 1$, the answer does not seem obvious.

Let us note that the counterpart of Problem 5.2 is one of the cornerstones of the theory of limits of dense graph sequences. More specifically, Lovász and Szegedy [14] characterized the set

$$\mathcal{T} := \left\{ (t(H, W))_H \text{ graphon} : W \text{ graphon} \right\} \subset \mathbb{R}^{\text{graphs}}. \quad (5.1)$$

Note that if W is a graphon associated to a set $A \subset \mathbb{R}^d$ (as we did in Section 2), then

$$\lambda_{v(H),d}(\mathcal{G}_A(H)) = \lambda_d(A)^{v(H)} \cdot t(H, W) \quad \text{for each } H.$$

Thus, for each d , the set \mathcal{S}_d is a subset of a suitable transformed (by the factors $\lambda_d(A)^{v(H)}$) set \mathcal{T} . It seems that when d is large, this set inclusion is actually not far from an equality.

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