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Abstract

We study convergence of a finite volume scheme for the Navier–Stokes–Fourier system describing the motion of compressible viscous and heat conducting fluids. The numerical flux uses upwinding with an additional numerical diffusion of order $\mathcal{O}(h^{\varepsilon+1})$, $0 < \varepsilon < 1$. The approximate solutions are piecewise constant functions with respect to the underlying mesh. We show that any uniformly bounded sequence of numerical solutions converges unconditionally to the solution of the Navier–Stokes–Fourier system. In particular, the existence of the solution to the Navier–Stokes–Fourier system is not a priori assumed.

Keywords: compressible Navier–Stokes–Fourier system, finite volume method, upwinding, convergence, Young measures, dissipative measure–valued solutions, weak–strong uniqueness

Contents

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1 Introduction

The time evolution of viscous compressible and heat conducting fluids is governed by the conservation of mass, momentum and energy. Altogether these conservation laws yield the well-known Navier–Stokes–Fourier system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1a}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S}(\mathbf{D}(\mathbf{u})), \tag{1.1b}$$

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e \mathbf{u}) - \mathrm{div}_x(\kappa \nabla_x \vartheta) = 2\mu |\mathbf{D}(\mathbf{u})|^2 + \lambda |\mathrm{div}_x \mathbf{u}|^2 - p \mathrm{div}_x \mathbf{u}, \tag{1.1c}$$

where $\varrho, \mathbf{u}, \vartheta, p, e$ are the density, velocity, temperature, pressure and internal energy, respectively. The pressure p satisfies the perfect gas law

$$p = \varrho \vartheta,$$

and the internal energy is

$$e = c_v \vartheta,$$

where $c_v > 0$ is the specific heat at constant volume. The constant $\kappa > 0$ denotes the heat conductivity coefficient. Further, we have denoted by

$$\mathbf{D}(\mathbf{u}) = \frac{\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u}}{2}$$

the symmetric velocity gradient and by

$$\mathbb{S}(\mathbf{D}(\mathbf{u})) = 2\mu \mathbf{D}(\mathbf{u}) + \lambda \mathrm{div}_x \mathbf{u} \mathbb{I}$$

the viscous stress tensor with the viscosity coefficients $\mu > 0$ and $\lambda \geq 0$. System (1.1) is solved in the time–space cylinder $(0, T) \times \Omega$. We prescribe the periodic boundary condition, which means $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is assumed to be a flat torus. To close the system we impose the initial conditions

$$\varrho(0) = \varrho_0, \ \mathbf{u}(0) = \mathbf{u}_0, \ \vartheta(0) = \vartheta_0, \ \text{ with } \varrho_0 > 0 \text{ and } \vartheta_0 > 0.$$

System (1.1) has numerous everyday applications, e.g., in aerodynamics, hydrodynamics, engineering or even in medicine. Therefore its numerical approximations have been widely studied in the past decades. Let us mention a few well-established and practical schemes, e.g., [2, 4, 5, 21, 26, 28, 29, 31, 33]. Despite of such variety of efficient numerical schemes, their convergence analysis is still open in general. Though there are some convergence (and even error estimate) results for numerical methods for the isentropic Navier-Stokes equations, see, e.g., [19, 20, 22, 24, 25] or [9, 10, 13], the convergence analysis of the full Navier–Stokes–Fourier system is considerably more involved and much less results are available in the literature. For a mixed finite element–finite volume method based on the Crouzeix–Raviart finite elements Feireisl, Karper and Novotný [8] proved the convergence to a weak solution for a rather specific state equation $p = a\rho^\gamma + b\rho + \rho\theta$, $a, b > 0$ and $\gamma > 3$. It is to be pointed out that the generalization of the result obtained in [8] to other schemes is still open, cf. also [23]. On the other hand, in our recent works [10, 11, 12, 13] we have proposed a new, rather general way for the convergence analysis via the concept of dissipative measure–valued (DMV) solutions.

Our approach bears some similarities with the recent works of Fjordholm et al. [16, 17, 18], who studied the convergence of entropy stable finite volume schemes to a measure–valued solution of the Euler equations. The main difference in using the concept of DMV solutions lies in the fact that we relax the energy conservation asking only the global energy to dissipate over time. Similarly to Fjordholm et al. we also require that the entropy inequality holds, cf. Definition 5.1.

The main goal of this paper is to demonstrate that the strategy proposed in [10, 13] can be extended to obtain the convergence for the full Navier–Stokes–Fourier system (1.1). To solve the latter numerically we apply a finite volume scheme with the numerical flux function based on upwinding to get a piecewise constant approximation of all unknown quantities. Under a realistic assumption that the numerical solutions have bounded temperature and density, we can prove the consistency of the finite volume scheme. This fact together with some suitable *a priori* estimates implies that the sequence of numerical solutions generates, up to a subsequence, a DMV solution. Note, that in contrast to the isentropic Navier–Stokes equations, we need to control also the gradients of the velocity and temperature, since they are now included in the support of the corresponding Young measure, cf. [3]. Furthermore, using the DMV-strong uniqueness principle for the solution of the Navier–Stokes–Fourier system, cf. [3], we get the strong convergence of the piecewise constant solutions to the strong (classical) solution on its lifespan, see Theorem 5.4. For any uniformly bounded sequence of numerical solutions we also obtain the global in time convergence to the strong (classical) solution of the Navier–Stokes–Fourier system (1.1) without a priori assuming the existence of its solution, see Theorem 5.6. Here *strong* means solutions in the standard energy spaces used by Valli and Zajackowski [32]. In particular, as shown in [32] these are *classical* solutions in the sense that all necessary derivatives are continuous.

The rest of the paper is organized as follows. In Section 2 we introduce the necessary notations and the numerical scheme. In Section 3 we show that the discrete solutions satisfy the global energy dissipation and the entropy inequality. The consistency formulation of the scheme is proved in Section 4. We present the main results on the convergence of our finite volume scheme in Section 5.

2 Numerical scheme

In this section we collect the necessary apparatus of the numerical analysis and introduce the finite volume method for the Navier–Stokes-Fourier system (1.1).

2.1 Space discretization

Mesh. Let $\mathcal{T}$ be a uniform quadrilateral mesh such that

$$\Omega = \bigcup_{K \in \mathcal{T}} K,$$

where K is a square ($d = 2$) or a cube ($d = 3$). For any $K \in \mathcal{T}$ we denote by $\mathbf{x}_K$ its center of mass and by $|K| = h^d$ its volume. Let $\mathcal{E}$ be the set of all faces, and $\mathcal{E}_i$, $i = 1, \ldots, d$, be the set of all faces that are orthogonal to the unit vector $\mathbf{e}_i$ of the i^{th} canonical direction. Moreover, we write $\mathcal{E}(K)$ as the set of all faces of an element K and $\mathcal{E}_i(K) = \mathcal{E}(K) \cap \mathcal{E}_i$. For any σ being the common face of elements K and L, we write $\sigma = K|L$. We further write $\sigma = \overrightarrow{K|L}$ if $\mathbf{x}_L = \mathbf{x}_K + h\mathbf{e}_i$ for any $i = 1, \ldots, d$. By $\mathbf{x}_\sigma$ we denote the center of mass of a generic face σ and by $|\sigma| = h^{d-1}$ its Lebesque measure.

Function space. The symbol Q_h stands for the set of piecewise constant functions on primary grid $\mathcal{T}$. We approximate the density, velocity and temperature by discrete functions ϱ_h, $\mathbf{u}_h$, $\vartheta_h \in Q_h$, respectively. Analogously, $s_h = s(\varrho_h, \vartheta_h)$ stands for a piecewise constant approximation of a function $s = s(\varrho, \vartheta)$ with respect to $\mathcal{T}$. Note that hereafter $\mathbf{v}_h \in Q_h$ means that every component of a vector–valued function $\mathbf{v}_h$ belongs to the set Q_h.

The standard projection operator associated to Q_h reads

$$\Pi_\mathcal{T} : \, L^1(\Omega) \to Q_h. \quad \Pi_\mathcal{T} \phi = \sum_{K \in \mathcal{T}} 1_K \frac{1}{|K|} \int_K \phi \, \mathrm{d}x.$$

For any $v_h \in Q_h$ we have

$$\int_\Omega v_h \, \mathrm{d}x = \sum_{K \in \mathcal{T}} |K| v_K, \quad v_K = v_h|_K.$$

Further, we use the following notations for the average and jump operators

$$\overline{v}(x) = \frac{v^{\text{in}}(x) + v^{\text{out}}(x)}{2}, \; [\![v]\!] = v^{\text{out}}(x) - v^{\text{in}}(x), \; \text{ where } v^{\text{out}}(x) = \lim_{\delta \to 0+} v(x + \delta \mathbf{n}), \; v^{\text{in}}(x) = \lim_{\delta \to 0+} v(x - \delta \mathbf{n}),$$

whenever $x \in \sigma \in \mathcal{E}$.

Discrete operators. For piecewise constant functions we define the discrete gradient and divergence operators in the following way

$$\nabla_h r_h(\mathbf{x}) = \sum_{K \in \mathcal{T}} (\nabla_h r_h)_K \, 1_K, \qquad (\nabla_h r_h)_K = \frac{|\sigma|}{|K|} \sum_{\sigma \in \mathcal{E}(K)} \overline{r_h} \mathbf{n},$$

$$\mathrm{div}_h \mathbf{v}_h(\mathbf{x}) = (\mathrm{div}_h \mathbf{v}_h)_K \, 1_K, \qquad (\mathrm{div}_h \mathbf{v}_h)_K = (\nabla_h \cdot \mathbf{v}_h)_K = \frac{|\sigma|}{|K|} \sum_{\sigma \in \mathcal{E}(K)} \overline{\mathbf{v}_h} \cdot \mathbf{n},$$

$$\nabla_h \mathbf{v}_h = (\nabla_h v_{1,h}, \ldots, \nabla_h v_{d,h})^T, \qquad \mathbf{D}_h(\mathbf{v}_h) = \left(\nabla_h \mathbf{v}_h + \nabla_h^T \mathbf{v}_h \right)/2,$$

for any r_h, $\mathbf{v}_h \in Q_h$. It is worth mentioning that due to the fact that $\int_{\partial K} \mathbf{n} \, \mathrm{d}S_x = 0$, we have

$$\int_{\partial K} \overline{r_h} \mathbf{n} \, \mathrm{d}S_x = \frac{1}{2} \int_{\partial K} [\![r_h]\!] \, \mathbf{n} \, \mathrm{d}S_x.$$

The discrete Laplace operator can be defined analogously

$$\Delta_h r_h(\mathbf{x}) = \sum_{K \in \mathcal{T}} (\Delta_h r_h)_K \, 1_K, \quad (\Delta_h r_h)_K = \frac{|\sigma|}{|K|} \sum_{\sigma \in \mathcal{E}(K)} \frac{[\![r_h]\!]}{h}, \qquad r_h \in Q_h.$$

In what follows we will also work with functions evaluated at the cell faces. Therefore it is convenient to introduce a dual grid associated to faces σ and the corresponding discrete function space.

Dual grid. For any $\sigma = K|L \in \mathcal{E}$, we define a dual cell $D_\sigma := D_{\sigma,K} \cup D_{\sigma,L}$, where $D_{\sigma,K}$ (resp. $D_{\sigma,L}$) is half of an element K (resp. L), see Figure 1 for an example of such a cell in two dimensions. We denote the set of all dual cells by $\mathcal{G}$. Furthermore, we define $\mathcal{G}_i = \{D_\sigma\}_{\sigma \in \mathcal{E}_i}, i = 1, \ldots, d$. Now we are able to define $W_h^{(i)}$, $i = 1, \ldots, d$, as the space of piecewise constant functions on the dual grid $\mathcal{G}_i$. By $\mathbf{q} = (q_1, \ldots, q_d) \in W_h := \left(W_h^{(1)}, \ldots, W_h^{(d)} \right)$ we mean that $q_i \in W_h^{(i)}$, for all $i = 1, \ldots, d$.

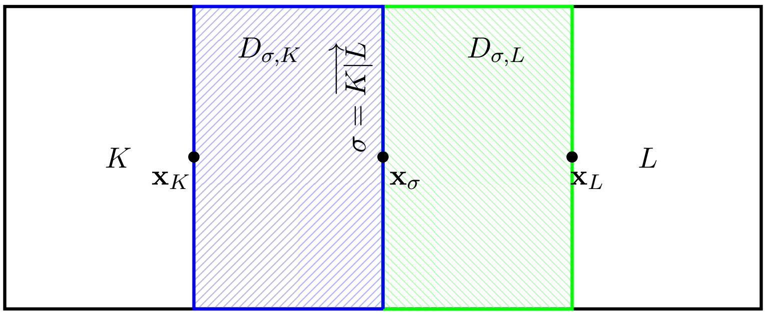

Figure 1: Dual grid

Accordingly, the associated projection of the functional spaces W_h is given by

$$\Pi_{\mathcal{E}}: \quad L^1(\Omega) \to W_h, \quad \Pi_{\mathcal{E}} = (\Pi_{\mathcal{E}}^{(1)}, \ldots, \Pi_{\mathcal{E}}^{(d)}), \quad \Pi_{\mathcal{E}}^{(i)} \phi = \sum_{\sigma \in \mathcal{E}_i} \frac{1_{D_\sigma}}{|D_\sigma|} \int_{D_\sigma} \phi \, \mathrm{d}x.$$

For any $r_h \in Q_h$ and $\mathbf{q}_h = (q_{1,h}, \ldots, q_{d,h}) \in W_h$ we define the following standard difference operators

$$\eth_{\mathcal{E}}^{(i)} r_h(\mathbf{x}) = \sum_{\sigma \in \mathcal{E}} 1_{D_\sigma} \left(\eth_{\mathcal{E}}^{(i)} r_h\right)_\sigma, \qquad \left(\eth_{\mathcal{E}}^{(i)} r_h\right)_\sigma = \frac{r_h|_L - r_h|_K}{h} \ \text{ for any } \ \sigma = \overrightarrow{K|L} \in \mathcal{E}_i,$$

$$\eth_{\mathcal{T}} q_{i,h}(\mathbf{x}) = \sum_{K \in \mathcal{T}} (\eth_{\mathcal{T}} q_{i,h})_K 1_K, \qquad (\eth_{\mathcal{T}} q_{i,h})_K = \frac{q_{i,h}|_{\sigma'} - q_{i,h}|_\sigma}{h} \ \text{ for all } \ \sigma, \sigma' \in \mathcal{E}_i(K) \text{ and } \mathbf{x}_{\sigma'} = \mathbf{x}_\sigma + h\mathbf{e}_i.$$

With the above notations, we further define

$$\nabla_{\mathcal{E}} r_h = \left(\eth_{\mathcal{E}}^{(1)}, \ldots, \eth_{\mathcal{E}}^{(d)}\right) r_h, \quad \mathrm{div}_{\mathcal{T}} \mathbf{q}_h = \sum_{i=1}^d \eth_{\mathcal{T}} q_{i,h}. \tag{2.1}$$

It is easy to observe that

$$\eth_{\mathcal{T}} \Pi_{\mathcal{E}}^{(i)} r_h = \Pi_{\mathcal{T}} \eth_{\mathcal{E}}^{(i)} r_h, \quad \nabla_h r_h = \eth_{\mathcal{T}} \Pi_{\mathcal{E}} r_h = \Pi_{\mathcal{T}} \nabla_{\mathcal{E}} r_h, \quad \Delta_h \vartheta_h = \mathrm{div}_{\mathcal{T}} \nabla_{\mathcal{E}} \vartheta_h. \tag{2.2}$$

Integration by parts. Let us start with recalling the following algebraic identity

$$\overline{u_h v_h} - \overline{u_h}\ \overline{v_h} = \frac{1}{4} [\![u_h]\!] [\![v_h]\!]$$

together with the product rule

$$[\![u_h v_h]\!] = \overline{u_h} [\![v_h]\!] + [\![u_h]\!] \overline{v_h}, \tag{2.3}$$

which are valid for any $u_h, v_h \in Q_h$. A direct application of the product rule (2.3) further implies the following lemma.

Lemma 2.1. *[13, Lemma 2.2] For any r_h, $\mathbf{v}_h \in Q_h$ it holds that*

$$\sum_{\sigma \in \mathcal{E}} \int_\sigma \left(\overline{r_h} [\![\mathbf{v}_h]\!] + \overline{\mathbf{v}_h} [\![r_h]\!]\right) \cdot \mathbf{n} \, \mathrm{d}S_x = 0. \tag{2.4}$$

Indeed, (2.4) indicates the Grad–Div duality for any $r_h, \mathbf{v}_h \in Q_h$, i.e.,

$$\begin{aligned}
&\int_\Omega \nabla_h r_h \cdot \mathbf{v}_h \, \mathrm{d}x = \sum_K \mathbf{v}_K \cdot \int_{\partial K} \overline{r_h} \mathbf{n} \, \mathrm{d}S_x = \sum_{K \in \mathcal{T}} \mathbf{v}_K \cdot \int_{\partial K} \frac{[\![r_h]\!]}{2} \mathbf{n} \, \mathrm{d}S_x = \sum_{\sigma \in \mathcal{E}} \int_\sigma \overline{\mathbf{v}_h} \cdot ([\![r_h]\!] \mathbf{n}) \, \mathrm{d}S_x \\
&= - \sum_{\sigma \in \mathcal{E}} \int_\sigma [\![\mathbf{v}_h]\!] \cdot (\overline{r_h} \mathbf{n}) \, \mathrm{d}S_x = - \sum_{K \in \mathcal{T}} r_K \int_{\partial K} \frac{[\![\mathbf{v}_h]\!]}{2} \cdot \mathbf{n} \, \mathrm{d}S_x = - \sum_{K \in \mathcal{T}} r_K \int_{\partial K} \overline{\mathbf{v}_h} \cdot \mathbf{n} \, \mathrm{d}S_x \\
&= - \int_\Omega r_h \mathrm{div}_h \mathbf{v}_h \, \mathrm{d}x.
\end{aligned}$$

It is also easy to observe the following discrete integration by parts formulae for all $r_h, \phi_h \in Q_h$ and $\mathbf{q}_h \in W_h$

$$\int_\Omega \Delta_h r_h \phi_h \,\mathrm{d}x = -\int_\Omega \nabla_{\mathcal{E}} r_h \cdot \nabla_{\mathcal{E}} \phi_h \,\mathrm{d}x = \int_\Omega r_h \Delta_h \phi_h \,\mathrm{d}x,$$

$$\int_\Omega q_{i,h} \eth_{\mathcal{E}}^{(i)} r_h \,\mathrm{d}x = -\int_\Omega r_h \eth_{\mathcal{T}} q_{i,h} \,\mathrm{d}x, \quad \text{for all} \quad i = 1, \dots, d.$$

Useful estimates. Next, we list some basic inequalities used in the numerical analysis. We assume the reader is fairly familiar with this matter, for which we refer to the monograph [6], and the article [20]. If $\phi \in C^1(\Omega)$, $h \in (0, h_0)$, $h_0 \ll 1$, we have

$$\left| [\![\Pi_{\mathcal{T}} \phi]\!] \big|_\sigma \right| \lesssim h\|\phi\|_{C^1}, \text{ for any } x \in \sigma \in \mathcal{E}, \ \|\phi - \Pi_{\mathcal{T}}\phi\|_{L^p(\Omega)} \lesssim h\|\phi\|_{C^1}, \ \|\Pi_{\mathcal{T}}\phi - \Pi_{\mathcal{E}}\Pi_{\mathcal{T}}\phi\|_{L^p(\Omega)} \lesssim h\|\phi\|_{C^1}. \tag{2.6}$$

Here and hereafter we denote $A \lesssim B$ if $A \leq cB$ for a positive constant c which is independent of the discretization parameter h. Furthermore, if $\phi \in C^2(\Omega)$ we have for all $1 < p \leq \infty$, $h \in (0, h_0)$, $h_0 \ll 1$

$$\left\|\nabla_x \phi - \nabla_{\mathcal{E}}\big(\Pi_{\mathcal{T}}\phi\big)\right\|_{L^p(\Omega)} \lesssim h, \quad \left\|\nabla_x \phi - \nabla_h\big(\Pi_{\mathcal{T}}\phi\big)\right\|_{L^p(\Omega)} \lesssim h, \quad \|\mathrm{div}_x \phi - \mathrm{div}_h(\Pi_{\mathcal{T}}\phi)\|_{L^p(\Omega)} \lesssim h. \tag{2.7}$$

Diffusive upwind flux. For a given a velocity $\mathbf{u}_h \in Q_h$ and a quantity $r_h \in Q_h$ the upwind numerical flux is defined at each face $\sigma \in \mathcal{E}$ as

$$Up[r_h, \mathbf{u}_h] = r_h^{\mathrm{up}} \mathbf{u}_h \cdot \mathbf{n} = r_h^{\mathrm{in}} [\overline{\mathbf{u}_h} \cdot \mathbf{n}]^+ + r_h^{\mathrm{out}} [\overline{\mathbf{u}_h} \cdot \mathbf{n}]^- = \overline{r}_h \ \overline{\mathbf{u}_h} \cdot \mathbf{n} - \frac{1}{2} |\overline{\mathbf{u}_h} \cdot \mathbf{n}| \, [\![r_h]\!] \,,$$

where

$$[f]^\pm := \frac{f \pm |f|}{2} \quad \text{and} \quad r^{\mathrm{up}} := \begin{cases} r^{\mathrm{in}} & \text{if } \overline{\mathbf{u}} \cdot \mathbf{n} \geq 0, \\ r^{\mathrm{out}} & \text{if } \overline{\mathbf{u}} \cdot \mathbf{n} < 0. \end{cases}$$

Now, we can define a numerical flux function

$$F_h(r_h, \mathbf{u}_h) = Up[r_h, \mathbf{u}_h] - h^\varepsilon \, [\![r_h]\!] \,, \quad 0 < \varepsilon < 1. \tag{2.8}$$

Let us point out that the $h^\varepsilon-$term introduced in the numerical flux actually acts as an artificial diffusion term of order $\mathcal{O}(h^{\varepsilon+1})$ in our finite volume scheme (2.9) defined below. Indeed,

$$\sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} h^\varepsilon \, [\![r_h]\!] = h^{\varepsilon+1} (\Delta_h r_h)_K$$

for $r_h \in \{\varrho_h, \varrho_h \mathbf{u}_h, \varrho_h \vartheta_h\}$. Note that the vector–valued flux function $\mathbf{F}_h(r_h, u_h)$ that is used in the momentum equation with $r_h = \varrho_h \mathbf{u}_h$ is defined componentwisely.

2.2 Time discretization

For a given time step $\Delta t \approx h > 0$ we denote the approximation of a function v_h at time $t^k = k\Delta t$ by v_h^k for $k = 1, \dots, N_T (= T/\Delta t)$. The time derivative is approximated by the backward finite difference

$$D_t v_h^k = \frac{v_h^k - v_h^{k-1}}{\Delta t}, \quad \text{for } k = 1, 2, \dots, N_T.$$

Furthermore, we introduce the functions $(\varrho_h, \mathbf{u}_h, \vartheta_h)$, piecewise constant in time, which are given by

$$\begin{aligned}
&\varrho_h(t, \cdot) = \varrho_h^0 \text{ for } t \in [0, \Delta t), \quad \varrho_h(t, \cdot) = \varrho_h^k \text{ for } t \in [k\Delta t, (k+1)\Delta t), \quad k = 1, 2, \dots, N_T, \\
&\mathbf{u}_h(t, \cdot) = \mathbf{u}_h^0 \text{ for } t \in [0, \Delta t), \quad \mathbf{u}_h(t, \cdot) = \mathbf{u}_h^k \text{ for } t \in [k\Delta t, (k+1)\Delta t), \quad k = 1, 2, \dots, N_T, \\
&\vartheta_h(t, \cdot) = \vartheta_h^0 \text{ for } t \in [0, \Delta t), \quad \vartheta_h(t, \cdot) = \vartheta_h^k \text{ for } t \in [k\Delta t, (k+1)\Delta t), \quad k = 1, 2, \dots, N_T,
\end{aligned}$$

and $p_h(t) = p(\varrho_h(t))$, $\quad s_h(t) = s(\varrho_h(t), \vartheta_h(t))$.

The discrete time derivative then reads

$$D_t v_h = \frac{v_h(t,\cdot) - v_h(t-\Delta t,\cdot)}{\Delta t}.$$

2.3 Numerical method for the Navier–Stokes–Fourier system

We are now ready to propose the following finite volume scheme for the compressible Navier–Stokes-Fourier system (1.1).

Definition 2.2 (Finite volume scheme)**.** Given the initial values $(\varrho_h^0, \mathbf{u}_h^0, \vartheta_h^0) = (\Pi_{\mathcal{T}}\varrho_0, \Pi_{\mathcal{T}}\mathbf{u}_0, \Pi_{\mathcal{T}}\vartheta_0)$, we seek a solution $\{(\varrho_h^k, \mathbf{u}_h^k, \vartheta_h^k)\}_{k=1}^{N_T} \in Q_h \times Q_h \times Q_h$ satisfying, for all $K \in \mathcal{T}$,

$$D_t \varrho_K^k + \sum_{\sigma\in\mathcal{E}(K)} \frac{|\sigma|}{|K|} F_h(\varrho_h^k, \mathbf{u}_h^k) = 0, \tag{2.9a}$$

$$D_t(\varrho_h^k \mathbf{u}_h^k)_K + \sum_{\sigma\in\mathcal{E}(K)} \frac{|\sigma|}{|K|} \mathbf{F}_h(\varrho_h^k \mathbf{u}_h^k, \mathbf{u}_h^k) + \nabla_h p_h^k = 2\mu(\mathrm{div}_h \mathbf{D}_h(\mathbf{u}_h^k))_K + \lambda \nabla_h(\mathrm{div}_h \mathbf{u}_h^k), \tag{2.9b}$$

$$c_v D_t(\varrho_h^k \vartheta_h^k)_K + c_v \sum_{\sigma\in\partial K} \frac{|\sigma|}{|K|} F_h(\varrho_h^k \vartheta_h^k, \mathbf{u}_h^k) - \kappa \Delta_h \vartheta_h^k = 2\mu |\mathbf{D}_h(\mathbf{u}_h^k)|_K^2 + \lambda |\mathrm{div}_h \mathbf{u}_h^k|_K^2 - p_K^k (\mathrm{div}_h \mathbf{u}_h^k)_K. \tag{2.9c}$$

For convenience of analysis we rewrite the above finite volume scheme into a weak formulation.

Definition 2.3 (Weak formulation)**.** The finite volume scheme (2.9) possesses an equivalent formulation

$$\int_\Omega D_t \varrho_h^k \phi_h \,\mathrm{d}x - \sum_{\sigma\in\mathcal{E}} \int_\sigma F_h(\varrho_h^k, \mathbf{u}_h^k) [\![\phi_h]\!] \,\mathrm{d}S_x = 0, \quad \text{for all } \phi_h \in Q_h, \tag{2.10a}$$

$$\begin{aligned}\int_\Omega D_t(\varrho_h^k \mathbf{u}_h^k)\cdot \boldsymbol{\phi}_h \,\mathrm{d}x &- \sum_{\sigma\in\mathcal{E}} \int_\sigma \mathbf{F}_h(\varrho_h^k \mathbf{u}_h^k, \mathbf{u}_h^k) \cdot [\![\boldsymbol{\phi}_h]\!] \,\mathrm{d}S_x - \int_\Omega p_h^k \mathrm{div}_h \boldsymbol{\phi}_h \,\mathrm{d}x \\ &= -2\mu \int_\Omega \mathbf{D}_h(\mathbf{u}_h^k) : \mathbf{D}_h(\boldsymbol{\phi}_h)\,\mathrm{d}x - \lambda \int_\Omega \mathrm{div}_h \mathbf{u}_h^k \,\mathrm{div}_h \boldsymbol{\phi}_h \,\mathrm{d}x, \quad \text{for all } \boldsymbol{\phi}_h \in Q_h.\end{aligned} \tag{2.10b}$$

$$\begin{aligned} c_v \int_\Omega D_t(\varrho_h^k \vartheta_h^k)\phi_h \,\mathrm{d}x &- c_v \sum_{\sigma\in\mathcal{E}} \int_\sigma F_h(\varrho_h^k \vartheta_h^k, \mathbf{u}_h^k) [\![\phi_h]\!] \,\mathrm{d}S_x + \int_\Omega \kappa \nabla_{\mathcal{E}} \vartheta_h^k \cdot \nabla_{\mathcal{E}} \phi_h \,\mathrm{d}x \\ &= \int_\Omega \left(2\mu |\mathbf{D}_h(\mathbf{u}_h^k)|^2 + \lambda |\mathrm{div}_h \mathbf{u}_h^k|^2 - p_h^k \mathrm{div}_h \mathbf{u}_h^k \right) \phi_h \,\mathrm{d}x, \quad \text{for all } \phi_h \in Q_h. \end{aligned} \tag{2.10c}$$

It is suitable to reformulate the convective terms in the following way, see [13, Lemma 2.5]. For reader's convenience we reproduce the proof.

Lemma 2.4. *For any $r_h, \mathbf{v}_h \in Q_h$, and $\phi \in C^1(\Omega)$, it holds*

$$\begin{aligned}&\int_\Omega r_h \mathbf{v}_h \cdot \nabla_x \phi \,\mathrm{d}x - \sum_{\sigma\in\mathcal{E}} \int_\sigma F_h[r_h, \mathbf{v}_h] \;\; [\![\Pi_{\mathcal{T}}\phi]\!] \,\mathrm{d}S_x \\ &= \sum_{\sigma\in\mathcal{E}} \int_\sigma \left(\frac{1}{2} |\overline{\mathbf{v}_h} \cdot \mathbf{n}| + h^\varepsilon + \frac{1}{4} [\![\mathbf{v}_h]\!] \cdot \mathbf{n} \right) [\![r_h]\!] \, [\![\Pi_{\mathcal{T}}\phi]\!] \,\mathrm{d}S_x + \int_\Omega r_h \mathbf{v}_h \cdot \big(\nabla_x \phi - \nabla_h \big(\Pi_{\mathcal{T}}\phi\big)\big) \,\mathrm{d}x.\end{aligned}$$

Proof. Using the basic equalities (2.2)–(2.4), we have

$$\int_\Omega r_h \mathbf{v}_h \cdot \nabla_x \phi \,\mathrm{d}x = \sum_{K\in\mathcal{T}} \int_K r_h \mathbf{v}_h \cdot \nabla_x \phi \;\, \mathrm{d}x$$

$$
\begin{aligned}
&= \sum_{K\in\mathcal{T}} \int_K r_h \mathbf{v}_h \cdot (\nabla_x \phi - \nabla_h(\Pi_{\mathcal{T}}\phi))\,\mathrm{d}x + \sum_{K\in\mathcal{T}} (r_h\mathbf{v}_h)_K \cdot \int_{\partial K} \mathbf{n}\,\overline{\Pi_{\mathcal{T}}\phi}\,\mathrm{d}S_x \\
&= \int_\Omega r_h \mathbf{v}_h \cdot (\nabla_x \phi - \nabla_h(\Pi_{\mathcal{T}}\phi))\,\mathrm{d}x - \sum_{\sigma\in\mathcal{E}} \int_\sigma [\![r_h\mathbf{v}_h]\!] \cdot \mathbf{n}\,\overline{\Pi_{\mathcal{T}}\phi}\,\mathrm{d}S_x \\
&= \int_\Omega r_h \mathbf{v}_h \cdot (\nabla_x \phi - \nabla_h(\Pi_{\mathcal{T}}\phi))\,\mathrm{d}x + \sum_{\sigma\in\mathcal{E}} \int_\sigma \overline{r_h\mathbf{v}_h} \cdot \mathbf{n}\,[\![\Pi_{\mathcal{T}}\phi]\!]\,\mathrm{d}S_x \\
&= \int_\Omega r_h \mathbf{v}_h \cdot (\nabla_x \phi - \nabla_h(\Pi_{\mathcal{T}}\phi))\,\mathrm{d}x + \sum_{\sigma\in\mathcal{E}} \int_\sigma (\overline{r_h\mathbf{v}_h} - \overline{r_h}\;\overline{\mathbf{v}_h}) \cdot \mathbf{n}\,[\![\Pi_{\mathcal{T}}\phi]\!]\,\mathrm{d}S_x \\
&\quad + \sum_{\sigma\in\mathcal{E}} \int_\sigma \overline{r_h}\;\overline{\mathbf{v}_h} \cdot \mathbf{n}\,[\![\Pi_{\mathcal{T}}\phi]\!]\,\mathrm{d}S_x \pm \sum_{\sigma\in\mathcal{E}} \int_\sigma \left(\frac{1}{2}|\overline{\mathbf{v}_h}\cdot\mathbf{n}| + h^\varepsilon\right) [\![r_h]\!]\,[\![\Pi_{\mathcal{T}}\phi]\!]\,\mathrm{d}S_x \\
&= \int_\Omega r_h \mathbf{v}_h \cdot (\nabla_x \phi - \nabla_h(\Pi_{\mathcal{T}}\phi))\,\mathrm{d}x + \sum_{\sigma\in\mathcal{E}} \int_\sigma \frac{1}{4}\,[\![r_h]\!]\,[\![\mathbf{v}_h]\!] \cdot \mathbf{n}\,[\![\Pi_{\mathcal{T}}\phi]\!]\,\mathrm{d}S_x \\
&\quad + \sum_{\sigma\in\mathcal{E}} \int_\sigma F_h[r_h,\mathbf{v}_h]\,[\![\Pi_{\mathcal{T}}\phi]\!]\,\mathrm{d}S_x + \sum_{\sigma\in\mathcal{E}} \int_\sigma \left(\frac{1}{2}|\overline{\mathbf{v}_h}\cdot\mathbf{n}| + h^\varepsilon\right) [\![r_h]\!]\,[\![\Pi_{\mathcal{T}}\phi]\!]\,\mathrm{d}S_x.
\end{aligned}
$$

□

Finally, we need a discrete analogue of the Sobolev–type inequality that can be proved exactly as [14, Theorem 11.23].

Lemma 2.5 (Sobolev-type inequality)**.** *Let the function $r \geq 0$ be such that*

$$
0 < \int_\Omega r\,\mathrm{d}x = c_M, \ \text{and} \ \int_\Omega r^\gamma\,\mathrm{d}x \leq c_E \ \text{for}\ \gamma > 1,
$$

where c_M and c_E are some positive constants. Then the following Poincaré–Sobolev type inequality holds true

$$
\|v_h\|_{L^6(\Omega)} \leq c\,\|\nabla_h v_h\|^2_{L^2(\Omega)} + c\left(\int_\Omega r|v_h|\,\mathrm{d}x\right)^2 \lesssim c\,\|\nabla_h v_h\|^2_{L^2(\Omega)} + c_M + c\int_\Omega r|v_h|^2\,\mathrm{d}x
$$

for any $v_h \in Q_h$, where the constant c depends on c_M and c_E but not on the mesh parameter h.

3 Stability

In this section we show the mass conservation, energy dissipation and entropy inequality for the numerical solutions obtained by the finite volume scheme (2.10). In what follows we assume $\varrho_h,\ \vartheta_h > 0$. Note, however, that the non-negativity of the discrete density follows from the renormalized continuity equation Lemma 3.2 in an analogous way as in [25].

3.1 Mass conservation

Setting $\phi_h = 1$ in (2.10a) we derive the mass conservation

$$
\int_\Omega \varrho_h(t)\,\mathrm{d}x = \int_\Omega \varrho_h(0)\,\mathrm{d}x = M_0 > 0,\ t \geq 0. \tag{3.1}
$$

3.2 Total energy dissipation

Theorem 3.1 (Energy balance)**.** *Let $(\varrho_h, \mathbf{u}_h, \vartheta_h)$ satisfy* (2.10)*. Then for any $k = 1, \ldots, N_T$ it holds*

$$D_t \int_\Omega \left(\frac{1}{2} \varrho_h^k |\mathbf{u}_h^k|^2 + c_v \varrho_h^k \vartheta_h^k \right) \mathrm{d}x + h^\varepsilon \sum_{\sigma \in \mathcal{E}} \int_\sigma \overline{\varrho_h^k} \left[\!\left[\mathbf{u}_h^k \right]\!\right]^2 \mathrm{d}S_x$$
$$+ \frac{\Delta t}{2} \int_\Omega \varrho_h^{k-1} |D_t \mathbf{u}_h^k|^2 \mathrm{d}x + \frac{1}{2} \sum_{\sigma \in \mathcal{E}} \int_\sigma (\varrho_h^k)^{\mathrm{up}} |\overline{\mathbf{u}_h^k} \cdot \mathbf{n}| \left[\!\left[\mathbf{u}_h^k \right]\!\right]^2 \mathrm{d}S_x = 0. \quad (3.2)$$

Proof. We start by recalling the kinetic energy balance, cf. [13, equation (3.4)],

$$D_t \int_\Omega \frac{1}{2} \varrho_h |\mathbf{u}_h^k|^2 \mathrm{d}x + 2\mu \int_\Omega |\mathbf{D}_h(\mathbf{u}_h^k)|^2 \mathrm{d}x + \lambda \int_\Omega |\mathrm{div}_h \mathbf{u}_h^k|^2 \mathrm{d}x - \int_\Omega p_h^k \mathrm{div}_h \mathbf{u}_h^k \mathrm{d}x$$
$$+ h^\varepsilon \sum_{\sigma \in \mathcal{E}} \int_\sigma \overline{\varrho_h^k} \left[\!\left[\mathbf{u}_h^k \right]\!\right]^2 \mathrm{d}S_x + \frac{\Delta t}{2} \int_\Omega \varrho_h^{k-1} |D_t \mathbf{u}_h^k|^2 \mathrm{d}x + \frac{1}{2} \sum_{\sigma \in \mathcal{E}} \int_\sigma (\varrho_h^k)^{\mathrm{up}} |\overline{\mathbf{u}_h^k} \cdot \mathbf{n}| \left[\!\left[\mathbf{u}_h^k \right]\!\right]^2 \mathrm{d}S_x = 0.$$

Setting $\phi_h = 1$ in (2.10c) we get

$$D_t \int_\Omega c_v \varrho_h^k \vartheta_h^k \mathrm{d}x = \int_\Omega \left(2\mu |\mathbf{D}_h(\mathbf{u}_h^k)|^2 + \lambda |\mathrm{div}_h \mathbf{u}_h^k|^2 - p_h^k \mathrm{div}_h \mathbf{u}_h^k \right) \mathrm{d}x.$$

Finally, we sum the previous two equations and finish the proof. □

Theorem 3.1 implies the energy dissipation

$$E_h(t) \le E_0, \quad (3.3)$$

where

$$E_h(t) := \int_\Omega \left(\frac{1}{2} \varrho_h(t) |\mathbf{u}_h(t)|^2 + c_v \varrho_h(t) \vartheta_h(t) \right) \mathrm{d}x \quad \text{and} \quad E_0 := E_h(0) = \int_\Omega \left(\frac{1}{2} \varrho_h^0 |\mathbf{u}_h^0|^2 + c_v \varrho_h^0 \vartheta_h^0 \right) \mathrm{d}x.$$

3.3 First a priori estimates

Let us summarize *a priori* estimates that we have obtained so far from (3.1) and (3.2).

$$\varrho_h \in L^\infty\left(0,T;L^1(\Omega)\right), \quad \varrho_h \mathbf{u}_h^2 \in L^\infty\left(0,T;L^1(\Omega)\right), \quad E_h \in L^\infty\left(0,T;L^1(\Omega)\right), \quad p_h \in L^\infty\left(0,T;L^1(\Omega)\right). \quad (3.4)$$

For simplicity, hereafter we denote by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{L^pL^q}$ the norms $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{L^p(0,T;L^q(\Omega))}$, respectively.

3.4 Entropy equation

The physical entropy for the perfect gas law is defined as a function of density ϱ and temperature ϑ as

$$s(\varrho, \vartheta) = \log\left(\frac{\vartheta^{c_v}}{\varrho} \right),$$

and can be rewritten in terms of density ϱ and pressure p as

$$s = s(\varrho, p) = \frac{1}{\gamma - 1} \log\left(\frac{p}{\varrho^\gamma} \right), \qquad \gamma = \frac{1}{c_v} + 1.$$

Then, it is easy to realize that

$$(\varrho, p) \mapsto -\varrho s(\varrho, p) = -\frac{\varrho}{\gamma - 1} \log\left(\frac{p}{\varrho^\gamma} \right)$$

is a convex function of (ϱ, p) for $\varrho > 0$ and $p > 0$. Moreover, it holds

$$\nabla_\varrho(-\varrho s) = c_v + 1 - s, \quad \nabla_p(-\varrho s) = -c_v/\vartheta. \quad (3.5)$$

Before deriving the discrete entropy inequality, we list two renormalized equations. We shall use the notation $\mathrm{co}\{A, B\} \equiv [\min\{A, B\}, \max\{A, B\}]$ in what follows.

Lemma 3.2. *[8, Section 4.1](Renormalized continuity equation) Let $(\varrho_h^k, \mathbf{u}_h^k)$ satisfy* (2.10a). *Then for any $\phi_h \in Q_h$ and any function B that is C^2 on the range of ϱ_h^k we have*

$$\int_\Omega D_t B(\varrho_h^k)\phi_h \,\mathrm{d}x - \sum_{\sigma\in\mathcal{E}} \int_\sigma Up[B(\varrho_h^k), \mathbf{u}_h^k] \llbracket \phi_h \rrbracket \,\mathrm{d}S_x + \int_\Omega \phi_h \left(B'(\varrho_h^k)\varrho_h^k - B(\varrho_h^k) \right) \mathrm{div}_h \mathbf{u}_h^k \,\mathrm{d}x$$
$$= -\int_\Omega \frac{\Delta t}{2} B''(\xi_{\varrho,h}^k)|D_t\varrho_h^k|^2\phi_h \,\mathrm{d}x - \sum_{\sigma\in\mathcal{E}} \int_\sigma \frac{B''(\eta_{\varrho,h}^k)}{2} \llbracket \varrho_h^k \rrbracket^2 |\overline{\mathbf{u}_h^k}\cdot\mathbf{n}|\phi_h \,\mathrm{d}S_x - h^\varepsilon \sum_{\sigma\in\mathcal{E}} \int_\sigma \llbracket \varrho_h^k \rrbracket \llbracket B'(\varrho_h^k)\phi_h \rrbracket \,\mathrm{d}S_x, \tag{3.6}$$

where $\xi_{\varrho,h}^k \in \mathrm{co}\{\varrho_h^{k-1}, \varrho_h^k\}$ and $\eta_{\varrho,h}^k \in \mathrm{co}\{\varrho_K^k, \varrho_L^k\}$ for any $\sigma(= K|L) \in \mathcal{E}_i$, $i = 1, \ldots, d$.

Lemma 3.3. *[23, Lemma 3.3](Renormalized internal energy equation) Let $(\varrho_h, \mathbf{u}_h, \vartheta_h)$ satisfy equation* (2.10c). *Then for any $\sigma \in K|L$ there exists $\xi_{\vartheta,h}^k \in \mathrm{co}\{\vartheta_h^{k-1}, \vartheta_h^k\}$ and $\eta_{\vartheta,h}^k \in \mathrm{co}\{\vartheta_K^k, \vartheta_L^k\}$, such that for any $\phi_h \in Q_h$, and any function χ that is C^2 on the range of ϑ_h^k it holds*

$$c_v \int_\Omega D_t \left(\varrho_h^k \chi(\vartheta_h^k) \right) \phi_h \,\mathrm{d}x - c_v \sum_{\sigma\in\mathcal{E}} \int_\sigma Up(\varrho_h^k\chi(\vartheta_h^k), \mathbf{u}_h^k) \llbracket \phi_h \rrbracket \,\mathrm{d}S_x + \sum_{\sigma\in\mathcal{E}} \int_\sigma \frac{\kappa}{h} \llbracket \vartheta_h^k \rrbracket \llbracket \chi'(\vartheta_h^k)\phi_h \rrbracket \,\mathrm{d}S_x$$
$$= \int_\Omega \left(2\mu|\mathbf{D}_h(\mathbf{u}_h^k)|^2 + \lambda|\mathrm{div}_h\mathbf{u}_h^k|^2 - p_h^k \mathrm{div}_h \mathbf{u}_h^k \right) \chi'(\vartheta_h^k)\phi_h \,\mathrm{d}x - \frac{c_v \Delta t}{2} \int_\Omega \chi''(\xi_{\vartheta,h}^k)\varrho_h^{k-1}|D_t\vartheta_h^k|^2\phi_h \,\mathrm{d}x$$
$$+ \frac{c_v}{2} \sum_{\sigma\in\mathcal{E}} \int_\sigma \chi''(\eta_{\vartheta,h}^k) \llbracket \vartheta_h^k \rrbracket^2 (\varrho_h^k)^{\mathrm{out}} [\overline{\mathbf{u}_h^k}\cdot\mathbf{n}]^- \phi_h \,\mathrm{d}S_x - c_v h^\varepsilon \sum_{\sigma\in\mathcal{E}} \int_\sigma \llbracket \varrho_h^k \rrbracket \left\llbracket \left(\chi(\vartheta_h^k) - \chi'(\vartheta_h^k)\vartheta_h^k \right) \phi_h \right\rrbracket \,\mathrm{d}S_x$$
$$- c_v h^\varepsilon \sum_{\sigma\in\mathcal{E}} \int_\sigma \llbracket \varrho_h^k \vartheta_h^k \rrbracket \llbracket \chi'(\vartheta^k)\phi_h \rrbracket \,\mathrm{d}S_x, \tag{3.7}$$

Now, we are ready to derive the discrete entropy equation for the numerical solution of scheme (2.10).

Lemma 3.4 (Entropy equation)**.** *Let $(\varrho_h, \mathbf{u}_h, \vartheta_h)$ be the solution of our finite volume scheme* (2.10) *such that ϱ_h^k, $\vartheta_h^k > 0$ for all $k = 1, \ldots, N_T$. Then, for any $\phi_h \in Q_h$ it holds*

$$\int_\Omega D_t \left(\varrho_h^k s_h^k \right) \phi_h \,\mathrm{d}x - \sum_{\sigma\in\mathcal{E}} \int_\sigma Up(\varrho_h^k s_h^k, \mathbf{u}_h^k) \llbracket \phi_h \rrbracket \,\mathrm{d}S_x + \int_\Omega \kappa \nabla_\mathcal{E} \vartheta_h^k \cdot \nabla_\mathcal{E}\left(\frac{\phi_h}{\vartheta_h^k} \right) \mathrm{d}x$$
$$- \int_\Omega \left(2\mu|\mathbf{D}_h(\mathbf{u}_h^k)|^2 + \lambda|\mathrm{div}_h\mathbf{u}_h^k|^2 \right) \frac{\phi_h}{\vartheta_h^k} \,\mathrm{d}x = \int_\Omega \left(D_1\phi_h + D_2\overline{\phi_h} + D_3 \cdot \nabla_\mathcal{E}\phi_h \right) \mathrm{d}x, \tag{3.8}$$

where

$$D_1 := \frac{\Delta t}{2\xi_{\varrho,h}^k}|D_t\varrho_h^k|^2 + \frac{h}{2\eta_{\varrho,h}^k}|\nabla_\mathcal{E}\varrho_h^k|^2|\overline{\mathbf{u}_h^k}\cdot\mathbf{n}| + \frac{c_v\Delta t}{2|\xi_{\vartheta,h}^k|^2}\varrho_h^{k-1}|D_t\vartheta_h^k|^2 - \frac{c_v h}{2|\eta_{\vartheta,h}^k|^2}|\nabla_\mathcal{E}\vartheta_h^k|^2(\varrho_h^k)^{\mathrm{out}}[\overline{\mathbf{u}_h^k}\cdot\mathbf{n}]^-,$$
$$D_2 := h^{\varepsilon+1}\nabla_\mathcal{E}\varrho_h^k \cdot \nabla_\mathcal{E}\left(\nabla_\varrho(-\varrho_h^k s_h^k) \right) + h^{\varepsilon+1}\nabla_\mathcal{E} p_h^k \cdot \nabla_\mathcal{E}\left(\nabla_p(-\varrho_h^k s_h^k) \right),$$
$$D_3 := h^{\varepsilon+1}\nabla_\mathcal{E}\varrho_h^k \cdot \overline{\nabla_\varrho(-\varrho_h^k s_h^k)} + h^{\varepsilon+1}\nabla_\mathcal{E} p_h^k \cdot \overline{\nabla_p(-\varrho_h^k s_h^k)}, \tag{3.9}$$

and $\xi_{\varrho,h}^k$, $\eta_{\varrho,h}^k$ and $\xi_{\vartheta,h}^k$, $\eta_{\vartheta,h}^k$ are given in Lemmas 3.2 and 3.3, respectively. Moreover, D_1, $D_2 \geq 0$.

Proof. Firstly, setting $B(\varrho) = \varrho\log(\varrho)$ in the renormalized density equation (3.6) implies

$$\int_\Omega D_t \left(\varrho_h^k \log(\varrho_h^k) \right) \phi_h \,\mathrm{d}x - \sum_{\sigma\in\mathcal{E}} \int_\sigma Up[\varrho_h^k\log(\varrho_h^k), \mathbf{u}_h^k] \llbracket \phi_h \rrbracket \,\mathrm{d}S_x + \int_\Omega \varrho_h^k \mathrm{div}_h \mathbf{u}_h^k \phi_h \,\mathrm{d}x$$

$$
= - \int_\Omega \frac{\Delta t}{2\xi^k_{\varrho,h}} |D_t \varrho_h^k|^2 \phi_h \, \mathrm{d}x - \sum_{\sigma\in\mathcal{E}} \int_\sigma \frac{\phi_h}{2\eta^k_{\varrho,h}} \left[\!\left[\varrho_h^k \right]\!\right]^2 |\overline{\mathbf{u}_h^k} \cdot \mathbf{n}| \, \mathrm{d}S_x - h^\varepsilon \sum_{\sigma\in\mathcal{E}} \int_\sigma \left[\!\left[\varrho_h^k \right]\!\right] \left[\!\left[\Big(\log(\varrho_h^k) + 1 \Big) \phi_h \right]\!\right] \mathrm{d}S_x. \tag{3.10}
$$

Next, we set $\chi(\vartheta) = \log(\vartheta)$ in (3.7) to get

$$
\begin{aligned}
& c_v \int_\Omega D_t \left(\varrho_h^k \log(\vartheta_h^k) \right) \phi_h \, \mathrm{d}x - c_v \sum_{\sigma\in\mathcal{E}} \int_\sigma Up(\varrho_h^k \log(\vartheta_h^k), \mathbf{u}_h^k) \, [\![\phi_h]\!] \, \mathrm{d}S_x + \sum_{\sigma\in\mathcal{E}} \int_\sigma \frac{\kappa}{h} \left[\!\left[\vartheta_h^k \right]\!\right] \left[\!\left[\frac{\phi_h}{\vartheta_h^k} \right]\!\right] \mathrm{d}S_x \\
&= \int_\Omega \left(2\mu |\mathbf{D}_h(\mathbf{u}_h^k)|^2 + \lambda |\mathrm{div}_h \mathbf{u}_h^k|^2 - p(\varrho_h^k) \mathrm{div}_h \mathbf{u}_h^k \right) \frac{\phi_h}{\vartheta_h^k} \, \mathrm{d}x + \frac{c_v \Delta t}{2} \int_\Omega \varrho_h^{k-1} \left| \frac{D_t \vartheta_h^k}{\xi^k_{\vartheta,h}} \right|^2 \phi_h \, \mathrm{d}x \\
&\quad - \frac{c_v}{2} \sum_{\sigma\in\mathcal{E}} \int_\sigma \left| \frac{[\![\vartheta_h^k]\!]}{\eta^k_{\vartheta,h}} \right|^2 (\varrho_h^k)^{\mathrm{out}} [\overline{\mathbf{u}_h^k} \cdot \mathbf{n}]^- \, \mathrm{d}S_x - c_v h^\varepsilon \sum_{\sigma\in\mathcal{E}} \int_\sigma \left[\!\left[\varrho_h^k \right]\!\right] \left[\!\left[\Big(\log(\vartheta_h^k) - 1 \Big) \phi_h \right]\!\right] \mathrm{d}S_x \\
&\quad - c_v h^\varepsilon \sum_{\sigma\in\mathcal{E}} \int_\sigma \left[\!\left[\varrho_h^k \vartheta_h^k \right]\!\right] \left[\!\left[\frac{\phi_h}{\vartheta_h^k} \right]\!\right] \mathrm{d}S_x.
\end{aligned} \tag{3.11}
$$

Subtracting (3.10) from (3.11) yields

$$
\begin{aligned}
& \int_\Omega D_t \left(\varrho_h^k s_h^k \right) \phi_h \, \mathrm{d}x - \sum_{\sigma\in\mathcal{E}} \int_\sigma Up(\varrho_h^k s_h^k, \mathbf{u}_h^k) \, [\![\phi_h]\!] \, \mathrm{d}S_x \\
&\qquad + \sum_{\sigma\in\mathcal{E}} \int_\sigma \frac{\kappa}{h} \left[\!\left[\vartheta_h^k \right]\!\right] \left[\!\left[\frac{\phi_h}{\vartheta_h^k} \right]\!\right] \mathrm{d}S_x - \int_\Omega \left(2\mu |\mathbf{D}_h(\mathbf{u}_h^k)|^2 + \lambda |\mathrm{div}_h \mathbf{u}_h^k|^2 \right) \frac{\phi_h}{\vartheta_h^k} \, \mathrm{d}x \\
&= \int_\Omega \frac{\Delta t}{2\xi^k_{\varrho,h}} |D_t \varrho_h^k|^2 \phi_h \, \mathrm{d}x + \sum_{\sigma\in\mathcal{E}} \int_\sigma \frac{\phi_h}{2\eta^k_{\varrho,h}} \left[\!\left[\varrho_h^k \right]\!\right]^2 |\overline{\mathbf{u}_h^k} \cdot \mathbf{n}| \, \mathrm{d}S_x \\
&\qquad + \frac{c_v \Delta t}{2} \int_\Omega \varrho_h^{k-1} \left| \frac{D_t \vartheta_h^k}{\xi^k_{\vartheta,h}} \right|^2 \phi_h \, \mathrm{d}x - \frac{c_v}{2} \sum_{\sigma\in\mathcal{E}} \int_\sigma \left| \frac{[\![\vartheta_h^k]\!]}{\eta^k_{\vartheta,h}} \right|^2 (\varrho_h^k)^{\mathrm{out}} [\overline{\mathbf{u}_h^k} \cdot \mathbf{n}]^- \phi_h \, \mathrm{d}S_x \\
&\qquad + h^\varepsilon \sum_{\sigma\in\mathcal{E}} \int_\sigma \left[\!\left[\varrho_h^k \right]\!\right] \left[\!\left[\Big(\log(\varrho_h^k) + 1 - c_v \log(\vartheta_h^k) + c_v \Big) \phi_h \right]\!\right] \mathrm{d}S_x - c_v h^\varepsilon \sum_{\sigma\in\mathcal{E}} \int_\sigma \left[\!\left[\varrho_h^k \vartheta_h^k \right]\!\right] \left[\!\left[\frac{\phi_h}{\vartheta_h^k} \right]\!\right] \mathrm{d}S_x.
\end{aligned}
$$

We finish the derivation of (3.8) by applying the product rule (2.3) on the last two terms, rewritten in a convenient way using the identities (3.5) and the notation of the discrete operator (2.1), such that

$$
\begin{aligned}
& h^\varepsilon \sum_{\sigma\in\mathcal{E}} \int_\sigma \left[\!\left[\varrho_h^k \right]\!\right] \left[\!\left[\Big(c_v + 1 - s_h^k \Big) \phi_h \right]\!\right] \mathrm{d}S_x + h^\varepsilon \sum_{\sigma\in\mathcal{E}} \int_\sigma \left[\!\left[\varrho_h^k \vartheta_h^k \right]\!\right] \left[\!\left[\left(-\frac{c_v}{\vartheta_h^k} \right) \phi_h \right]\!\right] \mathrm{d}S_x \\
&\quad = h^\varepsilon \sum_{\sigma\in\mathcal{E}} \int_\sigma \left[\!\left[\varrho_h^k \right]\!\right] \left[\!\left[\nabla_\varrho(-\varrho_h^k s_h^k) \phi_h \right]\!\right] \mathrm{d}S_x + h^\varepsilon \sum_{\sigma\in\mathcal{E}} \int_\sigma \left[\!\left[\varrho_h^k \vartheta_h^k \right]\!\right] \left[\!\left[\nabla_p(-\varrho_h^k s_h^k) \phi_h \right]\!\right] \mathrm{d}S_x \\
&\quad = h^{\varepsilon+1} \int_\Omega \nabla_\mathcal{E} \varrho_h^k \cdot \nabla_\mathcal{E} \left(\nabla_\varrho(-\varrho_h^k s_h^k) \phi_h \right) \mathrm{d}x + h^{\varepsilon+1} \int_\Omega \nabla_\mathcal{E} p_h^k \cdot \nabla_\mathcal{E} \left(\nabla_p(-\varrho_h^k s_h^k) \phi_h \right) \mathrm{d}x \\
&\quad = \int_\Omega \left(D_2 \overline{\phi_h} + D_3 \cdot \nabla_\mathcal{E} \phi_h \right) \mathrm{d}x.
\end{aligned}
$$

The term D_1 is obviously non–negative, and by the convexity of the entropy $-\varrho s(\varrho, p)$ we can conclude that the term D_2 is non–negative as well. Indeed, gradient of any convex sufficiently smooth function is a monotone map. □

3.5 Discrete entropy inequality

The discrete entropy inequality is now a direct consequence of Lemma 3.4. Indeed, we set $\phi_h = 1$ in the entropy equality (3.8) and get

$$\int_\Omega D_t\left(\varrho_h^k s_h^k\right)\,\mathrm{d}x = -\int_\Omega \kappa\nabla_{\mathcal{E}}\vartheta_h^k \cdot \nabla_{\mathcal{E}}\left(\frac{1}{\vartheta_h^k}\right)\,\mathrm{d}x + \int_\Omega \frac{1}{\vartheta_h^k}\left(2\mu|\mathbf{D}_h(\mathbf{u}_h^k)|^2 + \lambda|\mathrm{div}_h\mathbf{u}_h^k|^2\right)\,\mathrm{d}x + \mathcal{B}_h, \quad (3.12)$$

where $\mathcal{B}_h = \int_\Omega D_1 + D_2\,\mathrm{d}x \geq 0$ represents the numerical entropy production, cf. (3.9). The first two terms in (3.12) standing for the discrete counterpart of the physical entropy production are obviously non–negative. To exploit some useful estimates from the entropy production, it is crucial to keep the discrete entropy bounded. To this end we assume the following uniform bounds on the density and temperature:

$$\text{(A1)} \quad 0 < \underline{\varrho} \leq \varrho_h \leq \bar{\varrho} \quad \text{uniformly for all } h \to 0, \quad (3.13a)$$

$$\text{(A2)} \quad 0 < \underline{\vartheta} \leq \vartheta_h \leq \bar{\vartheta} \quad \text{uniformly for all } h \to 0. \quad (3.13b)$$

Clearly, the assumptions (A1) and (A2) imply

$$\underline{s} \leq s_h \leq \bar{s} \quad \text{uniformly for all } h \to 0. \quad (3.14)$$

3.6 Second a priori estimates

In what follows we derive the second *a priori* estimates from the energy equation and the entropy inequality. Firstly, from the energy equation (3.2), under the assumptions (3.13), we directly get the following estimates

$$h^\varepsilon \int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma [\![\mathbf{u}_h]\!]^2\,\mathrm{d}S_x \lesssim 1, \quad (3.15a)$$

$$\int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma |\overline{\mathbf{u}_h}\cdot\mathbf{n}|\,[\![\mathbf{u}_h]\!]^2\,\mathrm{d}S_x \lesssim 1. \quad (3.15b)$$

Secondly, the entropy inequality (3.12) and the assumptions (3.13) imply

$$-\int_0^T \int_\Omega \nabla_{\mathcal{E}}\vartheta_h \cdot \nabla_{\mathcal{E}}\left(\frac{1}{\vartheta_h}\right)\,\mathrm{d}x\,\mathrm{d}t \lesssim 1, \quad (3.16a)$$

$$\int_0^T \int_\Omega \left(2\mu|\mathbf{D}_h(\mathbf{u}_h)|^2 + \lambda|\mathrm{div}_h\mathbf{u}_h|^2\right)\,\mathrm{d}x\,\mathrm{d}t \lesssim 1, \quad (3.16b)$$

$$\int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma |\overline{\mathbf{u}_h}\cdot\mathbf{n}|\,[\![\varrho_h]\!]^2\,\mathrm{d}S_x \lesssim 1, \quad (3.16c)$$

and also

$$\int_0^T \int_\Omega D_1 + D_2\,\mathrm{d}x\,\mathrm{d}t \lesssim 1. \quad (3.16d)$$

Using Lemma 2.5 with (3.4), (3.16b) and (A1), we infer that

$$\|\mathbf{u}_h\|_{L^2L^6} \lesssim 1. \quad (3.16e)$$

Further, applying [8, Lemma 5.1] with $F(\vartheta_h) = \vartheta_h$, $G(\vartheta_h) = (\vartheta_h)^{-1}$, and (A2) we obtain

$$-\frac{\kappa}{h}\sum_{\sigma\in\mathcal{E}}\int_\sigma \left[\!\!\left[\vartheta_h^k\right]\!\!\right]\left[\!\!\left[\frac{1}{\vartheta_h^k}\right]\!\!\right]\,\mathrm{d}S_x \geq \frac{1}{4}\frac{\kappa}{h}\sum_{\sigma\in\mathcal{E}}\int_\sigma \left|\frac{[\![\vartheta_h]\!]}{\overline{\vartheta_h}}\right|^2\,\mathrm{d}S_x \gtrsim \frac{\kappa}{h}\sum_{\sigma\in\mathcal{E}}\int_\sigma [\![\vartheta_h]\!]^2\,\mathrm{d}S_x,$$

which combined with estimate (3.16a) gives the bound on the temperature gradient

$$\int_0^T \int_\Omega |\nabla_{\mathcal{E}} \vartheta_h|^2 \,\mathrm{d}x \,\mathrm{d}t = \int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma \frac{[\![\vartheta_h]\!]^2}{h} \,\mathrm{d}S_x \,\mathrm{d}t \lesssim 1. \tag{3.16f}$$

Thanks to the assumptions (3.13) we also have

$$\begin{aligned}\int_0^T \int_\Omega |D_3| \,\mathrm{d}x \,\mathrm{d}t &\lesssim h^{\varepsilon+1} \int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma \left| [\![\varrho_h]\!] \,\overline{(c_v + 1 - s_h)}\right| \mathrm{d}S_x \,\mathrm{d}t + h^{\varepsilon+1} \int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma \left| [\![\varrho_h \vartheta_h]\!] \,\overline{\left(\frac{-c_v}{\vartheta_h}\right)} \right| \mathrm{d}S_x \\ &\lesssim h^{\varepsilon+1} \int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma |\,[\![\varrho_h]\!]\,| + |[\![\varrho_h\vartheta_h]\!]| \,\mathrm{d}S_x \,\mathrm{d}t \lesssim h^{\varepsilon+1} \int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma \overline{\varrho_h} + \overline{\varrho_h \vartheta_h} \,\mathrm{d}S_x \,\mathrm{d}t \\ &\lesssim h^{\varepsilon},\end{aligned} \tag{3.16g}$$

where we have used the fact that $|\,[\![r_h]\!]\,| \leq 2\overline{r_h}$ for all $r_h \geq 0$.

4 Consistency

In this section, our aim is to show the consistency of the discrete continuity and momentum equations (2.10a) - (2.10b), and the discrete entropy equation (3.8), i.e. that there exist $\beta_i > 0$, $i = 1, 2, 3$, such that the numerical solution for $h \to 0$ satisfies

$$\begin{aligned}-\int_\Omega \varrho_h^0 \phi(0,\cdot) \,\mathrm{d}x &= \int_0^T \int_\Omega [\varrho_h \partial_t \phi + \varrho_h \mathbf{u}_h \cdot \nabla_x \phi] \,\mathrm{d}x \,\mathrm{d}t + \mathcal{O}(h^{\beta_1}), \\ -\int_\Omega \varrho_h^0 \mathbf{u}_h^0 \boldsymbol{\phi}(0,\cdot) \,\mathrm{d}x &= \int_0^T \int_\Omega [\varrho_h \mathbf{u}_h \cdot \partial_t \boldsymbol{\phi} + \varrho_h \mathbf{u}_h \otimes \mathbf{u}_h : \nabla_x \boldsymbol{\phi} + p_h \mathrm{div}_x \boldsymbol{\phi}] \,\mathrm{d}x \,\mathrm{d}t, \\ &\quad -2\mu \int_0^T \int_\Omega \mathbf{D}_h(\mathbf{u}_h) : \mathbf{D}(\boldsymbol{\phi}) \,\mathrm{d}x \,\mathrm{d}t - \lambda \int_0^T \int_\Omega \mathrm{div}_h \mathbf{u}_h \,\mathrm{div}_x \boldsymbol{\phi} \,\mathrm{d}x \,\mathrm{d}t + \mathcal{O}(h^{\beta_2}), \\ -\int_\Omega \varrho_h^0 s_h^0 \phi(0,\cdot) \,\mathrm{d}x &= \int_0^T \int_\Omega [\varrho_h s_h \partial_t \phi + \varrho_h s_h \mathbf{u}_h \cdot \nabla_x \phi] \,\mathrm{d}x - \int_0^T \int_\Omega \kappa \nabla_{\mathcal{E}} \vartheta_h \cdot \left(\frac{1}{\vartheta_h} \nabla_x \phi + \phi \nabla_{\mathcal{E}} \left(\frac{1}{\vartheta_h} \right) \right) \mathrm{d}x \\ &\quad + \int_0^T \int_\Omega \left(2\mu |\mathbf{D}_h(\mathbf{u}_h)|^2 + \lambda |\mathrm{div}_h \mathbf{u}_h|^2 \right) \frac{\phi}{\vartheta_h} \,\mathrm{d}x + \int_0^T \int_\Omega (D_1 + D_2) \phi \,\mathrm{d}x \,\mathrm{d}t + \mathcal{O}(h^{\beta_3}),\end{aligned}$$

for all test functions $\phi \in C^2([0,T] \times \Omega)$, $\boldsymbol{\phi} \in C^2([0,T] \times \Omega; \mathbb{R}^d)$ with $\phi(T) = 0 = \boldsymbol{\phi}(T)$, and $D_1, D_2 \geq 0$ given in Lemma 3.4.

To this end we proceed with each term step by step and estimate the consistency errors. We choose the corresponding piecewise constant test functions $\Pi_{\mathcal{T}} \phi$ and $\Pi_{\mathcal{T}} \boldsymbol{\phi}$ in equations (2.10a), (3.8) and (2.10b), respectively. For convenience, hereafter we use r_h for either ϱ_h, $\varrho_h u_{i,h}$ or $\varrho_h s_h$, and also $\Pi_{\mathcal{T}} \phi$ for $\Pi_{\mathcal{T}} \phi_i$, $i = 1, \ldots, d$.

4.1 Step 1 – time derivative terms

The time derivative term can be rewritten as

$$\begin{aligned}&\int_0^T \int_\Omega D_t r_h \Pi_{\mathcal{T}} \phi \,\mathrm{d}x \,\mathrm{d}t = \int_0^T \int_\Omega \frac{r_h(t) - r_h(t - \Delta t)}{\Delta t} \phi(t) \,\mathrm{d}x \,\mathrm{d}t \\ &= \frac{1}{\Delta t} \int_0^T \int_\Omega r_h(t) \phi(t) \,\mathrm{d}x \,\mathrm{d}t - \frac{1}{\Delta t} \int_{-\Delta t}^{T-\Delta t} \int_\Omega r_h(t) \phi(t + \Delta t) \,\mathrm{d}x \,\mathrm{d}t \\ &= -\int_0^T \int_\Omega r_h(t) D_t \phi(t) \,\mathrm{d}x \,\mathrm{d}t + \frac{1}{\Delta t} \int_{T-\Delta t}^T \int_\Omega r_h(t) \phi(t + \Delta t) \,\mathrm{d}x \,\mathrm{d}t - \frac{1}{\Delta t} \int_{-\Delta t}^0 \int_\Omega r_h(t) \phi(t + \Delta t) \,\mathrm{d}x \,\mathrm{d}t \\ &= -\int_0^T \int_\Omega r_h(t) \left(\partial_t \phi(t) + \frac{\Delta t}{2} \partial_t^2 \phi(t^*) \right) \mathrm{d}x \,\mathrm{d}t - \int_\Omega r_h^0 \phi(0) \,\mathrm{d}x, \text{ for a suitable } t^*.\end{aligned}$$

Using *a priori* estimates (3.4) and (3.14), we derive for r_h being ϱ_h, $\varrho_h u_{i,h}$ and $\varrho_h s_h$ that

$$\int_0^T \int_\Omega D_t r_h \Pi_{\mathcal{T}} \phi \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_\Omega r_h(t) \partial_t \phi(t) \,\mathrm{d}x \,\mathrm{d}t + \int_\Omega r_h^0 \phi(0) \,\mathrm{d}x \lesssim \Delta t \left\| r_h \right\|_{L^1 L^1} \left\| \phi \right\|_{C^2} \lesssim h.$$

4.2 Step 2 – convective terms

To deal with the convective terms, it is convenient to recall the identity from Lemma 2.4,

$$\int_0^T \int_\Omega r_h \mathbf{u}_h \cdot \nabla_x \phi \,\mathrm{d}x \,\mathrm{d}t - \int_0^T \sum_{\sigma \in \mathcal{E}} \int_\sigma F_h[r_h, \mathbf{u}_h] \left[\!\left[\Pi_{\mathcal{T}} \phi \right]\!\right] \mathrm{d}S_x \,\mathrm{d}t = \sum_{j=1}^4 E_j(r_h),$$

where the error terms can be bounded using the interpolation error estimates (2.6) and (2.7) as follows

$$\begin{aligned}
E_1(r_h) &= \frac{1}{2} \int_0^T \sum_{\sigma \in \mathcal{E}} \int_\sigma |\overline{\mathbf{u}_h} \cdot \mathbf{n}| \left[\!\left[r_h \right]\!\right] \left[\!\left[\Pi_{\mathcal{T}} \phi \right]\!\right] \mathrm{d}S_x \,\mathrm{d}t \lesssim h \left\| \phi \right\|_{C^1} \int_0^T \sum_{\sigma \in \mathcal{E}} \int_\sigma |\overline{\mathbf{u}_h} \cdot \mathbf{n}| \left| \left[\!\left[r_h \right]\!\right] \right| \mathrm{d}S_x \,\mathrm{d}t \\
E_2(r_h) &= \frac{1}{4} \int_0^T \sum_{\sigma \in \mathcal{E}} \int_\sigma \left[\!\left[\mathbf{u}_h \right]\!\right] \cdot \mathbf{n} \left[\!\left[r_h \right]\!\right] \left[\!\left[\Pi_{\mathcal{T}} \phi \right]\!\right] \mathrm{d}S_x \,\mathrm{d}t \lesssim h \left\| \phi \right\|_{C^1} \int_0^T \sum_{\sigma \in \mathcal{E}} \int_\sigma \left| \left[\!\left[\mathbf{u}_h \right]\!\right] \cdot \mathbf{n} \left[\!\left[r_h \right]\!\right] \right| \mathrm{d}S_x \,\mathrm{d}t \\
E_3(r_h) &= \int_0^T \int_\Omega r_h \mathbf{u}_h \cdot \Big(\nabla_x \phi - \nabla_h \big(\Pi_{\mathcal{T}} \phi \big) \Big) \,\mathrm{d}x \,\mathrm{d}t \lesssim h \left\| \phi \right\|_{C^2} \int_0^T \int_\Omega |r_h \mathbf{u}_h| \,\mathrm{d}x \,\mathrm{d}t \\
E_4(r_h) &= h^\varepsilon \int_0^T \sum_{\sigma \in \mathcal{E}} \int_\sigma \left[\!\left[r_h \right]\!\right] \left[\!\left[\Pi_{\mathcal{T}} \phi \right]\!\right] \mathrm{d}S_x \,\mathrm{d}t \lesssim h^{\varepsilon+1} \left\| \phi \right\|_{C^1} \int_0^T \sum_{\sigma \in \mathcal{E}} \int_\sigma \left| \left[\!\left[r_h \right]\!\right] \right| \mathrm{d}S_x \,\mathrm{d}t.
\end{aligned}$$

Error terms $E_1(r_h)$

Firstly, by setting $r_h = \varrho_h$ in $E_1(r_h)$ we derive

$$\begin{aligned}
E_1(\varrho_h) &\lesssim h \left\| \phi \right\|_{C^1} \int_0^T \sum_{\sigma \in \mathcal{E}} \int_\sigma |\overline{\mathbf{u}_h} \cdot \mathbf{n}| \left| \left[\!\left[\varrho_h \right]\!\right] \right| \mathrm{d}S_x \,\mathrm{d}t \\
&\lesssim h \left(\int_0^T \sum_{\sigma \in \mathcal{E}} \int_\sigma |\overline{\mathbf{u}_h} \cdot \mathbf{n}| \,\mathrm{d}S_x \,\mathrm{d}t \right)^{1/2} \left(\int_0^T \sum_{\sigma \in \mathcal{E}} \int_\sigma |\overline{\mathbf{u}_h} \cdot \mathbf{n}| \left[\!\left[\varrho_h \right]\!\right]^2 \mathrm{d}S_x \,\mathrm{d}t \right)^{1/2} \qquad (4.1) \\
&\lesssim h^{1/2} \left\| \mathbf{u}_h \right\|_{L^1 L^1}^{1/2} \lesssim h^{1/2} \left\| \mathbf{u}_h \right\|_{L^2 L^6}^{1/2} \lesssim h^{1/2},
\end{aligned}$$

where we have used the Hölder inequality with the estimates (3.16c) and (3.16e).
Secondly, for $r_h = \varrho_h u_{i,h}$ we control the consistency error $E_1(r_h)$ as follows

$$\begin{aligned}
E_1(\varrho_h u_{i,h}) &\lesssim h \left\| \phi \right\|_{C^1} \int_0^T \sum_{\sigma \in \mathcal{E}} \int_\sigma |\overline{\mathbf{u}_h} \cdot \mathbf{n}| \left| \left[\!\left[\varrho_h u_{i,h} \right]\!\right] \right| \mathrm{d}S_x \,\mathrm{d}t \\
&\lesssim h \int_0^T \sum_{\sigma \in \mathcal{E}} \int_\sigma |\overline{\mathbf{u}_h} \cdot \mathbf{n}| \left| \left[\!\left[\varrho_h \right]\!\right] \right| |\overline{u_{i,h}}| \,\mathrm{d}S_x \,\mathrm{d}t + h \int_0^T \sum_{\sigma \in \mathcal{E}} \int_\sigma |\overline{\mathbf{u}_h} \cdot \mathbf{n}| \left| \left[\!\left[u_{i,h} \right]\!\right] \right| |\overline{\varrho_h}| \,\mathrm{d}S_x \,\mathrm{d}t \\
&\lesssim h \left(\int_0^T \sum_{\sigma \in \mathcal{E}} \int_\sigma |\overline{\mathbf{u}_h}|^3 \,\mathrm{d}S_x \,\mathrm{d}t \right)^{1/2} \left(\int_0^T \sum_{\sigma \in \mathcal{E}} \int_\sigma |\overline{\mathbf{u}_h} \cdot \mathbf{n}| \left[\!\left[\varrho_h \right]\!\right]^2 \mathrm{d}S_x \,\mathrm{d}t \right)^{1/2} \\
&+ h \left(\int_0^T \sum_{\sigma \in \mathcal{E}} \int_\sigma |\overline{\mathbf{u}_h}|^2 \,\mathrm{d}S_x \,\mathrm{d}t \right)^{1/2} \left(\int_0^T \sum_{\sigma \in \mathcal{E}} \int_\sigma \left[\!\left[\mathbf{u}_h \right]\!\right]^2 \mathrm{d}S_x \,\mathrm{d}t \right)^{1/2} \\
&\lesssim h^{1/2} + h^{\frac{1-\varepsilon}{2}}.
\end{aligned}$$

Here we have used the Hölder inequality, product rule (2.3), the estimates (3.15a), (3.16c), and the interpolation inequality

$$\|\mathbf{u}_h\|_{L^3L^3} \stackrel{<}{\sim} \|\mathbf{u}_h\|_{L^\infty L^2}^{1/2} \|\mathbf{u}_h\|_{L^2L^6}^{1/2} \stackrel{<}{\sim} 1$$

with (3.4), (3.16e) and (A1).

Note that for any $f \in C^1((0,\infty))$ there exists $z_h^* \in \operatorname{co}\{z_h^{\text{out}}, z_h^{\text{in}}\}$ such that the following estimate holds

$$|\,[\![f(z_h)]\!]\,| = |\nabla f(z_h^*)\,[\![z_h]\!]\,| \lesssim |\,[\![z_h]\!]\,|. \tag{4.2}$$

Hence, setting $r_h = \varrho_h s_h$ in $E_1(r_h)$ and using (4.2) with $f(z_h) = \log(z_h)$, $z_h \in \{\varrho_h, \vartheta_h\}$, we finally get

$$\begin{aligned}
E_1(\varrho_h s_h) &\stackrel{<}{\sim} h\,\|\phi\|_{C^1} \int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma |\overline{\mathbf{u}_h}\cdot\mathbf{n}|\,|\,[\![\varrho_h s_h]\!]\,|\,\mathrm{d}S_x\,\mathrm{d}t \\
&\stackrel{<}{\sim} h \int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma |\overline{\mathbf{u}_h}\cdot\mathbf{n}|\,|\,[\![\varrho_h]\!]\,|\,|\overline{s_h}|\,\mathrm{d}S_x\,\mathrm{d}t + h \int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma |\overline{\mathbf{u}_h}\cdot\mathbf{n}|\,|\,[\![c_v \log\vartheta_h - \log\varrho_h]\!]\,|\,|\overline{\varrho_h}|\,\mathrm{d}S_x\,\mathrm{d}t \\
&\stackrel{<}{\sim} h \int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma |\overline{\mathbf{u}_h}\cdot\mathbf{n}|\,|\,[\![\varrho_h]\!]\,|\,\mathrm{d}S_x\,\mathrm{d}t + h \int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma |\overline{\mathbf{u}_h}\cdot\mathbf{n}|\,(|\,[\![\vartheta_h]\!]\,| + |\,[\![\varrho_h]\!]\,|)\,\mathrm{d}S_x\,\mathrm{d}t \\
&\stackrel{<}{\sim} 2h \int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma |\overline{\mathbf{u}_h}\cdot\mathbf{n}|\,|\,[\![\varrho_h]\!]\,|\,\mathrm{d}S_x\,\mathrm{d}t + h \left(\int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma |\overline{\mathbf{u}_h}|^2\,\mathrm{d}S_x\,\mathrm{d}t\right)^{1/2} \left(\int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma [\![\vartheta_h]\!]^2\,\mathrm{d}S_x\,\mathrm{d}t\right)^{1/2} \\
&\stackrel{<}{\sim} h^{1/2} + h\,\|\mathbf{u}_h\|_{L^2L^2} \stackrel{<}{\sim} h^{1/2},
\end{aligned}$$

where we have again used the product rule (2.3), the Hölder inequality, and the estimates (4.1), (3.16c), (3.16f) and (3.4) with the assumptions (3.14) and (A1).

Error terms $E_2(r_h)$

To deal with the second error terms we first set $r_h = \varrho_h$ and obtain

$$E_2(\varrho_h) \stackrel{<}{\sim} h\,\|\phi\|_{C^1} \left(\int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma [\![\mathbf{u}_h]\!]^2\,\mathrm{d}S_x\,\mathrm{d}t\right)^{1/2} \left(\int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma [\![\varrho_h]\!]^2\,\mathrm{d}S_x\,\mathrm{d}t\right)^{1/2} \stackrel{<}{\sim} h^{\frac{1-\varepsilon}{2}}$$

due to (3.15a) and (3.13a).

Then, inserting $r_h = \varrho_h u_{i,h}$ into $E_2(r_h)$ and taking into account the estimates (3.4), (3.15a) with (A1) we get in an analogous way as before

$$\begin{aligned}
E_2(\varrho_h u_{i,h}) &\stackrel{<}{\sim} h\,\|\phi\|_{C^1} \int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma \left|\,[\![\mathbf{u}_h]\!]\cdot\mathbf{n}\big(\,[\![\varrho_h]\!]\,\overline{u_{i,h}} + \overline{\varrho_h}\,[\![u_{i,h}]\!]\,\big)\right|\,\mathrm{d}S_x\,\mathrm{d}t \\
&\stackrel{<}{\sim} h \int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma |\,[\![\varrho_h]\!]\,|\,|\,[\![\mathbf{u}_h]\!]\cdot\mathbf{n}|\,|\overline{\mathbf{u}_h}|\,\mathrm{d}S_x\,\mathrm{d}t + h \int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma \overline{\varrho_h}\,[\![\mathbf{u}_h]\!]^2\,\mathrm{d}S_x\,\mathrm{d}t \\
&\stackrel{<}{\sim} h \left(\int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma [\![\mathbf{u}_h]\!]^2\,\mathrm{d}S_x\,\mathrm{d}t\right)^{1/2} \left(\int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma |\overline{\mathbf{u}_h}|^2\,\mathrm{d}S_x\,\mathrm{d}t\right)^{1/2} + h^{1-\varepsilon} \\
&\stackrel{<}{\sim} h^{1-\frac{1}{2}-\frac{\varepsilon}{2}}\,\|\mathbf{u}_h\|_{L^2L^2} + h^{1-\varepsilon} \stackrel{<}{\sim} h^{\frac{1-\varepsilon}{2}} + h^{1-\varepsilon}.
\end{aligned}$$

Finally, for $r_h = \varrho_h s_h$ we deduce, by (4.2) with $f(z_h) = \log(z_h)$, $z_h \in \{\varrho_h, \vartheta_h\}$, and (3.15a), (2.1), (A1), the bound

$$
\begin{aligned}
E_2(\varrho_h s_h) &\lesssim h \|\phi\|_{C^1} \int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma \left| [\![\mathbf{u}_h]\!] \cdot \mathbf{n}\big([\![\varrho_h]\!] \, \overline{s_h} + \overline{\varrho_h} \, [\![s_h]\!] \big) \right| \mathrm{d}S_x \, \mathrm{d}t \\
&\lesssim h \int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma | [\![\mathbf{u}_h]\!] \cdot \mathbf{n}| \big(| [\![\varrho_h]\!] | + | [\![c_v \log \vartheta_h - \log \varrho_h]\!] \, \overline{\varrho_h}| \big) \, \mathrm{d}S_x \, \mathrm{d}t \\
&\lesssim h \left(\int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma [\![\mathbf{u}_h]\!]^2 \, \mathrm{d}S_x \, \mathrm{d}t \right)^{1/2} \left[\left(\int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma [\![\varrho_h]\!]^2 \, \mathrm{d}S_x \, \mathrm{d}t \right)^{1/2} + \left(\int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma [\![\vartheta_h]\!]^2 \, \mathrm{d}S_x \, \mathrm{d}t \right)^{1/2} \right] \\
&\lesssim h^{\frac{1-\varepsilon}{2}} + h^{\frac{3-\varepsilon}{2}}.
\end{aligned}
$$

Error terms $E_3(r_h)$

The estimates of the third error terms are straightforward due to (3.4), (3.13) and (3.14). Indeed,

$$
\begin{aligned}
E_3(\varrho_h) &\lesssim h \|\phi\|_{C^2} \int_0^T \int_\Omega |\varrho_h \mathbf{u}_h| \, \mathrm{d}x \, \mathrm{d}t \lesssim h \|\mathbf{u}_h\|_{L^2 L^2} \lesssim h, \\
E_3(\varrho_h u_{i,h}) &\lesssim h \|\phi\|_{C^2} \int_0^T \int_\Omega |\varrho_h u_{i,h} \mathbf{u}_h| \, \mathrm{d}x \, \mathrm{d}t \lesssim h \left\| \varrho_h |\mathbf{u}_h|^2 \right\|_{L^\infty L^1} \lesssim h, \\
E_3(\varrho_h s_h) &\lesssim h \|\phi\|_{C^2} \int_0^T \int_\Omega |\varrho_h s_h \mathbf{u}_h| \, \mathrm{d}x \, \mathrm{d}t \lesssim h \|\mathbf{u}_h\|_{L^2 L^2} \lesssim h.
\end{aligned}
$$

Error terms $E_4(r_h)$

Finally, we treat the fourth error terms. For $r_h = \varrho_h$ the argumentation is simple and analogous as above. For $r_h = \varrho_h s_h$ the term is not present, i.e. $E_4(\varrho_h s_h) = 0$. Thus we only concentrate on a slightly more involved estimate for $r_h = \varrho_h u_{i,h}$,

$$
\begin{aligned}
E_4(\varrho_h u_{i,h}) &\lesssim h^{\varepsilon+1} \|\phi\|_{C^1} \int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma | [\![\varrho_h u_{i,h}]\!] | \, \mathrm{d}S_x \, \mathrm{d}t \lesssim h^{\varepsilon+1} \int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma | [\![\varrho_h]\!] \, \overline{u_{i,h}}| + |\overline{\varrho_h} \, [\![u_{i,h}]\!] | \, \mathrm{d}S_x \, \mathrm{d}t \\
&\lesssim h^{\varepsilon+1} \left(\int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma |\overline{\mathbf{u}_h}|^2 \, \mathrm{d}S_x \, \mathrm{d}t \right)^{1/2} \left(\int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma [\![\varrho_h]\!]^2 \, \mathrm{d}S_x \, \mathrm{d}t \right)^{1/2} \\
&\quad + h^{\varepsilon+1} \left(\int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma \overline{\varrho_h}^2 \, \mathrm{d}S_x \, \mathrm{d}t \right)^{1/2} \left(\int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma [\![\mathbf{u}_h]\!]^2 \, \mathrm{d}S_x \, \mathrm{d}t \right)^{1/2} \\
&\lesssim h^{\varepsilon} + h^{(\varepsilon+1)/2},
\end{aligned}
$$

where we have used the assumption (A1), and bounds (3.4), (3.15a).

Collecting the above estimates of $E_i(r_h), i = 1, \ldots, 4$ for $r_h \in \{\varrho_h, \varrho_h u_{i,h}, \varrho_h s_h\}$, we know that there exists a positive $\beta > 0$ such that

$$
\int_0^T \int_\Omega r_h \mathbf{u}_h \cdot \nabla_x \phi \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma F_h[r_h, \mathbf{u}_h] \, [\![\Pi_{\mathcal{T}} \phi]\!] \, \mathrm{d}S_x \, \mathrm{d}t = \sum_{j=1}^4 E_j(r_h) \lesssim h^\beta,
$$

provided $\varepsilon \in (0,1)$.

4.3 Step 3 – κ–term in the entropy equation (3.8)

Using the product rule (2.3) we can write

$$
\begin{aligned}
&\int_0^T\int_\Omega \kappa\nabla_{\mathcal{E}}\vartheta_h\cdot\left(\frac{1}{\vartheta_h}\nabla_x\phi+\phi\nabla_{\mathcal{E}}\left(\frac{1}{\vartheta_h}\right)\right)\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_\Omega \kappa\nabla_{\mathcal{E}}\vartheta_h\cdot\nabla_{\mathcal{E}}\left(\frac{\Pi_{\mathcal{T}}\phi}{\vartheta_h}\right)\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_0^T\int_\Omega \kappa\nabla_{\mathcal{E}}\vartheta_h\cdot\left(\frac{1}{\vartheta_h}\nabla_x\phi+\phi\nabla_{\mathcal{E}}\left(\frac{1}{\vartheta_h}\right)\right)\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_0^T\int_\Omega \kappa\nabla_{\mathcal{E}}\vartheta_h\cdot\left((\nabla_{\mathcal{E}}(\Pi_{\mathcal{T}}\phi))\overline{\left(\frac{1}{\vartheta_h}\right)}+\left(\nabla_{\mathcal{E}}\left(\frac{1}{\vartheta_h}\right)\right)\overline{\Pi_{\mathcal{T}}\phi}\right)\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_0^T\int_\Omega \kappa\nabla_{\mathcal{E}}\vartheta_h\cdot\left(\frac{1}{\vartheta_h}\nabla_x\phi-\overline{\left(\frac{1}{\vartheta_h}\right)}(\nabla_{\mathcal{E}}(\Pi_{\mathcal{T}}\phi))\right)\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega \kappa\nabla_{\mathcal{E}}\vartheta_h\cdot\left(\nabla_{\mathcal{E}}\left(\frac{1}{\vartheta_h}\right)\right)\left(\phi-\overline{\Pi_{\mathcal{T}}\phi}\right)\,\mathrm{d}x\,\mathrm{d}t\\
&=: I_1+I_2,
\end{aligned}
$$

where the residual terms I_1 and I_2 shall be controlled in what follows. Applying the Hölder inequality, interpolation estimates (2.6), (2.7) and (3.16f) yields

$$
\begin{aligned}
I_1&=\int_0^T\int_\Omega \kappa\nabla_{\mathcal{E}}\vartheta_h\cdot\left(\frac{1}{\vartheta_h}\nabla_x\phi-\overline{\left(\frac{1}{\vartheta_h}\right)}\nabla_{\mathcal{E}}(\Pi_{\mathcal{T}}\phi)\pm\overline{\left(\frac{1}{\vartheta_h}\right)}\nabla_x\phi\right)\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_0^T\int_\Omega \kappa\nabla_{\mathcal{E}}\vartheta_h\cdot\nabla_x\phi\left(\frac{1}{\vartheta_h}-\overline{\left(\frac{1}{\vartheta_h}\right)}\right)\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega \kappa\overline{\left(\frac{1}{\vartheta_h}\right)}\nabla_{\mathcal{E}}\vartheta_h\cdot(\nabla_x\phi-\nabla_{\mathcal{E}}(\Pi_{\mathcal{T}}\phi))\,\mathrm{d}x\,\mathrm{d}t\\
&\stackrel{<}{\sim} h\left\|\nabla_{\mathcal{E}}\vartheta_h\right\|_{L^2L^2}\left\|\nabla_{\mathcal{E}}\left(\frac{1}{\vartheta_h}\right)\right\|_{L^2L^2}\|\phi\|_{C^1}+h\left\|\nabla_{\mathcal{E}}\vartheta_h\right\|_{L^2L^2}\|\phi\|_{C^2}\stackrel{<}{\sim} h.
\end{aligned}
$$

Recalling (4.2) with $f(\vartheta_h)=\log(\vartheta_h)$ we could infer from (3.16f) the bound

$$
\left\|\nabla_{\mathcal{E}}\left(\frac{1}{\vartheta_h}\right)\right\|_{L^2L^2}\stackrel{<}{\sim}\|\nabla_{\mathcal{E}}\vartheta_h\|_{L^2L^2}\lesssim 1.
$$

By an analogous argument we have

$$
I_2=\int_0^T\int_\Omega \kappa\nabla_{\mathcal{E}}\vartheta_h\cdot\left(\nabla_{\mathcal{E}}\left(\frac{1}{\vartheta_h}\right)\right)\left(\phi-\overline{\Pi_{\mathcal{T}}\phi}\right)\,\mathrm{d}x\,\mathrm{d}t\stackrel{<}{\sim} h\left\|\nabla_{\mathcal{E}}\vartheta_h\right\|_{L^2L^2}\left\|\nabla_{\mathcal{E}}\left(\frac{1}{\vartheta_h}\right)\right\|_{L^2L^2}\|\phi\|_{C^1}\stackrel{<}{\sim} h.
$$

Thus we have shown the consistency of the κ–term

$$
\int_0^T\int_\Omega \kappa\nabla_{\mathcal{E}}\vartheta_h\cdot\left(\frac{1}{\vartheta_h}\nabla_x\phi+\phi\nabla_{\mathcal{E}}\left(\frac{1}{\vartheta_h}\right)\right)\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_\Omega \kappa\nabla_{\mathcal{E}}\vartheta_h\cdot\left(\nabla_{\mathcal{E}}\left(\frac{\Pi_{\mathcal{T}}\phi}{\vartheta_h}\right)\right)\,\mathrm{d}x\,\mathrm{d}t\stackrel{<}{\sim} h.
$$

4.4 Step 4 – dissipation terms

Applying the estimate (3.16b) with (A2) for the dissipation terms in the entropy equation (3.8) we immediately get

$$
\begin{aligned}
&\int_0^T\int_\Omega\left(2\mu|\mathbf{D}_h(\mathbf{u}_h)|^2+\lambda|\mathrm{div}_h\mathbf{u}_h|^2\right)\frac{\Pi_{\mathcal{T}}\phi}{\vartheta_h}\,\mathrm{d}x-\int_0^T\int_\Omega\left(2\mu|\mathbf{D}_h(\mathbf{u}_h)|^2+\lambda|\mathrm{div}_h\mathbf{u}_h|^2\right)\frac{\phi}{\vartheta_h}\,\mathrm{d}x\\
&\stackrel{<}{\sim} h\|\phi\|_{C^1}\int_0^T\int_\Omega\left(2\mu|\mathbf{D}_h(\mathbf{u}_h)|^2+\lambda|\mathrm{div}_h\mathbf{u}_h|^2\right)\frac{1}{\vartheta_h}\,\mathrm{d}x\stackrel{<}{\sim} h.
\end{aligned}
$$

4.5 Step 5 – viscosity terms

The interpolation error estimate (2.7) and the *a priori* bound (3.16b) are enough to control the viscosity terms in the momentum equation. Indeed, we have

$$\begin{aligned}&\int_0^T\int_\Omega \mathbf{D}_h(\mathbf{u}_h):\mathbf{D}(\boldsymbol{\phi})\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_\Omega \mathbf{D}_h(\mathbf{u}_h):\mathbf{D}_h(\Pi_{\mathcal{T}}\boldsymbol{\phi})\,\mathrm{d}x\,\mathrm{d}t\\ &=\int_0^T\int_\Omega \mathbf{D}_h(\mathbf{u}_h):\big(\mathbf{D}(\boldsymbol{\phi})-\mathbf{D}_h(\Pi_{\mathcal{T}}\boldsymbol{\phi})\big)\,\mathrm{d}x\,\mathrm{d}t\lesssim \|\mathbf{D}_h(\mathbf{u}_h)\|_{L^2L^2}\,h\,\|\boldsymbol{\phi}\|_{C^2}\lesssim h,\end{aligned}$$

and analogously, for the divergence term

$$\begin{aligned}&\int_0^T\int_\Omega \mathrm{div}_h\mathbf{u}_h\,\mathrm{div}_h(\Pi_{\mathcal{T}}\boldsymbol{\phi})\,\mathrm{d}x-\int_0^T\int_\Omega \mathrm{div}_h\mathbf{u}_h\,\mathrm{div}_x\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t\\ &=\int_0^T\int_\Omega \mathrm{div}_h\mathbf{u}_h\,\big(\mathrm{div}_h\,(\Pi_{\mathcal{T}}\boldsymbol{\phi})-\mathrm{div}_x\boldsymbol{\phi}\big)\,\mathrm{d}x\,\mathrm{d}t\lesssim \|\mathrm{div}_h\mathbf{u}_h\|_{L^2L^2}\,h\,\|\boldsymbol{\phi}\|_{C^2}\lesssim h.\end{aligned}$$

4.6 Step 6 – pressure term

The pressure term in the momentum equation is controlled, thanks to the interpolation estimate (2.7) and the *a priori* estimate (3.4) for the pressure, as

$$\int_0^T\int_\Omega p_h\mathrm{div}_h(\Pi_{\mathcal{T}}\boldsymbol{\phi})\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_\Omega p_h\mathrm{div}_x\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t\lesssim \|p_h\|_{L^\infty L^1}\,h\,\|\boldsymbol{\phi}\|_{C^2}\lesssim h.$$

4.7 Step 7 – entropy production terms D_1, D_2 and D_3

In an analogous way we bound the three entropy production terms in the entropy equation,

$$\int_0^T\int_\Omega (D_1\Pi_{\mathcal{T}}\phi+D_2\overline{\Pi_{\mathcal{T}}\phi})\,\mathrm{d}x-\int_0^T\int_\Omega (D_1+D_2)\phi\,\mathrm{d}x\lesssim h\,\|\phi\|_{C^1}\,\big(\,\|D_1\|_{L^1L^1}+\|D_2\|_{L^1L^1}\,\big)\lesssim h,$$

and

$$\int_0^T\int_\Omega D_3\cdot\nabla_{\mathcal{E}}(\Pi_{\mathcal{T}}\phi)\,\mathrm{d}x\lesssim \|D_3\|_{L^1L^1}\,\|\phi\|_{C^2}\lesssim h^\varepsilon,$$

using the *a priori* estimates (3.16d) and (3.16g), respectively.

Let us summarize the above calculations leading to the desired consistency formulation of the numerical approximation of the continuity and momentum equations as well as the discrete entropy equation.

Lemma 4.1 (Consistency of the continuity and momentum equations)**.** *Let $(\varrho_h,\mathbf{u}_h,\vartheta_h)$, $h\in(0,h_0)$, $h_0\ll 1$ be the numerical solution obtained by our finite volume scheme* (2.10) *with $\Delta t\approx h$ and $0<\varepsilon<1$. Then there exists $\beta>0$ such that*

$$-\int_\Omega \varrho_h^0\phi(0,\cdot)\,\mathrm{d}x=\int_0^T\int_\Omega[\varrho_h\partial_t\phi+\varrho_h\mathbf{u}_h\cdot\nabla_x\phi]\,\mathrm{d}x\,\mathrm{d}t+\mathcal{O}(h^\beta),\tag{4.5}$$

for any $\phi\in C^2([0,T]\times\Omega)$, $\phi(T)=0$;

$$\begin{aligned}-\int_\Omega \varrho_h^0\mathbf{u}_h^0\boldsymbol{\phi}(0,\cdot)\,\mathrm{d}x=&\int_0^T\int_\Omega[\varrho_h\mathbf{u}_h\cdot\partial_t\boldsymbol{\phi}+\varrho_h\mathbf{u}_h\otimes\mathbf{u}_h:\nabla_x\boldsymbol{\phi}+p_h\mathrm{div}_x\boldsymbol{\phi}]\,\mathrm{d}x\,\mathrm{d}t\\ &-\mu\int_0^T\int_\Omega \mathbf{D}_h(\mathbf{u}_h):\mathbf{D}(\boldsymbol{\phi})\,\mathrm{d}x\,\mathrm{d}t-\lambda\int_0^T\int_\Omega \mathrm{div}_h\mathbf{u}_h\,\mathrm{div}_x\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t+\mathcal{O}(h^\beta),\end{aligned}\tag{4.6}$$

for any $\boldsymbol{\phi}\in C^2([0,T]\times\Omega;\mathbb{R}^d)$, $\boldsymbol{\phi}(T)=0$.

Lemma 4.2 (Consistency of the entropy equation)**.** *Let* $(\varrho_h, \mathbf{u}_h, \vartheta_h)$*,* $h \in (0, h_0)$*,* $h_0 \ll 1$ *be the numerical solution obtained by our finite volume scheme* (2.10) *with* $\Delta t \approx h$ *and* $0 < \varepsilon < 1$*. Then there exists* $\beta > 0$ *such that for any* $\phi \in C^2([0,T] \times \Omega)$*,* $\phi(T) = 0$*, it holds that*

$$-\int_\Omega \varrho_h^0 s_h^0 \phi(0,\cdot)\,\mathrm{d}x = \int_0^T \int_\Omega [\varrho_h s_h \partial_t \phi + \varrho_h s_h \mathbf{u}_h \cdot \nabla_x \phi]\,\mathrm{d}x - \int_0^T \int_\Omega \kappa \nabla_\mathcal{E} \vartheta_h \cdot \left(\frac{1}{\vartheta_h} \nabla_x \phi + \phi \nabla_\mathcal{E} \left(\frac{1}{\vartheta_h} \right) \right) \mathrm{d}x$$
$$+ \int_0^T \int_\Omega \left(2\mu |\mathbf{D}_h(\mathbf{u}_h)|^2 + \lambda |\mathrm{div}_h \mathbf{u}_h|^2 \right) \frac{\phi}{\vartheta_h}\,\mathrm{d}x + \int_0^T \int_\Omega (D_1 + D_2)\phi\,\mathrm{d}x\,\mathrm{d}t + \mathcal{O}(h^\beta), \quad (4.7)$$

where $D_1 + D_2 \in L^1((0,T) \times \Omega)$ *are the non–negative numerical entropy production terms, cf.* (3.9).

It should be pointed out here again that the numerical scheme (2.10) is energy dissipative, cf. (3.3), which means

$$\int_\Omega \left(\frac{1}{2} \varrho_h |\mathbf{u}_h|^2 + c_v \varrho_h \vartheta_h \right) \mathrm{d}x \leq \int_\Omega \left(\frac{1}{2} \varrho_h^0 |\mathbf{u}_h^0|^2 + c_v \varrho_h^0 \vartheta_h^0 \right) \mathrm{d}x. \quad (4.8)$$

5 Convergence of the finite volume method

The aim of this section is to show the convergence of our finite volume method (2.9) to the strong solution of the Navier–Stokes–Fourier system (1.1) on the lifespan of the latter. We begin with the definition of the DMV solution of (1.1) which plays an essential role in the proof of the main result, see also [3, Definition 2.3].

Definition 5.1 (DMV solution)**.** A parametrized family of probability measures $\{\mathcal{V}_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$ is the *dissipative measure–valued* solution to the Navier–Stokes–Fourier system (1.1) with the initial condition $\{\mathcal{V}_{0,x}\}_{x\in\Omega}$ if the following hold:

- the mapping
$$\mathcal{V}_{t,x} : (t,x) \in (0,T) \times \Omega \mapsto \mathcal{P}(\mathcal{F}) \quad \text{is weakly-(*) measurable,}$$
with $\mathcal{P}$ being the space of probability measures defined on the phase space
$$\mathcal{F} = \left\{ \varrho, \vartheta, \mathbf{u}, \mathbf{D}_u, \mathbf{D}_\vartheta \;\middle|\; \varrho \geq 0,\ \vartheta \geq 0,\ \mathbf{u} \in \mathbb{R}^d,\ \mathbf{D}_u \in \mathbb{R}^{d\times d}_{\mathrm{sym}},\ \mathbf{D}_\vartheta \in \mathbb{R}^d \right\};$$
- $\{\mathcal{V}_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$ complies with the compatibility condition
$$-\int_0^T \int_\Omega \langle \mathcal{V}_{t,x}; \mathbf{u} \rangle \cdot \mathrm{div}_x \mathbb{T}\ \mathrm{d}x\ \mathrm{d}t = \int_0^T \int_\Omega \langle \mathcal{V}_{t,x}; \mathbf{D}_u \rangle : \mathbb{T}\ \mathrm{d}x\ \mathrm{d}t, \quad \text{for any } \mathbb{T} \in C^1([0,T]\times\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$$
$$-\int_0^T \int_\Omega \langle \mathcal{V}_{t,x}; \vartheta \rangle \mathrm{div}_x \boldsymbol{\varphi}\ \mathrm{d}x\ \mathrm{d}t = \int_0^T \int_\Omega \langle \mathcal{V}_{t,x}; \mathbf{D}_\vartheta \rangle \boldsymbol{\varphi}\ \mathrm{d}x\ \mathrm{d}t, \quad \text{for any } \boldsymbol{\varphi} \in C^1([0,T]\times\Omega; \mathbb{R}^d); \quad (5.1)$$
- conservation of mass
$$\left[\int_\Omega \langle \mathcal{V}_{t,x}; \varrho \rangle\, \varphi(t,x)\,\mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [\langle \mathcal{V}_{t,x}; \varrho \rangle\, \partial_t \varphi(t,x) + \langle \mathcal{V}_{t,x}; \varrho\mathbf{u} \rangle \cdot \nabla_x \varphi(t,x)]\,\mathrm{d}x\,\mathrm{d}t \quad (5.2)$$
for a.a. $\tau \in [0,T]$ and any $\varphi \in C^1([0,T]\times\Omega)$;
- balance of momentum
$$\left[\int_\Omega \langle \mathcal{V}_{t,x}; \varrho\mathbf{u} \rangle \cdot \boldsymbol{\varphi}(t,x)\,\mathrm{d}x \right]_{t=0}^{t=\tau}$$
$$= \int_0^\tau \int_\Omega [\langle \mathcal{V}_{t,x}; \varrho\mathbf{u} \rangle \cdot \partial_t \boldsymbol{\varphi}(t,x) + \langle \mathcal{V}_{t,x}; \varrho\mathbf{u}\otimes\mathbf{u} \rangle : \nabla_x \boldsymbol{\varphi}(t,x) + \langle \mathcal{V}_{t,x}; p(\varrho,\vartheta) \rangle\, \mathrm{div}_x \boldsymbol{\varphi}(t,x)]\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; \mathbb{S}(\mathbf{D}_u) \rangle : \nabla_x \boldsymbol{\varphi}(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega \nabla_x \boldsymbol{\varphi} : \mathrm{d}\nu_C \quad (5.3)$$

for a.a. $\tau \in [0,T]$ and any $\boldsymbol{\varphi} \in C^1([0,T]\times\Omega;\mathbb{R}^d)$, where $\nu_C \in \mathcal{M}([0,T]\times\Omega;\mathbb{R}^{d\times d})^1$ is called *concentration defect measure*;

- energy inequality

$$\int_\Omega \left\langle \mathcal{V}_{\tau,x}; \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) \right\rangle \mathrm{d}x \le \int_\Omega \left\langle \mathcal{V}_{0,x}; \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) \right\rangle \mathrm{d}x \tag{5.4}$$

for a.a. $\tau \in [0,T]$;

- entropy inequality

$$\begin{aligned}
&\left[\int_\Omega \langle \mathcal{V}_{t,x}; \varrho s(\varrho,\vartheta)\rangle \varphi(t,x)\,\mathrm{d}x\right]_{t=0}^{t=\tau} \\
&\ge \int_0^\tau \int_\Omega \left[\langle \mathcal{V}_{t,x}; \varrho s(\varrho,\vartheta)\rangle \partial_t\varphi(t,x) + \left\langle \mathcal{V}_{t,x}; \varrho s(\varrho,\vartheta)\mathbf{u} - \frac{\kappa\nabla_x\vartheta}{\vartheta}\right\rangle \cdot \nabla_x\varphi(t,x)\right] \mathrm{d}x\,\mathrm{d}t \qquad (5.5)\\
&+ \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \frac{1}{\vartheta}\left(\mathbb{S}(\mathbf{D}_u) : \mathbf{D}_u + \frac{\kappa|\mathbf{D}_\vartheta|^2}{\vartheta}\right)\right\rangle \varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

for a.a. $\tau \in [0,T]$ and any $\varphi \in C^1([0,T]\times\Omega)$, $\varphi \ge 0$;

- The *dissipation defect* given by

$$\mathcal{D}(\tau) = \int_\Omega \left\langle \mathcal{V}_{0,x}; \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) \right\rangle \mathrm{d}x - \int_\Omega \left\langle \mathcal{V}_{\tau,x}; \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) \right\rangle \mathrm{d}x \ge 0$$

and the concentration defect measure ν_C from (5.3) satisfy

$$\int_0^T \psi(t) \int_\Omega \mathrm{d}|\nu_C| \lesssim \int_0^T \psi(t)\mathcal{D}(t)\ \mathrm{d}t \tag{5.6}$$

for any $\psi \in C([0,T])$, $\psi \ge 0$.

□

We refer the reader to, e.g., [1, 27] for more details on the Young measure.

Remark 5.2. It should be noted that in [3, Definition 2.3] an additional compatibility condition of Korn–Poincaré–type inequality, cf. [3, (2.13)], was required for the case of no slip/no flux boundary conditions for the velocity and the heat flux, respectively. This condition was needed in the proof of the DMV–strong uniqueness principle, cf. [3, Sections 4.1, 5.1.2]. In the case of space–periodic boundary conditions the Korn–Poincaré inequality does not hold. Nevertheless, the DMV–strong uniqueness principle can be obtained in an analogous way as in [3] provided the density is bounded from below.

5.1 Convergence to a dissipative measure-valued solution

In view of the assumptions (3.13) and *a priori* estimates (3.4), (3.15) and (3.16) we may deduce, at least for a subsequence, that the numerical solutions $\{\mathbf{U}_h\}_{h>0} = \{(\varrho_h, \mathbf{u}_h, \vartheta_h, \mathbf{D}_h(\mathbf{u}_h), \nabla_{\mathcal{E}}\vartheta_h)\}_{h>0}$ in the limit for $h \to 0$ generate a Young measure $\{\mathcal{V}_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$, whose support is contained in the set

$$\mathrm{supp}[\mathcal{V}_{t,x}] \subset \left\{\varrho, \vartheta, \mathbf{u}, \mathbf{D}_u, \mathbf{D}_\vartheta \;\middle|\; 0 < \underline{\varrho} \le \varrho \le \overline{\varrho},\ 0 < \underline{\vartheta} \le \vartheta \le \overline{\vartheta},\ \mathbf{u} \in \mathbb{R}^d,\ \mathbf{D}_u \in \mathbb{R}^{d\times d}_{\mathrm{sym}},\ \mathbf{D}_\vartheta \in \mathbb{R}^d\right\}$$

for a.a. $(t,x) \in (0,T)\times\Omega$. More specifically,

[1] The symbol ν_C stands for a tensor–valued signed Borel measure and the term $\int_0^\tau\int_\Omega \nabla_x\boldsymbol{\varphi} : \mathrm{d}\nu_C$ is understood as the value of the functional ν_C over the continuous function $\nabla_x\boldsymbol{\varphi}$.

- the mapping $\mathcal{V}_{t,x} : (t,x) \in (0,T) \times \Omega \mapsto \mathcal{P}(\mathcal{F})$ is weakly-(*) measurable
- $G(\mathbf{U}_h) \to \{G(\mathbf{U})\}$ weakly-(*) in $L^\infty((0,T)\times\Omega)$ and

$$\{G(\mathbf{U})\}(t,x) = \int_{\mathcal{F}} G(\mathbf{U}) \mathrm{d}\mathcal{V}_{t,x} \equiv \langle \mathcal{V}_{t,x}; G(\mathbf{U}) \rangle \text{ for a.a. } (t,x) \in (0,T)\times\Omega,$$

for any $G \in C_c(\mathcal{F})$, $\mathbf{U} = (\varrho, \vartheta, \mathbf{u}, \mathbf{D}_u, \mathbf{D}_\vartheta) \in \mathcal{F}$.

This, in particular, means that all nonlinearities appearing in the consistency formulation (4.5) – (4.6) are weakly precompact in the Lebesgue space $L^1((0,T)\times\Omega)$, and hence passing to the limit with $h \to 0$ yields (5.2) – (5.3) and $\nu_C \equiv 0$. The compatibility condition (5.1) is a direct consequence of (3.16), since

$$\mathbf{D}_h(\mathbf{u}_h) \to \mathbf{D}(\mathbf{u}) \text{ weakly in } L^2((0,T)\times\Omega;\mathbb{R}^{d\times d}), \quad \nabla_{\mathcal{E}}\vartheta_h \to \nabla_x\vartheta \text{ weakly in } L^2((0,T)\times\Omega;\mathbb{R}^d).$$

Now we recall [7, Lemma 2.1] which shall be to used to pass to the limit in the entropy equality.

Lemma 5.3. *Let*

$$|F(\mathbf{U})| \leq G(\mathbf{U}) \ \textit{for all} \ \mathbf{U} \in \mathcal{F}.$$

Then

$$|\{F(\mathbf{U})\} - \langle \mathcal{V}_{t,x}; F(\mathbf{U})\rangle| \leq \{G(\mathbf{U})\} - \langle \mathcal{V}_{t,x}; G(\mathbf{U})\rangle \ \textit{in} \ \mathcal{M}([0,T]\times\Omega).$$

We consider the limit in the entropy equation (4.7). For the nonlinear discrete entropy production terms

$$\mathcal{R}_h + \mathcal{P}_h := \frac{1}{\vartheta_h}\left(2\mu|\mathbf{D}_h(\mathbf{u}_h)|^2 + \lambda|\mathrm{div}_h\mathbf{u}_h|^2\right) + \frac{\kappa|\nabla_h\vartheta_h|^2}{\vartheta_h^2} + \mathcal{P}_h \geq 0,$$

where $\mathcal{P}_h := D_1 + D_2 \geq 0$, cf. (3.12), we can only assert that

$$\mathcal{R}_h + \mathcal{P}_h \to \{\mathcal{R} + \mathcal{P}\} \text{ weakly-(*) in } \mathcal{M}([0,T]\times\Omega;\mathbb{R}).$$

We apply Lemma 5.3 for $F(\mathbf{U}) \equiv 0$ and

$$G(\mathbf{U}) = \frac{1}{\vartheta}\left(2\mu|\mathbf{D}_u|^2 + \lambda|\mathrm{tr}\,\mathbf{D}_u|^2\right) + \frac{\kappa|\mathbf{D}_\vartheta|^2}{\vartheta^2} = \frac{1}{\vartheta}\left(S(\mathbf{D}_u) : \mathbf{D}_u + \frac{\kappa|\mathbf{D}_\vartheta|^2}{\vartheta}\right)$$

to get

$$0 \leq \{\mathcal{R}\} - \langle \mathcal{V}_{t,x}; \mathcal{R}\rangle .$$

Consequently, passing to the limit in the entropy equation (4.7) with non-negative test function we derived the entropy inequality (5.5). Similarly, passing to the limit in the discrete energy inequality (4.8) directly yields (5.4). Note that the inequality (5.6) is satisfied since $\nu_c \equiv 0$. Summing up the preceding discussion, we can state the following result.

Theorem 5.4 (Convergence to DMV solution)**.** *Let the initial data satisfy the assumptions*

$$0 < \underline{\varrho} \leq \varrho_{0,h} \leq \overline{\varrho}, \ 0 < \underline{\vartheta} \leq \vartheta_{0,h} \leq \overline{\vartheta}, \ \|\mathbf{u}_{0,h}\|_{L^2} \leq \overline{u},$$

for some positive constants $\underline{\varrho}$, $\overline{\varrho}$, $\underline{\vartheta}$, $\overline{\vartheta}$, $\overline{u}$. *Let* $(\varrho_h, \vartheta_h, \mathbf{u}_h)$ *be the solution of the finite volume scheme* (2.10) *with* $0 < \varepsilon < 1$, *such that the assumptions* (3.13) *hold, i.e.,*

$$0 < \underline{\varrho} \leq \varrho_h(t) \leq \overline{\varrho}, \ 0 < \underline{\vartheta} \leq \vartheta_h(t) \leq \overline{\vartheta} \ \textit{ uniformly for } \ h \to 0 \ \textit{ and all } \ t \in (0,T).$$

Then the family $\{\varrho_h, \vartheta_h, \mathbf{u}_h, \mathbf{D}_h(\mathbf{u}_h), \nabla_h\vartheta_h\}_{h>0}$ *generates a Young measure* $\{\mathcal{V}_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$ *that is a DMV solution of the Navier–Stokes–Fourier system* (1.1) *in the sense of Definition 5.1.*

5.2 Convergence to the strong solution

Having shown the family of approximate solutions computed by our finite volume scheme (2.10) generates the DMV solution of the limit system (1.1), we may use the DMV–strong uniqueness principle established in [3, Theorem 6.1] to get the following result.

Theorem 5.5. *Let $\kappa > 0$, $\mu > 0$, and $\lambda \geq 0$ be constant. Let the thermodynamic functions p, e, and s comply with the perfect gas constitutive relations*

$$p(\varrho, \vartheta) = \varrho\vartheta,\ e(\varrho, \vartheta) = c_v\vartheta,\ s(\varrho, \vartheta) = \log\left(\frac{\vartheta^{c_v}}{\varrho}\right),\ c_v > 1.$$

Assume that $\{\mathcal{V}_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$ is a DMV solution of the Navier–Stokes–Fourier system (1.1) in the sense of Definition 5.1 such that

$$\mathcal{V}_{t,x}\left\{0 < \underline{\varrho} \leq \varrho \leq \overline{\varrho},\ \vartheta \leq \overline{\vartheta},\ |\mathbf{u}| \leq \overline{u}\right\} = 1 \text{ for a.a. } (t,x) \in (0,T)\times\Omega \tag{5.7}$$

for some constants $\underline{\varrho}$, $\overline{\varrho}$, $\overline{\vartheta}$, and $\overline{\mathbf{u}}$. Assume further that

$$\mathcal{V}_{0,x} = \delta_{\varrho_0(x),\vartheta_0(x),\mathbf{u}_0(x)} \quad \text{for a.a. } x \in \Omega,$$

where $(\varrho_0, \vartheta_0, \mathbf{u}_0)$ belong to the regularity class

$$\varrho_0, \vartheta_0 \in W^{3,2}(\Omega),\ \varrho,\ \vartheta > 0 \ \text{ in } \ \Omega,\ \mathbf{u}_0 \in W_0^{3,2}(\Omega;\mathbb{R}^3). \tag{5.8}$$

Then

$$\mathcal{V}_{t,x} = \delta_{[\tilde{\varrho}(t,x),\tilde{\vartheta}(t,x),\tilde{\mathbf{u}}(t,x),\mathbf{D}(\tilde{\mathbf{u}})(t,x),\nabla_x\tilde{\vartheta}(t,x)]} \text{ for a.a. } (t,x) \in (0,T)\times\Omega,$$

where $(\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}})$ is a strong (classical) solution to the Navier–Stokes–Fourier system with the initial data $(\varrho_0, \vartheta_0, \mathbf{u}_0)$.

Proof. Under the regularity assumption (5.8), the strong solution exists locally in time, say on $[0, T_{\max})$, see, e.g., Valli and Zajackowski [32]. Thus we can use the DMV–strong uniqueness principle on $[0, T_{\max})$. On the other hand, hypothesis (5.7) implies that the no–blow up criterion of Sun, Wang, and Zhang [30] applies yielding $T_{max} = T$. As a matter of fact, the results of Sun, Wang, and Zhang [30] have been established on a bounded domain with suitable boundary conditions. However their extension to the space–periodic case is straightforward. In particular, the assumption on the uniform bound of the velocity makes it possible to handle general viscosity coefficients (cf. Remark 3 in [30]). □

In order to use the above result we additionally need that the DMV solution has also bounded velocity, cf. (3.13) and (5.7). Then, as a consequence of Theorem 5.5 and the DMV–strong uniqueness on $(0,T)\times\Omega$, we can show that the DMV solution coincides with the global strong solution.

Theorem 5.6 (Convergence to strong solution)**.** *In addition to the hypotheses of Theorem 5.5, suppose that the Navier–Stokes–Fourier system (1.1) is endowed with the initial data $(\varrho_0, \vartheta_0, \mathbf{u}_0)$ satisfying* (5.8)*. Let $(\varrho_h, \vartheta_h, \mathbf{u}_h)$ be the solution of the finite volume scheme* (2.10) *with $0 < \varepsilon < 1$, satisfying the assumptions* (3.13) *and, in addition,*

$$|\mathbf{u}_h(t)| \leq \overline{u} \text{ uniformly for } \ h \to 0 \ \text{ and all } \ t \in (0,T).$$

Then

$$\varrho_h \to \varrho \text{ (strongly) in } L^p\left((0,T)\times\Omega\right),\ \vartheta_h \to \vartheta \text{ (strongly) in } L^p\left((0,T)\times\Omega\right),$$
$$\mathbf{u}_h \to \mathbf{u} \text{ (strongly) in } L^p\left((0,T)\times\Omega;\mathbb{R}^d\right),\ p \in [1,\infty),$$

where ϱ, ϑ, and $\mathbf{u}$ is a strong (classical) solution of the Navier–Stokes–Fourier system.

Remark 5.7. We have constructed solution having periodic boundary conditions. When considering a polyhedral domain, the existence of smooth solutions remains open and may be a delicate task. To avoid this problem, one has to approximate a smooth domain by a family of polyhedral domains analogously as in [15]. Clearly, such a problem does not occur for periodic boundary conditions.

6 Conclusions

In the present paper we have studied a long–standing open problem of rigorous convergence analysis of finite volume schemes for multidimensional compressible flows. We have proved that the bounded numerical solutions generated by the finite volume method (2.9) converge to the global strong solution of the Navier–Stokes–Fourier system (1.1) describing motion of viscous compressible and heat conducting fluids. To this goal we have applied a rather general technique using the dissipative measure–valued solutions. Indeed, realising that for the numerical solutions the conservation of mass (3.1) and the discrete energy dissipation (3.3) hold, we have derived the first *a priori* estimates (3.4). To proceed further the discrete entropy inequality (3.12) has played a fundamental role. In order to control the discrete entropy we had to assume boundedness of the discrete density and temperature, cf. (3.13). This has allowed us, together with the entropy inequality, to obtain the second *a priori* estimates (3.15) and (3.16). Equipped with the above bounds we have shown in Section 4 the consistency of our finite volume method.

Consequently, the numerical solutions were shown to generate, up to a subsequence, the Young measure that represents a dissipative measure–valued solution of the Navier–Stokes–Fourier system, see Section 5. Using the DMV–strong uniqueness principle, cf. [3, Theorem 6.1], we have obtained the strong convergence of the finite volume solutions towards the strong (classical) solution of the Navier–Stokes–Fourier system (1.1) on the lifespan of the latter. Assuming moreover that the numerical solution emanating from the initial data satisfying (5.8) has also bounded velocity, we were able to use the DMV–strong uniqueness result stated in Theorem 5.5 to show the strong convergence to the global in time strong (classical) solution of (1.1) without assuming its existence a priori, cf. Theorem 5.6.

As far as we know this is the first rigorous convergence proof for the finite volume method applied to the Navier–Stokes–Fourier system. The numerical flux (2.8) in our scheme is based on the upwinding with an additional numerical diffusion of order $\mathcal{O}(h^{\varepsilon+1})$, $0 < \varepsilon < 1$. In fact, the additional numerical diffusion is only a technical tool. Consequently, our result implies the convergence of any finite volume method with a numerical diffusion larger than that of our diffusive upwinding.

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