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Abstract

Let I be an F_σ-ideal on natural numbers. We characterize the ultrafilters which are generic over the model $L(\mathbb{R})$ for the poset of I-positive sets of natural numbers ordered by inclusion.

1 Introduction

The study of ultrafilters on natural numbers is a traditional subfield of set theory. Ultrafilters are useful in model theory, topology, Ramsey theory, and many other contexts. It turns out that certain properties $\phi(U)$ of ultrafilters U provide what is often called "complete combinatorics" for U in the sense that for nearly all properties ψ of ultrafilters invariant under permutations of the underlying set, $\phi(U)$ either implies $\psi(U)$ or $\neg\psi(U)$. Such a sweeping statement can never be literally correct because of obstacles connected with Gödel's incompleteness theorems. In an effort to formalize this intuition, mathematicians proved a number of theorems showing that a given property $\phi(U)$ implies that U is generic over a suitable inner model for a natural partially ordered set [13, 11, 6, 5, 4, 3, 2]. The oldest result of this form states that any Ramsey ultrafilter is generic over the inner model $L(\mathbb{R})$ for the poset $\mathcal{P}(\omega)$ modulo finite, if certain mild large cardinal assumptions are met [7].

In this note, we provide a simple characterization of genericity for a large and interesting class of ultrafilters. Let I be an F_σ ideal on ω. A natural way to obtain an ultrafilter disjoint from I is to use Kuratowski–Zorn lemma to find a maximal ideal J extending I and then let $U = \mathcal{P}(\omega) \setminus J$. Another natural way is to force such an ultrafilter; this has the advantage of keeping much tighter control over its properties than what is possible with the Kuratowski–Zorn lemma. The natural partial order to use in the forcing construction is the

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poset $\mathcal{P}(\omega)$ modulo I, or perhaps its combinatorial presentation, the poset $P(I)$ of I-positive subsets of ω ordered by inclusion. The poset has been studied in numerous papers [12]. Elementary density arguments show that the poset is σ-closed and a generic subset of $P(I)$ is in fact a P-point ultrafilter on ω disjoint from I. It turns out that genericity for the poset $P(I)$ is fairly easy to characterize. To state it succintly, we use the following terminology. A *simplicial complex* on ω is a collection of finite subsets of ω closed under subset. If $\mathcal{K}$ is a simplicial complex on ω, a $\mathcal{K}$-*set* is a set $a \subset \omega$ such that every finite subset of a is in $\mathcal{K}$. This is the main result of this note:

Theorem 1.1. *Assume that there are infinitely many Woodin cardinals with a measurable cardinal above them all. Let I be an F_σ-ideal on ω and let U be an ultrafilter on ω disjoint from I. The following are equivalent:*

1. *U is a $P(I)$-generic set over the model $L(\mathbb{R})$;*
2. *U is a P-point, and for every simplicial complex $\mathcal{K}$ on ω, U contains either a $\mathcal{K}$-set or a set all of whose $\mathcal{K}$-subsets belong to I.*

The large cardinal assumption is necessary but benign; we use only one consequence of it: all subsets of 2^ω in the model $L(\mathbb{R})$ are universally Baire [8].

The notation of the paper follows the set theoretic standard of [10] to which we refer the reader for all undefined notions. The first author was supported by the GACR project 17-33849L and RVO: 67985840.

2 The proof

The implication (1)→(2) is much easier; in particular, it does not need the large cardinal assumptions. Assume that U is a $P(I)$-generic set over the model $L(\mathbb{R})$. To prove that U is a P-point, use the following claim:

Claim 2.1. *Let $a = \{a_i : i \in \omega\}$ be a countable subset of $\mathcal{P}(\omega)$. The set $D_a = \{b \in P(I)$: either there is a finite set $c \subset \omega$ such that $b \cap \bigcap_{i\in c} a_i \in I$ or for all $i \in \omega$ the set $b \setminus a_i$ is finite$\}$ is dense in $P(I)$.*

Proof. Let $b \in P(I)$ be any condition. To find a condition $d \subset b$ in the set D_a, either there is a finite set $c \subset \omega$ such that $b \cap \bigcap_{i\in c} a_i \in I$; in such a case, $d = b$ works. Otherwise, express I as a countable union $I = \bigcup_n I_n$ of closed subsets of $\mathcal{P}(\omega)$ closed under subsets, and for each $n \in \omega$ use the fact that $b \cap \bigcap_{i\in n} a_i \notin I_n$ to find a finite set $e_n \subset b \cap \bigcap_{i\in n} a_i$ such that no superset of e_n belongs to I_n. Let $d = \bigcup_n e_n$ and check that $d \subset b$ is an I-positive set such that $d \setminus a_i$ is finite for all $i \in \omega$. In other words, $d \in D_a$ as required. □

Now let $a = \{a_i : i \in \omega\}$ is a countable subset of U. Since the set D_a from the claim is dense in $P(I)$ and belongs to the model $L(\mathbb{R})$, by the genericity assumption there must be a condition $b \in U \cap D_a$. For every finite set $c \subset \omega$, the set $b \cap \bigcap_{i\in c} a_i$ belongs to U and therefore not to I. Thus, it must be the case that for all $i \in \omega$ the set $b \setminus a_i$ is finite. The P-point property of U has just been verified.

To prove that for every simplicial complex $\mathcal{K}$, U contains either a $\mathcal{K}$-set or a set such that all of its $\mathcal{K}$-subsets belong to I, we need a similar simple claim.

Claim 2.2. *Let $\mathcal{K}$ be a simplicial complex on ω. The set $D_\mathcal{K} = \{b \in P(I)\colon$ either b is a $\mathcal{K}$-set or all its $\mathcal{K}$-subsets belong to $I\}$ is dense in $P(I)$.*

Proof. Let $b \in P(I)$ be an arbitrary condition. To find a condition $c \leq b$ in the set $D_\mathcal{K}$, either every $\mathcal{K}$-subset of b is in $I$and then $c = b$ works. Or, there is an I-positive $\mathcal{K}$-set $c \subset b$, and then such c will work. □

Now, let $\mathcal{K}$ be a simplicial complex on ω. The set $D_\mathcal{K} \subset P(I)$ is dense and belongs to the model $L(\mathbb{R})$; thus, by the genericity assumption, there must be a condition $b \in U \cap D_\mathcal{K}$. Either b is a $\mathcal{K}$-set or every $\mathcal{K}$-subset of b is in I–in each case, (2) has been confirmed.

The implication (1)→(2) is the heart of the matter. We need to set up some simple terminology. Fix the F_σ-ideal I. For an ultrafilter U disjoint from I, the symbol $\Phi(U)$ denotes the statement that for every simplicial complex $\mathcal{K}$ on ω, U contains either a $\mathcal{K}$-set or a set all of whose $\mathcal{K}$-subsets belong to I. Regardless of the status of $\Phi(U)$, we define the *diagonalization poset* (or *Mathias–Prikry poset* [9]) $Q(U)$ to consist of all pairs $q = \langle a_q, b_q\rangle$ where $a_q \subset \omega$ is a finite set and $b_q \in U$. The ordering is defined by $r \leq q$ if $a_q \subset a_r$, $b_r \subset b_q$, and $a_r \setminus a_q \subset b_q$. The poset $Q(U)$ adds a generic subset of ω denoted by $\dot{a}_{gen}$; this is the union of the first coordinates of the conditions in the generic filter. The generic filter is reconstructed from $\dot{a}_{gen}$ as the set of all conditions $q \in Q(U)$ such that $a_q \subset \dot{a}_{gen}$ and $\dot{a}_{gen} \setminus a_q \subset b_q$. Now, suppose that M is a transitive model of set theory and U is a nonprincipal ultrafilter on ω in the model M. We say that an infinite set $a \subset \omega$ *diagonalizes* U if for all elements $b \in U$ in the model M, the set $a \setminus b$ is finite. Moreover, we say that a is $Q(U)$-*generic* over M if the set of all conditions $q \in Q(U)$ such that $a_q \subset a$ and $a \setminus a_q \subset b_q$ is a $Q(U)$-generic filter over the model M. The following key claim is an improvement of ideas from [9] and [1, Proposition 4].

Claim 2.3. *Let M be a countable transitive model of a large fragment of ZFC. Suppose that $U \in M$ and $M \models \Phi(U)$. The following are equivalent for every infinite set $a \subset \omega$:*

1. *$a \notin I$ and a diagonalizes U;*
2. *a is a disjoint union of two sets separately $Q(U)$-generic over M.*

Proof. (2) implies (1) is the easier direction. To see it, observe that $Q(U) \Vdash \dot{a}_{gen} \notin I$: express $I = \bigcup_n I_n$ as a countable union of closed subsets of $\mathcal{P}(\omega)$ closed under subset, for each condition $q \in Q$ and every number $n \in \omega$ find a finite set $e \subset b_q$ such that no superset of e belongs to I_n, and let $r = \langle a_q \cup e, b_q\rangle \leq q$. By the choice of the finite set e, the condition $r \in Q(U)$ forces $\dot{a}_{gen} \notin I_n$ as desired.

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Thus, any $Q(U)$-generic set does not belong to I and diagonalizes U, and these two properties are preserved under finite unions.

(1) implies (2) is the more difficult direction. Let $D \subset Q(U)$ be an open dense set in the model M. Let $O_D = \{t \in 2^{<\omega}\colon$ there is a condition $q \in D$ such that $a_q = \{n \in a \cap \operatorname{dom}(t)\colon t(n) = 1\}$ and $a \setminus \operatorname{dom}(t) \subset b_q\}$. We will show that the set $O_D \subset 2^{<\omega}$ is dense. To see how this proves the claim, by a Baire category argument on the Cantor space find a function $x \in 2^{<\omega}$ such that both x and $1-x$ have initial segments in all sets O_D for all open dense sets $D \subset Q(U)$ in the model M. Let $a_0 = \{n \in a\colon x(n) = 0\}$ and $a_1 = \{n \in a\colon x(n) = 1\}$. It is immediate from the definitions of the sets O_D that the sets a_0, a_1 are $Q(U)$-generic over the model M.

To see why the set O_D is dense, let $s \in 2^{<\omega}$ be any binary string; we have to find an extension $t \supset s$ in the set O_D. Write $d = \{n \in \operatorname{dom}(s)\colon n \in a$ and $s(n) = 1\}$ and let $\mathcal{K}$ be the set of all finite subsets $c \subset \omega \setminus \operatorname{dom}(t)$ such that there exists no condition $q \in D$ such that a_q is a union of d and some subset of c. Clearly, $\mathcal{K}$ is a simplicial complex in the model M. Arguing in the model M, U cannot contain a $\mathcal{K}$-set: if $b \in U$ were a $\mathcal{K}$-set, then by the density of the set D there would have to be a condition $q \leq \langle d, b\rangle$ which would then contradict the assumption that b is a $\mathcal{K}$-set. Now, by the property $\Phi(U)$, it must be the case that there is $b \in U$ such that every $\mathcal{K}$-subset of b belongs to I. This inclusion holds in M, but by Mostowski absoluteness transfers to V without change. Since a diagonalizes U and is I-positive, the set $a \cap b \setminus \operatorname{dom}(s)$ is I-positive and therefore not a $\mathcal{K}$-set. Thus, there exists a condition $q \in D$ such that a_q is a union of d and some subset of $a \cap b$. Since a diagonalizes the ultrafilter U, it is the case that the set $a \setminus b_q$ is finite. Let $t \in 2^{<\omega}$ be any binary string such that $\operatorname{dom}(t)$ includes all points in $\operatorname{dom}(s)$, a_q and in $a \setminus b_q$, and let $t(n) = 1$ if and only if $n \in a_q$. Then $s \subset t$ and $t \in O_D$ as desired. □

Now we are ready to conclude the proof. Suppose that U is a P-point ultrafilter disjoint from I and $\Phi(U)$ holds. Let $D \subset P(I)$ be a dense set in the model $L(\mathbb{R})$; we must show that $U \cap D \neq 0$. Since U is invariant under finite changes, we may assume that D is invariant under finite changes. Identify 2^ω with $\mathcal{P}(\omega)$ via the characteristic functions; thus, D becomes a subset of 2^ω. The large cardinal assumption shows that the set D, as well as all other subsets of 2^ω in $L(\mathbb{R})$, is universally Baire [8]: for some cardinal κ there are trees $T, S \subset (2 \times \kappa)^{<\omega}$ such that T, S project to complementary sets in all generic extensions by posets of size continuum, and T projects onto D.

Claim 2.4. *$Q(U) \Vdash \dot{a}_{gen}$ belongs to the projection of $\check{T}$.*

Proof. Suppose towards contradiction that this fails; then there must be a condition $q \in Q(U)$ which forces the generic set $\dot{a}_{gen}$ to be in the projection of the tree $\check{S}$. Let M be a countable elementary submodel of a large structure containing all objects named so far, in particular U and q. Identify M with its transitive collapse. Since U is a P-point ultrafilter, there is a set $a \in U$ which diagonalizes $U \cap M$. Use the density of the set D to find an I-positive set $b \subset a$

such that $b \in D$. Use Claim 2.3 to find an I-positive set $c \subset b$ which is $Q(U)$-generic. Perform some finite changes on the set c if necessary to make sure that the $Q(U)$-generic ultrafilter over M derived from c meets the condition q. By the forcing theorem then, $M[c] \models c$ is in the projection of the tree S. At the same time, the set c belongs to the set D and therefore belongs to the projection of the tree T. A contradiction. □

Now, let M be a countable elementary submodel of a large structure containing all objects named so far, in particular the ultrafilter U. Since U is a P-point, there is a set $a \in U$ which diagonalizes $U \cap M$. By Claim 2.3, there are sets $a_0, a_1 \subset \omega$ such that $a = a_0 \cup a_1$ and a_0, a_1 are $Q(U)$-generic over the model M. One of these sets must belong to U; for definiteness, assume it is a_0. By Claim 2.4 and the forcing theorem, $M[a_0] \models a_0$ belongs to the projection of the tree T. This means that a_0 indeed belongs to the projection of the tree T which is the set D. In particular, $a_0 \in U \cap D$ as required.

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