INSTITUTE OF MATHEMATICS

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Amiran Gogatishvili
Martin Křepela
Rastislav Oľhava
Luboš Pick

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WEIGHTED INEQUALITIES FOR DISCRETE ITERATED HARDY OPERATORS

AMIRAN GOGATISHVILI[1], MARTIN KŘEPELA[2], RASTISLAV OĽHAVA[3,4], AND LUBOŠ PICK[5]

Abstract. We characterize a three-weight inequality for an iterated discrete Hardy-type operator. In the case when the domain space is a weighted space ℓ^p with $p \in (0,1]$, we develop characterizations which enable us to reduce the problem to another one with $p=1$. This, in turn, makes it possible to establish an equivalence of the weighted discrete inequality to an appropriate inequality for iterated Hardy-type operators acting on measurable functions defined on $\mathbb{R}$, for all cases of involved positive exponents.

1. Introduction

In this paper we focus on a three-weight inequality for the composition of a discrete supremal and integral Hardy operator. Let us denote by $\mathbb{R}_+^{\mathbb{Z}}$ the space of all double-infinite sequences of positive (nonnegative) real numbers. We are interested in the question under what conditions on given $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{R}_+^{\mathbb{Z}}$ there exist constants $C_1, C_2 \in (0,\infty)$ such that the inequalities

$$\left(\sum_{n\in\mathbb{Z}}\left(\sup_{i\geq n} u_i \sum_{k\leq i} a_k\right)^q w_n\right)^{\frac{1}{q}} \leq C_1 \left(\sum_{n\in\mathbb{Z}} a_n^p v_n\right)^{\frac{1}{p}} \tag{1.1}$$

E:d-gop

and

$$\left(\sum_{n\in\mathbb{Z}}\left(\sup_{i\geq n} u_i \sum_{k\geq i} a_k\right)^q w_n\right)^{\frac{1}{q}} \leq C_2 \left(\sum_{n\in\mathbb{Z}} a_n^p v_n\right)^{\frac{1}{p}} \tag{1.2}$$

E:d-antigop

hold for every sequence $\mathbf{a} \in \mathbb{R}_+^{\mathbb{Z}}$. We study several aspects of such an inequality including its relationship to an analogous one for integral operators.

Before continuing, let us recall that (1.1) being satisfied for all $\mathbf{a} \in \mathbb{R}_+^{\mathbb{Z}}$ is equivalent to

$$\left(\sum_{n\in\mathbb{Z}}\left(\sup_{i\leq n} \overline{u}_i \sum_{k\geq i} a_k\right)^q \overline{w}_n\right)^{\frac{1}{q}} \leq C_1 \left(\sum_{n\in\mathbb{Z}} a_n^p \overline{v}_n\right)^{\frac{1}{p}}, \tag{1.3}$$

E:dual-trivka

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also being satisfied for all $\mathbf{a} \in \mathbb{R}_+^{\mathbb{Z}}$. This is obvious by the index change $\overline{u}_n = u_{-n}$, $\overline{v}_n = v_{-n}$ and $\overline{w}_n = w_{-n}$. Analogously, the inequality

$$\left(\sum_{n\in\mathbb{Z}} \left(\sup_{i\leq n} \overline{u}_i \sum_{k\leq i} a_k \right)^q \overline{w}_n \right)^{\frac{1}{q}} \leq C_1 \left(\sum_{n\in\mathbb{Z}} a_n^p \overline{v}_n \right)^{\frac{1}{p}} \tag{1.4}$$

E:dual-antigop-tri

is equivalent to (1.2). It is common to refer to (1.3) and (1.4) as to the *dual versions* of (1.1) and (1.2), respectively. In contrast, inequalities (1.1) and (1.2) (hence also (1.3) and (1.4)) are essentially different.

The success that the theory of weighted inequalities has seen in last three decades can be credited greatly to a clever combination of classical techniques such as symmetrization or interpolation with new methods such as discretization (the blocking technique), antidiscretization, reduction theorems, and the use of supremum operators.

The research of problems in mathematical physics often leads to the investigation of certain Sobolev-type embeddings. Under certain circumstances, these can be quite successfully attacked by classical symmetrization techniques. After performing this step, one often faces some kind of an inequality involving operators acting on monotone functions. Handling monotone functions is, however, in general substantially more difficult than working with general nonnegative functions.

There are several possibilities how to continue at this stage. One of the important ones is the use of the so-called reduction theorems, in which the inequality involving monotone functions is equivalently replaced with an inequality (or inequalities) involving general nonnegative functions.

For certain types of technically difficult inequalities involving monotone functions, stronger tools have to be used. One of such tools that has proved its merit beyond any doubt, is discretization. Discretization techniques replace weighted inequalities involving integrals with those involving sums. The basic advantage of this step is that discrete inequalities can be effectively manipulated with the help of the so-called blocking technique (see the comprehensive treatment in [GE98]. The drawback is the fact that verification of the discretized conditions on weight functions in practice is virtually impossible. So here we face the danger of replacing one mystery with another one without making much progress. For this reason, a substantial effort has been spent in order to develop *antidiscretization* techniques (the pivotal paper in this direction is [GP03]). After performing antidiscretization, one gets manageable and easily verifiable conditions for weighted inequalities that could not be obtained otherwise. Let us note that this approach brought a significant progress to theory of function spaces and the study of properties of operators on function spaces and several long-standing open problems were solved thanks to it. A particular impact could be seen, for instance, to classical Lorentz spaces or to Orlicz spaces (see, for instance, [ACS17, Sla15, GKPS17, Mus16, Mus19, CM19] and more).

One of the most important topics intensively studied in the recent theory of weighted inequalities is that of handling *iterated* operators. The reason stems from the wide field of applications, see for example [GM17b, GM17a, Kře17b, GKPS17, ACS17] and the references therein.

One of the basic problems in the theory of weighted inequalities is the comparison of discrete inequalities to their continuous analogues. Consider, for example, a classical discrete

Hardy-type inequality

$$\left(\sum_{n\in\mathbb{Z}}\left(\sum_{i\geq n}a_i\right)^q w_n\right)^{\frac{1}{q}} \leq C_3\left(\sum_{n\in\mathbb{Z}}a_n^p v_n\right)^{\frac{1}{p}}, \tag{1.5}$$

E:discrete_hardy

which is supposed to hold for all $\mathbf{a}\in\mathbb{R}_+^{\mathbb{Z}}$ with the same constant C_3, and where $\mathbf{v},\mathbf{w}\in\mathbb{R}_+^{\mathbb{Z}}$ are fixed sequences (weights). Compare this to its "continuous" analogue

$$\left(\int_0^\infty\left(\int_t^\infty f(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \leq C_4\left(\int_0^\infty f(t)^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}} \tag{1.6}$$

E:integral_hardy

which is to hold with a constant C_4 for all positive measurable functions f on $\mathbb{R}$. In here, the weights v, w are fixed positive measurable functions. The relation between the two inequalities is materialized through setting

$$v(t)=\sum_{n\in\mathbb{Z}}v_n\chi_{[n,n+1)}(t), \qquad w(t)=\sum_{n\in\mathbb{Z}}w_n\chi_{[n,n+1)}(t)$$

for all $t\in\mathbb{R}$. While (1.5) and (1.6) are rather easily seen to be equivalent for $p\geq 1$, the situation is dramatically different when $p\in(0,1)$. In that case it is not difficult to realize that (1.6) cannot hold for any nontrivial weights, because one can always find a function f for which the right-hand side of (1.6) is finite but which is at the same time not locally integrable, hence turning the left hand side to infinity. On the other hand, (1.5) can still be satisfied for a wide variety of nontrivial weight sequences. One of our principal goals in this paper is to show that, nevertheless, an appropriate continuous analogue can be found even for $p\in(0,1)$. To achieve this result, we combine a certain scaling argument with a powerful technique based on a somewhat surprising equivalence of several weighted inequalities. We then employ the fact that the case $p=1$ is a meeting point of the separated worlds. It is worth to illustrate this technique in more detail. The point of departure is a chain of elementary inequalities, namely

$$\sup_{i\geq n}a_i \leq \sum_{i\geq n}a_i \leq \left(\sum_{i\geq n}a_i^p\right)^{\frac{1}{p}}. \tag{1.7}$$

E:elm

This is obviously true for every $p\in(0,1]$, $n\in\mathbb{Z}$ and $\mathbf{a}\in\mathbb{R}_+^{\mathbb{Z}}$. It immediately follows from (1.7) that if $p\in(0,1]$ and the sequences $\mathbf{v},\mathbf{w}$ are such that the inequality

$$\left(\sum_{n\in\mathbb{Z}}\left(\sum_{i\geq n}a_i^p\right)^{\frac{q}{p}} w_n\right)^{\frac{1}{q}} \leq C_3\left(\sum_{n\in\mathbb{Z}}a_n^p v_n\right)^{\frac{1}{p}} \tag{1.8}$$

E:discrete_hardy_p

holds for every $\mathbf{a}\in\mathbb{R}_+^{\mathbb{Z}}$, then so does (1.5). In turn, (1.5) implies that

$$\left(\sum_{n\in\mathbb{Z}}\left(\sup_{i\geq n}a_i\right)^q w_n\right)^{\frac{1}{q}} \leq C_3\left(\sum_{n\in\mathbb{Z}}a_n^p v_n\right)^{\frac{1}{p}} \tag{1.9}$$

E:discrete_hardy_su

holds for all $\mathbf{a}\in\mathbb{R}_+^{\mathbb{Z}}$ as well. The surprising part of the method is that the implication (1.9)$\Rightarrow$(1.8) holds as well, therefore the three inequalities are in fact equivalent. It is important to notice that all this is possible only in the case when $p\in(0,1]$, for p bigger than 1 the equivalence fails. The technique just described is not entirely new. Similar ideas were used, albeit in a somewhat hidden form, in the proof of [CGMP08, Theorem 3.1]. An analogous idea works also for continuous-type problems, again for $p \in (0,1]$ only, as shown in [GP07]. The special role of the case $p = 1$ (the "meeting point" of intervals of parameters in which things are considerably different) can be also seen for instance in [SS96, Sin94].

We will present several characterizations of the inequality (1.1), quite different in nature. In the first theorem we state the equivalence of (1.1) to an appropriate integral inequality for functions on $\mathbb{R}$.

We will denote by $\mathcal{M}_+$ the collection of all nonnegative measurable functions on $\mathbb{R}$.

⟨T:main-prop1⟩ **Theorem 1.1.** *Let $p \in [1,\infty)$ and $q \in (0,\infty)$. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}_+^{\mathbb{Z}}$. Define*

$$u = \sum_{n\in\mathbb{Z}} u_n \chi_{[n,n+1)}, \qquad v = \sum_{n\in\mathbb{Z}} v_n \chi_{[n,n+1)}, \qquad w = \sum_{n\in\mathbb{Z}} w_n \chi_{[n,n+1)}.$$

Then (1.1) holds for every sequence $\mathbf{a} \in \mathbb{R}_+^{\mathbb{Z}}$ if and only if

$$\left(\int_{\mathbb{R}} \left(\sup_{s\geq t} u(s) \int_{-\infty}^{s} f(y)\,\mathrm{d}y \right)^q w(t)\,\mathrm{d}t \right)^{\frac{1}{q}} \leq C_1 \left(\int_{\mathbb{R}} f(t)^p v(t)\,\mathrm{d}t \right)^{\frac{1}{p}} \tag{1.10}$$ E:spoj-gop

holds for every $f \in \mathcal{M}_+$.

Similarly, (1.2) holds for every sequence $\mathbf{a} \in \mathbb{R}_+^{\mathbb{Z}}$ if and only if

$$\left(\int_{\mathbb{R}} \left(\sup_{s\geq t} u(s) \int_{s}^{\infty} f(y)\,\mathrm{d}y \right)^q w(t)\,\mathrm{d}t \right)^{\frac{1}{q}} \leq C_2 \left(\int_{\mathbb{R}} f(t)^p v(t)\,\mathrm{d}t \right)^{\frac{1}{p}} \tag{1.11}$$ E:spoj-antigop

holds for every $f \in \mathcal{M}_+$.

In Section 2 below we give the main results concerning characterizations of (1.1) and (1.2). Section 3 contains some auxiliary results and, above all, the equivalent characterizations for the case $p \in (0,1]$. In the final section we give the remaining proofs of the main results.

2. Discrete iterated Hardy operators

This section contains the main results concerning boundedness of iterated Hardy-type operators on weighted sequence spaces.

From now on we are going to use the following notation. Let $\mathbf{u} \in \mathbb{R}_+^{\mathbb{Z}}$. For $n \in \mathbb{Z}$ we define

$$\overset{\uparrow}{u}_n = \sup_{k\leq n} u_k, \quad \overset{\downarrow}{u}_n = \sup_{k\geq n} u_k.$$

The sequences $\overset{\uparrow}{\mathbf{u}}$ and $\overset{\downarrow}{\mathbf{u}}$ are called the *increasing* and *decreasing upper envelope of* $\mathbf{u}$, respectively. Next, define

$$\underset{\uparrow}{u}_n = \inf_{k\geq n} u_k, \quad \underset{\downarrow}{u}_n = \inf_{k\leq n} u_k.$$

The sequences $\underset{\uparrow}{\mathbf{u}}$ and $\underset{\downarrow}{\mathbf{u}}$ are called the *increasing* and *decreasing lower envelope of* $\mathbf{u}$, respectively. If $\mathbf{a}, \mathbf{b} \in \mathbb{R}_+^{\mathbb{Z}}$ satisfy $a_n \leq b_n$ for all $n \in \mathbb{Z}$, we write $\mathbf{a} \leq \mathbf{b}$. Furthermore, the notation $A \lesssim B$ means that there exists a constant $C \in (0,\infty)$ depending only on p and q and such that $A \leq CB$. We write $A \approx B$ if $A \lesssim B \lesssim A$.

n-discrete-p-big⟩ **Theorem 2.1.** *Let $p, q \in (0,\infty)$ and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}_+^{\mathbb{Z}}$. Then the least constant C_1 such that (1.1) holds for all $\mathbf{a} \in \mathbb{R}_+^{\mathbb{Z}}$ admits the following estimates.*

(i) *If* $1 < p \le q$, *then*

$$C_1 \approx \sup_{n\in\mathbb{Z}} \left(\overset{\downarrow}{u}{}_n^q \sum_{i\le n} w_i + \sum_{i\ge n} \overset{\downarrow}{u}{}_i^q w_i \right)^{\frac{1}{q}} \left(\sum_{k\le n} v_k^{\frac{1}{1-p}} \right)^{\frac{p-1}{p}}.$$

(ii) *If* $p > 1$ *and* $q < p$, *then*

$$C_1 \approx \left(\sum_{n\in\mathbb{Z}} \left(\sum_{i\ge n} \overset{\downarrow}{u}{}_i^q w_i \right)^{\frac{q}{p-q}} \overset{\downarrow}{u}{}_n^q w_n \left(\sum_{k\le n} v_k^{\frac{1}{1-p}} \right)^{\frac{(p-1)q}{p-q}} \right)^{\frac{p-q}{pq}} + \left(\sum_{n\in\mathbb{Z}} \left(\sum_{i\le n} w_i \right)^{\frac{q}{p-q}} w_n \sup_{k\ge n} \overset{\downarrow}{u}{}_k^{\frac{pq}{p-q}} \left(\sum_{j\le k} v_j^{\frac{1}{1-p}} \right)^{\frac{(p-1)q}{p-q}} \right)^{\frac{p-q}{pq}}.$$

(iii) *If* $0 < p \le 1$ *and* $p \le q$, *then*

$$C_1 \approx \sup_{n\in\mathbb{Z}} \left(\overset{\downarrow}{u}{}_n^q \sum_{i\le n} w_i + \sum_{k\ge n} \overset{\downarrow}{u}{}_k^q w_k \right)^{\frac{1}{q}} \sup_{j\le n} v_j^{-\frac{1}{p}}.$$

(iv) *If* $0 < q < p \le 1$, *then*

$$C_1 \approx \left(\sum_{n\in\mathbb{Z}} \left(\sum_{i\ge n} w_i \overset{\downarrow}{u}{}_i^q \right)^{\frac{q}{p-q}} \overset{\downarrow}{u}{}_n^q w_n \sup_{k\le n} v_k^{\frac{q}{q-p}} + \sum_{n\in\mathbb{Z}} \left(\sum_{i\le n} w_i \right)^{\frac{q}{p-q}} w_n \sup_{k\ge n} \overset{\downarrow}{u}{}_k^{\frac{pq}{p-q}} v_k^{\frac{q}{q-p}} \right)^{\frac{p-q}{pq}}.$$

⟨T:main-antigop⟩ **Theorem 2.2.** *Let* $p, q \in (0,\infty)$ *and* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}_+^{\mathbb{Z}}$. *Then the least constant* C_2 *such that* (1.2) *holds for all* $\mathbf{a} \in \mathbb{R}_+^{\mathbb{Z}}$ *admits the following estimates.*

(i) *If* $1 < p \le q$, *then*

$$C_2 \approx \sup_{n\in\mathbb{Z}} \left(\sum_{i\le n} w_i \sup_{i\le j\le n} u_j^q \right)^{\frac{1}{q}} \left(\sum_{k\ge n} v_k^{\frac{1}{1-p}} \right)^{\frac{p-1}{p}}.$$

(ii) *If* $p > 1$ *and* $q < p$, *then*

$$C_2 \approx \left(\sum_{n\in\mathbb{Z}} \left(\sum_{i\ge n} w_i \right)^{\frac{q}{p-q}} w_n^{\frac{q}{p-q}} \sup_{k\ge n} u_k^{\frac{pq}{p-q}} \left(\sum_{m\le k} v_m^{\frac{1}{1-p}} \right)^{\frac{(p-1)q}{p-q}} \right)^{\frac{p-q}{pq}} + \left(\sum_{n\in\mathbb{Z}} \left(\sum_{i\le n} w_i \sup_{i\le j\le n} u_j^q \right)^{\frac{q}{p-q}} w_n \sup_{k\ge n} u_k^q \left(\sum_{m\ge k} v_m^{\frac{1}{1-p}} \right)^{\frac{(p-1)q}{p-q}} \right)^{\frac{p-q}{pq}}.$$

(iii) *If* $0 < p \le 1$ *and* $p \le q$, *then*

$$C_2 \approx \sup_{n\in\mathbb{Z}} \left(u_n^q \sum_{i\le n} w_i + \sum_{k\ge n} u_k^q w_k \right)^{\frac{1}{q}} \sup_{j\le n} v_j^{-\frac{1}{p}}.$$

(iv) *If* $0<q<p\le 1$, *then*

$$C_2 \approx \left(\sum_{n\in\mathbb{Z}}\left(\sum_{i\le n} w_i\right)^{\frac{q}{p-q}} w_n \sup_{k\ge n} u_k^{\frac{q}{p-q}} \sup_{m\ge k} v_m^{\frac{q}{q-p}}\right)^{\frac{p-q}{pq}} + \left(\sum_{n\in\mathbb{Z}}\left(\sum_{i\le n} w_i \sup_{i\le j\le n} u_j^q\right)^{\frac{q}{p-q}} w_n \sup_{k\ge n} u_k^q \sup_{m\ge k} v_m^{\frac{q}{q-p}}\right)^{\frac{p-q}{pq}}.$$

3. Equivalence theorems for $p\in(0,1]$

In this section, after presenting some auxiliary results, we show an equivalence principle for supremal and integral Hardy operators in the case $p\in(0,1]$. These results establish a link between discrete and continuous Hardy-type inequalities for such p, but they are of independent interest.

The first preliminary result is an extension of [Sin03, Theorem 3.1] concerning "transferring monotonicity" to the weight sequence on the right-hand side. In here, we use the following notation, for $\mathbf{a}\in\mathbb{R}_+^{\mathbb{Z}}$,

$$S\mathbf{a}_n = \sup_{j\le n} a_j, \qquad I\mathbf{a}_j = \sum_{j\le n} a_j.$$

Hence $S\mathbf{a}, I\mathbf{a}\in\overline{\mathbb{R}}_+^{\mathbb{Z}}$ and $S\mathbf{a}_n$, $I\mathbf{a}_n$ are the n-th entries of $S\mathbf{a}$ and $I\mathbf{a}$, respectively.

⟨L:Sinn-trik⟩ **Lemma 3.1.** *Let* $\mathbf{u}\in\mathbb{R}_+^{\mathbb{Z}}$. *Let* $\varphi:\mathbb{R}_+^{\mathbb{Z}}\to\mathbb{R}_+$ *be a functional such that there exists a sequence* $\mathbf{c}\in\mathbb{R}_+^{\mathbb{Z}}$ *with a finite number of non-zero entries for which* $\varphi(\mathbf{c})>0$. *In addition to this, assume that* φ *satisfies*

$$S\mathbf{a}\le S\mathbf{b} \quad\Longrightarrow\quad \varphi(\mathbf{a})\le\varphi(\mathbf{b}) \tag{3.1}$$ E:SiS

or

$$I\mathbf{a}\le I\mathbf{b} \quad\Longrightarrow\quad \varphi(\mathbf{a})\le\varphi(\mathbf{b}) \tag{3.2}$$ E:SiI

for all $\mathbf{a},\mathbf{b}\in\mathbb{R}_+^{\mathbb{Z}}$. *Then we have*

$$\sup_{\mathbf{a}\in\mathbb{R}_+^{\mathbb{Z}}} \frac{\varphi(\mathbf{a})}{\sum_{n\in\mathbb{Z}} a_n u_n} = \sup_{\mathbf{a}\in\mathbb{R}_+^{\mathbb{Z}}} \frac{\varphi(\mathbf{a})}{\sum_{n\in\mathbb{Z}} a_n \underset{\downarrow}{u}_n}. \tag{3.3}$$ E:zmono

Proof. The assertion involving an operator satisfying (3.2) follows from the proof of [Sin03, Theorem 3.1]. The proof for the case (3.1) is rather similar but we give it here for the sake of completeness.

The inequality "$\le$" is obvious since $\underset{\downarrow}{\mathbf{u}}\le\mathbf{u}$. We have to show "$\ge$". First assume that $\underset{\downarrow}{\mathbf{u}}$ is identically zero. By the properties of φ, there exists a finite set of indices $M\subset\mathbb{Z}$ and a sequence $\mathbf{c}\in\mathbb{R}_+^{\mathbb{Z}}$ such that $\varphi(\mathbf{c})>0$ and $c_n=0$ unless $n\in M$. Let $\varepsilon>0$. Since $\liminf_{n\to-\infty} u_n=0$, there exists $N\in\mathbb{Z}$ such that $N\le\min M$ and $u_N<\varepsilon$. Define $b_N=\max_{n\in M} c_n$ and $b_n=0$ for all $n\in\mathbb{Z}\setminus\{N\}$. The sequence $\mathbf{b}=\{b_n\}_{n\in\mathbb{Z}}$ satisfies $S\mathbf{b}\ge S\mathbf{c}$, thus also $\varphi(\mathbf{b})\ge\varphi(\mathbf{c})$. Moreover, we have

$$\sum_{n\in\mathbb{Z}} b_n u_n = b_N u_N < \varepsilon \max_{n\in M} c_n.$$

Hence,

$$\frac{\varphi(\mathbf{b})}{\sum_{n\in\mathbb{Z}} b_n u_n} > \frac{\varphi(\mathbf{b})}{\varepsilon \max_{n\in M} c_n} \geq \frac{\varphi(\mathbf{c})}{\varepsilon \max_{n\in M} c_n},$$

and therefore

$$\sup_{\mathbf{a}\in\mathbb{R}_+^{\mathbb{N}}} \frac{\varphi(\mathbf{a})}{\sum_{n\in\mathbb{Z}} a_n u_n} > \frac{\varphi(\mathbf{c})}{\varepsilon \max_{n\in M} c_n}.$$

Since $\varepsilon > 0$ was arbitrary, we have

$$\sup_{\mathbf{a}\in\mathbb{R}_+^{\mathbb{N}}} \frac{\varphi(\mathbf{a})}{\sum_{n\in\mathbb{Z}} a_n u_n} = \infty,$$

so the inequality "$\geq$" in (3.3) is obviously satisfied.

In the following, we assume that $\underset{\downarrow}{\mathbf{u}}$ is not identically zero, hence

$$\lim_{n\to-\infty} \underset{\downarrow}{u}_n = \liminf_{n\to-\infty} u_n > 0. \tag{3.4}$$ E:nenula

Let $\varepsilon > 0$. By definition of the envelope and (3.4), there exists an index $n_0 \in \mathbb{Z}$ such that

$$u_{n_0} \leq (1+\varepsilon)\underset{\downarrow}{u}_{n_0}.$$

Now we define a sequence $\{n_k\}$ recursively. At first, we construct the "positive part" with indices $k > 0$ as follows. If $k \in \mathbb{N}$, n_{k-1} is defined and $n_{k-1} < \infty$, define

$$n_k = \inf\left\{ j \in \mathbb{Z} \,\middle|\, j > n_{k-1},\, u_j \leq (1+\varepsilon)\underset{\downarrow}{u}_j \right\},$$

where $\inf \emptyset = \infty$. In this way, we get a strictly increasing sequence of indices $\{n_k\}_{n=0}^{K}$ which is either finite with $K \in \mathbb{N}$ and $n_K = \infty$, or infinite with $K = \infty$. Furthermore, we construct the "negative part" with indices $k < 0$. If $k \in \mathbb{Z}$, $k < 0$, is such that n_{k+1} is already defined, put

$$n_k = \sup\left\{ j \in \mathbb{Z} \,\middle|\, j < n_{k+1},\, u_j \leq (1+\varepsilon)\underset{\downarrow}{u}_j \right\}.$$

In this case, the set over which the supremum is taken is nonempty, by the definition of $\underset{\downarrow}{\mathbf{u}}$ and (3.4). Hence, altogether we obtain a strictly increasing sequence of indices $\{n_k\}_{n=-\infty}^{K}$ such that

$$u_{n_k} \leq (1+\varepsilon)\underset{\downarrow}{u}_{n_k} \tag{3.5}$$ E:blizko

and

$$\underset{\downarrow}{u}_n = \underset{\downarrow}{u}_{n_k} \quad \text{for all } n \in \{n_k, \ldots, n_{k+1}-1\} \tag{3.6}$$ E:kst

for all $k \in \mathbb{Z}$ such that $k < K$. To verify (3.6), suppose that if $\underset{\downarrow}{u}_{n_k} > \underset{\downarrow}{u}_j$ for some $j \in \mathbb{Z}$, $j > n_k$. Without loss of generality, j is the smallest index with this property. Then necessarily $\underset{\downarrow}{u}_j = u_j$ by definition of the envelope, and thus $n_{k+1} \leq j$ by definition of $\{n_k\}$.

Let us note that if $\underset{\downarrow}{\mathbf{u}}$ contains no infinite constant subsequence, the above construction may be performed with $\varepsilon = 0$ ($K = \infty$ is then guaranteed).

Fix $\mathbf{a} \in \mathbb{R}_+^{\mathbb{Z}}$ arbitrary. We define a sequence $\mathbf{b}$ by setting

$$b_n = \begin{cases} \sup\limits_{n_k \leq j < n_{k+1}} a_n & \text{if } n = n_k \text{ for some } k \in \mathbb{Z},\ k < K, \\ 0 & \text{else.} \end{cases} \tag{3.7}$$ E:defb

It follows that $S\mathbf{a} \le S\mathbf{b}$. Indeed, for each $n \in \mathbb{Z}$ there exists $k \in \mathbb{Z}$, $k < K$, such that $n_k \le n < n_{k+1}$ and we have, for each n,

$$S\mathbf{a}_n = \sup_{j\le n} a_j \le \max\left\{\sup_{j<n_k} a_j, \sup_{n_k\le j<n_{k+1}} a_j\right\} = S\mathbf{b}_n.$$

Moreover, by (3.7), (3.6) and (3.5) one has

$$\sum_{n\in\mathbb{Z}} b_n u_n = \sum_{k\le K-1} u_{n_k} \sup_{n_k\le j<n_{k+1}} a_j \le (1+\varepsilon)\sum_{k\le K-1} \sup_{n_k\le j<n_{k+1}} a_j \underset{\downarrow}{u}_j \le (1+\varepsilon)\sum_{n\in\mathbb{Z}} a_n \underset{\downarrow}{u}_n. \quad (3.8)$$ E:a-vs-b

By the properties of φ, $S\mathbf{a} \le S\mathbf{b}$ implies $\varphi(\mathbf{a}) \le \varphi(\mathbf{b})$. From this and (3.8) we obtain

$$\frac{\varphi(\mathbf{a})}{\sum_{n\in\mathbb{Z}} a_n \underset{\downarrow}{u}_n} \le \frac{(1+\varepsilon)\varphi(\mathbf{a})}{\sum_{n\in\mathbb{Z}} b_n u_n} \le \frac{(1+\varepsilon)\varphi(\mathbf{b})}{\sum_{n\in\mathbb{Z}} b_n u_n} \le (1+\varepsilon)\sup_{\mathbf{b}\in\mathbb{R}_+^{\mathbb{Z}}} \frac{\varphi(\mathbf{b})}{\sum_{n\in\mathbb{Z}} b_n u_n}.$$

Since $\mathbf{a} \in \mathbb{R}_+^{\mathbb{Z}}$ and $\varepsilon > 0$ were arbitrary, we get the desired inequality

$$\sup \frac{\varphi(\mathbf{a})}{\sum_{n\in\mathbb{Z}} a_n \underset{\downarrow}{u}_n} \le \sup_{\mathbf{b}\in\mathbb{R}_+^{\mathbb{Z}}} \frac{\varphi(\mathbf{b})}{\sum_{n\in\mathbb{Z}} b_n u_n}.$$

□

⟨R:dua1⟩ *Remark* 3.2. For $\mathbf{a} \in \mathbb{R}_+^{\mathbb{Z}}$, define

$$S^*\mathbf{a}_n = \sup_{j\ge n} a_j \quad \text{and} \quad I^*\mathbf{a}_j = \sum_{j\ge n} a_j.$$

Lemma 3.1 holds unchanged if we replace S by S^* in (3.1) as well as I by I^* in (3.2), and $\underset{\downarrow}{u}$ by $\underset{\uparrow}{u}$ in (3.3). To check this, it suffices to perform the index change $\overline{a}_n = a_{-n}$, $n \in \mathbb{Z}$.

In what follows we are going to use a blocking technique (see [GE98]). To this end, we need the following definition.

Let $\mathbf{w} \in \mathbb{R}_+^{\mathbb{Z}}$ and $n_0 \in \mathbb{Z}$. The *block partition with respect to* $\mathbf{w}$ *starting at* n_0 is the sequence $\{n_k\}_{k=0}^K$ defined recursively by

$$n_1 = n_0 + 1,$$
$$n_k = \inf\left\{ j \in \mathbb{Z} \;\middle|\; j > n_{k-1},\ \sum_{i\ge j} w_i \ge 2\sum_{i=n_{k-1}}^{j-1} w_i \right\} \quad \text{for } k \ge 2.$$

In here, $K \in \mathbb{Z}$ if $\sum_{n\in\mathbb{Z}} w_n < \infty$, and $K = \infty$ otherwise. Furthermore, define

$$\mathbb{K} = \left\{ k \in \{1,\dots,K-1\} \;\middle|\; n_k < n_{k+1} - 1 \right\}.$$

By the construction, for all $k \in \mathbb{K}$ it holds that

$$\sum_{i=n_k}^{n_{k+1}-2} w_i < 2\sum_{i=n_{k-1}}^{n_k-1} w_i.$$

The blocking technique relies on the following well-known proposition (see [GE98, Kře17a, GP03]).

⟨P:dya⟩ **Proposition 3.3.** *Let $0 < \alpha < \infty$. Then there exists a constant $C \in (0, \infty)$ such that for any $k_{\min}, k_{\max} \in \mathbb{Z} \cup \{\pm\infty\}$, $k_{\min} < k_{\max}$, and any $\mathbf{b}, \mathbf{c} \in \mathbb{R}_+^{\mathbb{Z}}$ satisfying $b_{k+1} \geq 2b_k$ for all $k \in \mathbb{Z}$, $k_{\min} \leq k < k_{\max}$, one has*

$$\sum_{k=k_{\min}}^{k_{\max}} \left(\sum_{m=k}^{k_{\max}} c_m \right)^{\alpha} b_k \leq C \sum_{k=k_{\min}}^{k_{\max}} c_k^{\alpha} b_k,$$

$$\sum_{k=k_{\min}}^{k_{\max}} \sup_{k \leq m \leq k_{\max}} c_m b_k \leq C \sum_{k=k_{\min}}^{k_{\max}} c_k b_k.$$

The constant C depends only on α.

As the least (optimal) constants are expressed as suprema in the results below, the convention $0^{-\alpha} = \infty$, $\infty^{-\alpha} = 0$ $(\alpha > 0)$, $0 \cdot \infty = 0$ is in charge.

The first result obtained by the blocking technique involves a simple Hardy inequality. It may be recovered by examining the characterizations in [GE98, Theorem 7.7]. Here we present a direct proof since we are going to use its elements further on.

⟨L:prop24⟩ **Lemma 3.4.** *Let $p \in (0, 1]$, $q \in (0, \infty)$ and $\mathbf{v}, \mathbf{w} \in \mathbb{R}_+^{\mathbb{Z}}$. Define*

$$A_{(3.9)} = \sup_{\mathbf{a} \in \mathbb{R}_+^{\mathbb{Z}}} \left(\sum_{n \in \mathbb{Z}} w_n \sup_{j \geq n} a_j^q \right)^{\frac{1}{q}} \left(\sum_{n \in \mathbb{Z}} a_n^p v_n \right)^{-\frac{1}{p}}, \tag{3.9} \boxed{10}$$

$$A_{(3.10)} = \sup_{\mathbf{a} \in \mathbb{R}_+^{\mathbb{Z}}} \left(\sum_{n \in \mathbb{Z}} w_n \Big[\sum_{j \geq n} a_j \Big]^q \right)^{\frac{1}{q}} \left(\sum_{n \in \mathbb{Z}} a_n^p v_n \right)^{-\frac{1}{p}}, \tag{3.10} \boxed{11}$$

$$A_{(3.11)} = \sup_{\mathbf{a} \in \mathbb{R}_+^{\mathbb{Z}}} \left(\sum_{n \in \mathbb{Z}} w_n \Big[\sum_{j \geq n} a_j^p \Big]^{\frac{q}{p}} \right)^{\frac{1}{q}} \left(\sum_{n \in \mathbb{Z}} a_n^p v_n \right)^{-\frac{1}{p}}. \tag{3.11} \boxed{12}$$

Then the quantities $A_{(3.9)}$, $A_{(3.10)}$ and $A_{(3.11)}$ are equivalent, and, moreover, the equivalence constants depend only on p and q.

Proof. Since $p \in (0, 1]$, the inequalities $A_{(3.9)} \leq A_{(3.10)} \leq A_{(3.11)}$ follow from (1.7). We will prove $A_{(3.11)} \leq C A_{(3.9)}$ with an appropriate constant C.

By Remark 3.2, we may assume that $\mathbf{v}$ is increasing. Let $\mathbf{a} \in \mathbb{R}_+^{\mathbb{Z}}$ be such that $\sum_{n \in \mathbb{Z}} a_n^p v_n \in (0, \infty)$. Fix an arbitrary $n_0 \in \mathbb{Z}$. Let $\{n_k\}_{k=0}^K$ be the block partition with respect to $\mathbf{w}$ starting at n_0. We have

$$\begin{aligned}
\sum_{n\geq n_0} w_n \Big[\sum_{j\geq n} a_j^p\Big]^{\frac{q}{p}} &= \sum_{k=0}^{K-1} \sum_{n=n_k}^{n_{k+1}-1} w_n \Big[\sum_{j\geq n} a_j^p\Big]^{\frac{q}{p}} \\
&= \sum_{k=0}^{K-2} w_{n_{k+1}-1} \Big[\sum_{j\geq n_{k+1}-1} a_j^p\Big]^{\frac{q}{p}} + \sum_{k\in\mathbb{K}} \sum_{n=n_k}^{n_{k+1}-2} w_n \Big[\sum_{j\geq n} a_j^p\Big]^{\frac{q}{p}} \\
&\lesssim \sum_{k=0}^{K-2} w_{n_{k+1}-1} \Big[\sum_{j\geq n_{k+1}-1} a_j^p\Big]^{\frac{q}{p}} + \sum_{k\in\mathbb{K}} \sum_{n=n_{k-1}}^{n_k-1} w_n \Big[\sum_{j\geq n_k-1} a_j^p\Big]^{\frac{q}{p}} \\
&\lesssim \sum_{k=0}^{K-2} \sum_{n=n_k}^{n_{k+1}-1} w_n \Big[\sum_{j\geq n_{k+1}-1} a_j^p\Big]^{\frac{q}{p}} \\
&\lesssim \sum_{k=0}^{K-2} \sum_{n=n_k}^{n_{k+1}-1} w_n \Big[\sum_{j=n_{k+1}-1}^{n_{k+2}-2} a_j^p\Big]^{\frac{q}{p}}.
\end{aligned}$$

Here we used the properties of the block partition on the third line, and Proposition 3.3 on the fifth. Now define the sequence $\mathbf{b}\in\mathbb{R}_+^{\mathbb{Z}}$ by

$$b_n = \begin{cases} \Big[\displaystyle\sum_{j=n_k-1}^{n_{k+1}-2} a_j^p\Big]^{\frac{1}{p}} & \text{if } n = n_k-1 \text{ for some } k\in\{1,\dots,K-1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathbf{v}$ is increasing, we have

$$\sum_{n\in\mathbb{Z}} b_n^p v_n = \sum_{k=1}^{K-1} \sum_{j=n_k-1}^{n_{k+1}-2} a_j^p v_j \leq \sum_{n\in\mathbb{Z}} a_n^p v_n.$$

Altogether, we obtain the following chain of relations in which $C\in(0,\infty)$ depends only on p and q,

$$\begin{aligned}
\bigg(\sum_{n\geq n_0} w_n \Big[\sum_{j\geq n} a_j^p\Big]^{\frac{q}{p}}\bigg)^{\frac{1}{q}} \bigg(\sum_{n\in\mathbb{Z}} a_n^p v_n\bigg)^{-\frac{1}{p}} &\leq C\bigg(\sum_{k=0}^{K-2} \sum_{n=n_k}^{n_{k+1}-1} w_n \Big[\sum_{j=n_{k+1}-1}^{n_{k+2}-2} a_j^p\Big]^{\frac{q}{p}}\bigg)^{\frac{1}{q}} \bigg(\sum_{n\in\mathbb{Z}} a_n^p v_n\bigg)^{-\frac{1}{p}} \\
&= C\bigg(\sum_{k=0}^{K-2} \sum_{n=n_k}^{n_{k+1}-1} w_n b_{n_{k+1}-1}^q\bigg)^{\frac{1}{q}} \bigg(\sum_{n\in\mathbb{Z}} a_n^p v_n\bigg)^{-\frac{1}{p}} \\
&\leq C\bigg(\sum_{k=0}^{K-2} \sum_{n=n_k}^{n_{k+1}-1} w_n \sup_{j\geq n} b_j^q\bigg)^{\frac{1}{q}} \bigg(\sum_{n\in\mathbb{Z}} b_n^p v_n\bigg)^{-\frac{1}{p}} \\
&\leq C \sup_{\mathbf{b}\in\mathbb{R}_+^{\mathbb{Z}}} \bigg(\sum_{n\in\mathbb{Z}} w_n \sup_{j\geq n} b_j^q\bigg)^{\frac{1}{q}} \bigg(\sum_{n\in\mathbb{Z}} b_n^p v_n\bigg)^{-\frac{1}{p}}.
\end{aligned}$$

Since n_0 was arbitrary, we have

$$\left(\sum_{n\in\mathbb{Z}} w_n \Big[\sum_{j\geq n} a_j^p\Big]^{\frac{q}{p}}\right)^{\frac{1}{q}} \left(\sum_{n\in\mathbb{Z}} a_n^p v_n\right)^{-\frac{1}{p}} \leq C \sup_{\mathbf{b}\in\mathbb{R}_+^{\mathbb{Z}}} \left(\sum_{n\in\mathbb{Z}} w_n \sup_{j\geq n} b_j^q\right)^{\frac{1}{q}} \left(\sum_{n\in\mathbb{Z}} b_n^p v_n\right)^{-\frac{1}{p}}$$

with the same C. If $\sum_{n\in\mathbb{Z}} a_n^p v_n = 0$, the inequality holds trivially. If $\sum_{n\in\mathbb{Z}} a_n^p v_n = \infty$, both sides of the inequality are either zero (when $\mathbf{w}$ is constant zero) or infinite. Hence, we may take the supremum over $\mathbf{a}\in\mathbb{R}_+^{\mathbb{Z}}$ on the left-hand side, which yields $A_{(3.11)} \leq CA_{(3.9)}$. □

An analogous statement to the preceding lemma in the case when $q=\infty$ holds, too. It can be easily proved by interchanging the suprema.

⟨L:inft⟩ **Lemma 3.5.** *Let $p\in(0,1]$ and $\mathbf{v},\mathbf{w}\in\mathbb{R}_+^{\mathbb{Z}}$. Then*

$$\sup_{\mathbf{a}\in\mathbb{R}_+^{\mathbb{Z}}} \sup_{n\in\mathbb{Z}} u_n \sum_{j\geq n} a_j \left(\sum_{i\in\mathbb{Z}} a_i^p v_i\right)^{-\frac{1}{p}} = \sup_{\mathbf{a}\in\mathbb{R}_+^{\mathbb{Z}}} \sup_{n\in\mathbb{Z}} u_n \sup_{j\geq n} a_j \left(\sum_{i\in\mathbb{Z}} a_i^p v_i\right)^{-\frac{1}{p}} = \sup_{n\in\mathbb{Z}} u_n \sup_{j\geq n} v_j^{-\frac{1}{p}}.$$

⟨R:dua2⟩ *Remark* 3.6. As usual, both Lemmas 3.4 and 3.5 have their "dual versions", in which the suprema or sums over $j\geq n$ are replaced by their respective counterparts over $j\leq n$. We omit the details.

We are now in a position to prove a similar equivalence for the more complicated iterated Hardy operators.

⟨T:ekv-antigop⟩ **Theorem 3.7.** *Let $p\in(0,1]$, $q\in(0,\infty)$ and $\mathbf{v},\mathbf{w}\in\mathbb{R}_+^{\mathbb{Z}}$. Define*

$$A_{(3.12)} = \sup_{\mathbf{a}\in\mathbb{R}_+^{\mathbb{Z}}} \left(\sum_{n\in\mathbb{Z}} w_n \Big[\sup_{j\geq n} u_j \sup_{i\geq j} a_i\Big]^q\right)^{\frac{1}{q}} \left(\sum_{n\in\mathbb{Z}} v_n a_n^p\right)^{-\frac{1}{p}}, \tag{3.12}$$ ag1

$$A_{(3.13)} = \sup_{\mathbf{a}\in\mathbb{R}_+^{\mathbb{Z}}} \left(\sum_{n\in\mathbb{Z}} w_n \Big[\sup_{j\geq n} u_j \sum_{i\geq j} a_i\Big]^q\right)^{\frac{1}{q}} \left(\sum_{n\in\mathbb{Z}} v_n a_n^p\right)^{-\frac{1}{p}}, \tag{3.13}$$ ag2

$$A_{(3.14)} = \sup_{\mathbf{a}\in\mathbb{R}_+^{\mathbb{Z}}} \left(\sum_{n\in\mathbb{Z}} w_n \Big[\sup_{j\geq n} u_j^p \sum_{i\geq j} a_i^p\Big]^{\frac{q}{p}}\right)^{\frac{1}{q}} \left(\sum_{n\in\mathbb{Z}} v_n a_n^p\right)^{-\frac{1}{p}}. \tag{3.14}$$ ag3

Then $A_{(3.12)}$, $A_{(3.13)}$ and $A_{(3.14)}$ are mutually equivalent, and, moreover, the equivalence constants depend only on p and q.

Proof. Due to (1.7), only $A_{(3.14)} \leq CA_{(3.12)}$ needs proving. Let $n_0\in\mathbb{Z}$ and let $\{n_k\}_{k=0}^K$ be the block partition with respect to $\mathbf{w}$ starting at n_0. Without loss of generality we may assume that $K\geq 3$. Let $\mathbf{a}\in\mathbb{R}_+^{\mathbb{Z}}$ be such that $\sum_{n\in\mathbb{Z}} a_n^p v_n \in (0,\infty)$. Analogously as in Lemma

3.4 we have

$$\begin{aligned}
\sum_{n\geq n_0} w_n \sup_{j\geq n} u_j^q \Big[\sum_{i\geq j} a_i^p\Big]^{\frac{q}{p}} &= \sum_{k=0}^{K-1}\sum_{n=n_k}^{n_{k+1}-1} w_n \sup_{j\geq n} u_j^q \Big[\sum_{i\geq j} a_i^p\Big]^{\frac{q}{p}} \\
&= \sum_{k=0}^{K-2} w_{n_{k+1}-1} \sup_{j\geq n_{k+1}-1} u_j^q \Big[\sum_{i\geq j} a_i^p\Big]^{\frac{q}{p}} \\
&\quad + \sum_{k\in\mathbb{K}}\sum_{n=n_k}^{n_{k+1}-2} w_n \sup_{n\leq j\leq n_{k+1}-2} u_j^q \Big[\sum_{i\geq j} a_i^p\Big]^{\frac{q}{p}} \\
&\lesssim \sum_{k=0}^{K-2}\sum_{n=n_k}^{n_{k+1}-1} w_n \sup_{j\geq n_{k+1}-1} u_j^q \Big[\sum_{i\geq j} a_i^p\Big]^{\frac{q}{p}} \\
&\lesssim \sum_{k=0}^{K-2}\sum_{n=n_k}^{n_{k+1}-1} w_n \sup_{n_{k+1}-1\leq j\leq n_{k+2}-2} u_j^q \Big[\sum_{i\geq j} a_i^p\Big]^{\frac{q}{p}} \\
&\lesssim \sum_{k=0}^{K-2}\sum_{n=n_k}^{n_{k+1}-1} w_n \sup_{n_{k+1}-1\leq j\leq n_{k+2}-2} u_j^q \Big[\sum_{i=j}^{n_{k+2}-2} a_i^p\Big]^{\frac{q}{p}} \\
&\quad + \sum_{k=0}^{K-3}\sum_{n=n_k}^{n_{k+1}-1} w_n \sup_{n_{k+1}-1\leq j\leq n_{k+2}-2} u_j^q \Big[\sum_{i\geq n_{k+2}-1} a_i^p\Big]^{\frac{q}{p}} \\
&= B_1 + B_2.
\end{aligned}$$

If $k\in\{0,\dots,K-2\}$ and

$$\sum_{n=n_{k+1}-1}^{n_{k+2}-2} a_n^p v_n > 0, \tag{3.15}$$

E:podm

find $c_{n_{k+1}-1},\dots,c_{n_{k+2}-2}\geq 0$ such that

$$\sum_{n=n_{k+1}-1}^{n_{k+2}-2} c_n^p v_n = \sum_{n=n_{k+1}-1}^{n_{k+2}-2} a_n^p v_n$$

and

$$\begin{aligned}
&\sup_{\mathbf{b}\in\mathbb{R}_+^{\mathbb{Z}}} \sup_{n_{k+1}-1\leq j\leq n_{k+2}-2} u_j \sup_{j\leq i\leq n_{k+2}-2} b_i \Bigg(\sum_{m=n_{k+1}-1}^{n_{k+2}-2} b_m^p v_m\Bigg)^{-\frac{1}{p}} \\
&\qquad\qquad \leq 2 \sup_{n_{k+1}-1\leq j\leq n_{k+2}-2} u_j \sup_{j\leq i\leq n_{k+2}-2} c_i \Bigg(\sum_{n=n_{k+1}-1}^{n_{k+2}-2} c_n^p v_n\Bigg)^{-\frac{1}{p}}.
\end{aligned}$$

For all other indices $n\in\mathbb{Z}$ such that $n\notin\{n_{k+1}-1,\dots,n_{k+2}-2\}$ and all $k\in\{0,\dots,K-2\}$ satisfying (3.15) we define $c_n=0$. In this way we obtain a sequence $\mathbf{c}\in\mathbb{R}_+^{\mathbb{Z}}$ which moreover satisfies

$$\sum_{n\in\mathbb{Z}} c_n^p v_n \leq \sum_{n\geq n_0} a_n^p v_n.$$

Using Lemma 3.5 we get

$$\begin{aligned} B_1 &\le \sum_{k=0}^{K-2} \sum_{n=n_k}^{n_{k+1}-1} w_n \Bigg[\sup_{\mathbf{b}\in\mathbb{R}_+^{\mathbb{Z}}} \sup_{n_{k+1}-1\le j\le n_{k+2}-2} u_j \sum_{i=j}^{n_{k+2}-2} b_i \Bigg(\sum_{m=n_{k+1}-1}^{n_{k+2}-2} b_m^p v_m \Bigg)^{-\frac{1}{p}} \Bigg]^q \Bigg(\sum_{n=n_{k+1}-1}^{n_{k+2}-2} a_n^p v_n \Bigg)^{\frac{q}{p}} \\ &\lesssim \sum_{k=0}^{K-2} \sum_{n=n_k}^{n_{k+1}-1} w_n \Bigg[\sup_{\mathbf{b}\in\mathbb{R}_+^{\mathbb{Z}}} \sup_{n_{k+1}-1\le j\le n_{k+2}-2} u_j \sup_{j\le i\le n_{k+2}-2} b_i \Bigg(\sum_{m=n_{k+1}-1}^{n_{k+2}-2} b_m^p v_m \Bigg)^{-\frac{1}{p}} \Bigg]^q \Bigg(\sum_{n=n_{k+1}-1}^{n_{k+2}-2} a_n^p v_n \Bigg)^{\frac{q}{p}} \\ &\lesssim \sum_{k=0}^{K-2} \sum_{n=n_k}^{n_{k+1}-1} w_n \Big[\sup_{n_{k+1}-1\le j\le n_{k+2}-2} u_j \sup_{j\le i\le n_{k+2}-2} c_i \Big]^q \\ &\lesssim \sum_{n\ge n_0} w_n \Big[\sup_{j\ge n} u_j \sup_{i\ge j} c_i \Big]^q. \end{aligned}$$

Hence,

$$\begin{aligned} B_1 \Bigg(\sum_{n\ge n_0} a_n^p v_n \Bigg)^{-\frac{q}{p}} &\lesssim \sum_{n\ge n_0} w_n \Big[\sup_{j\ge n} u_j \sup_{i\ge j} c_i \Big]^q \Bigg(\sum_{n\in\mathbb{Z}} a_n^p v_n \Bigg)^{-\frac{q}{p}} \\ &\lesssim \sum_{n\ge n_0} w_n \Big[\sup_{j\ge n} u_j \sup_{i\ge j} c_i \Big]^q \Bigg(\sum_{n\in\mathbb{Z}} c_n^p v_n \Bigg)^{-\frac{q}{p}} \\ &\le A_{(3.12)}^q. \end{aligned}$$

Lemma 3.4 further yields

$$\begin{aligned} B_2 \Bigg(\sum_{n\ge n_0} a_n^p v_n \Bigg)^{-\frac{q}{p}} &\lesssim \sum_{k=0}^{K-3} \sum_{n=n_k}^{n_{k+1}-1} w_n \sup_{n_{k+1}-1\le j\le n_{k+2}-2} u_j^q \Big[\sup_{i\ge n_{k+2}-1} a_i \Big]^q \Bigg(\sum_{n\in\mathbb{Z}} a_n^p v_n \Bigg)^{-\frac{q}{p}} \\ &\le \sum_{n\in\mathbb{Z}} w_n \Big[\sup_{j\ge n} u_j \sup_{i\ge j} a_i \Big]^q \Bigg(\sum_{n\in\mathbb{Z}} a_n^p v_n \Bigg)^{-\frac{q}{p}} \\ &\le A_{(3.12)}^q. \end{aligned}$$

Finally, we get

$$\begin{aligned} &\Bigg(\sum_{n\ge n_0} w_n \Big[\sup_{j\ge n} u_j \sum_{i\le j} a_i \Big]^q \Bigg)^{\frac{1}{q}} \Bigg(\sum_{n\in\mathbb{Z}} a_n^p v_n \Bigg)^{-\frac{1}{p}} \\ &\le \Bigg(\sum_{n\ge n_0} w_n \Big[\sup_{j\ge n} u_j \sum_{i\le j} a_i \Big]^q \Bigg)^{\frac{1}{q}} \Bigg(\sum_{n\ge n_0} a_n^p v_n \Bigg)^{-\frac{1}{p}} \\ &\le C A_{(3.12)}. \end{aligned}$$

Since n_0 may be arbitrarily small, we obtain, with the same constant C, the desired inequality $A_{(3.13)} \le C A_{(3.12)}$. The cases when $\sum_{n\in\mathbb{Z}} a_n^p v_n$ is either zero or infinite can be treated as in the end of the proof of Lemma 3.4. □

⟨T:ekv-gop⟩

Theorem 3.8. *Let $p \in (0,1]$, $q \in (0,\infty)$ and $\mathbf{v},\mathbf{w} \in \mathbb{R}_+^{\mathbb{Z}}$. Define*

$$A_{(3.16)} = \sup_{a\in\mathbb{R}_+^{\mathbb{Z}}} \left(\sum_{n\in\mathbb{Z}} w_n \Big[\sup_{j\ge n} u_j \sup_{i\le j} a_i\Big]^q\right)^{\frac{1}{q}} \left(\sum_{n\in\mathbb{Z}} v_n a_n^p\right)^{-\frac{1}{p}}, \tag{3.16}$$ g1

$$A_{(3.17)} = \sup_{a\in\mathbb{R}_+^{\mathbb{Z}}} \left(\sum_{n\in\mathbb{Z}} w_n \Big[\sup_{j\ge n} u_j \sum_{i\le j} a_i\Big]^q\right)^{\frac{1}{q}} \left(\sum_{n\in\mathbb{Z}} v_n a_n^p\right)^{-\frac{1}{p}}, \tag{3.17}$$ g2

$$A_{(3.18)} = \sup_{a\in\mathbb{R}_+^{\mathbb{Z}}} \left(\sum_{n\in\mathbb{Z}} w_n \Big[\sup_{j\ge n} u_j^p \sum_{i\le j} a_i^p\Big]^{\frac{q}{p}}\right)^{\frac{1}{q}} \left(\sum_{n\in\mathbb{Z}} v_n a_n^p\right)^{-\frac{1}{p}}. \tag{3.18}$$ g3

Then $A_{(3.16)}$, $A_{(3.17)}$ and $A_{(3.18)}$ are equivalent, and, moreover, the equivalence constants depend only on q.

Proof. The proof is essentially the same as that of Theorem 3.7. The only minor difference is that, with $\{n_k\}_{k=0}^K$ being the block partition with respect to $\mathbf{w}$ starting at n_0 and $\mathbf{a} \in \mathbb{R}_+^{\mathbb{Z}}$ being a sequence such that $\sum_{n\in\mathbb{Z}} a_n^p v_n \in (0,\infty)$, we get the following estimate:

$$\begin{aligned}\sum_{n\ge n_0} w_n \Big[\sup_{j\ge n} u_j \sum_{i\le j} a_i\Big]^q &\lesssim \sum_{k=0}^{K-2} \sum_{n=n_k}^{n_{k+1}-1} w_n \Big[\sup_{n_{k+1}-1\le j\le n_{k+2}-2} u_j \sum_{i=n_{k+1}-2}^{j} a_i\Big]^q \\ &\quad + \sum_{k=1}^{K-2} \sum_{n=n_k}^{n_{k+1}-1} w_n \sup_{n_{k+1}-1\le j\le n_{k+2}-2} u_j^q \Big[\sum_{i\le n_{k+1}-2} a_i\Big]^q.\end{aligned}$$

Both terms can then be treated as in Theorem 3.7. A slight difference concerns the second one for which we just have to use the "dual version" of Lemma 3.4 (see Remark 3.6) instead of the standard one. □

Remark 3.9. It goes without saying that Theorems 3.7 and 3.8 may be restated in a "dual form" by replacing each symbol "$\le$" in their statements by "$\ge$" and vice versa.

At this point we may apply the obtained results to establish an interesting characterization of a discrete inequality by a continuous one in the case $p \in (0,1]$.

⟨T:main-prop2⟩ **Corollary 3.10.** *Let $p \in (0,1]$ and $q \in (0,\infty)$. Let $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{R}_+^{\mathbb{Z}}$. Define $\mathbf{u},\mathbf{v}$ and $\mathbf{w}$ as in Theorem 1.1. Then* (1.1) *holds for every sequence $\mathbf{a} \in \mathbb{R}_+^{\mathbb{Z}}$ if and only if*

$$\left(\int_{\mathbb{R}} \left(\sup_{s\ge t} u(s)^p \int_{-\infty}^{s} f(y)\,\mathrm{d}y\right)^{\frac{q}{p}} w(t)\,\mathrm{d}t\right)^{\frac{p}{q}} \le C_1^p \int_{\mathbb{R}} f(t)v(t)\,\mathrm{d}t$$

holds for every $f \in \mathcal{M}_+$.

Similarly, (1.2) *holds for every sequence $\mathbf{a} \in \mathbb{R}_+^{\mathbb{Z}}$ if and only if*

$$\left(\int_{\mathbb{R}} \left(\sup_{s\ge t} u(s)^p \int_{s}^{\infty} f(y)\,\mathrm{d}y\right)^{\frac{q}{p}} w(t)\,\mathrm{d}t\right)^{\frac{p}{q}} \le C_2^p \int_{\mathbb{R}} f(t)v(t)\,\mathrm{d}t$$

holds for every $f \in \mathcal{M}_+$.

4. Proofs

Let us start by proving Theorem 1.1 from the introduction.

Proof of Theorem 1.1. Suppose that (1.1) holds and let $f \in \mathcal{M}_+$. Set $a_n = \int_n^{n+1} f$ for $n \in \mathbb{Z}$. Then we get, using the Hölder inequality,

$$\left(\sum_{n\in\mathbb{Z}} a_n^p v_n\right)^{\frac{1}{p}} \le \left(\sum_{n\in\mathbb{Z}} \int_n^{n+1} f(t)^p v_n \,\mathrm{d}t\right)^{\frac{1}{p}} = \left(\int_0^\infty f(t)^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}} \qquad (4.1)\ \texttt{bla1}$$

and

$$\begin{aligned}\left(\sum_{n\in\mathbb{Z}}\left(\sup_{i\ge n} u_i \sum_{k\le i} a_k\right)^q w_n\right)^{\frac{1}{q}} &= \left(\sum_{n\in\mathbb{Z}}\left(\sup_{i\ge n} u_i \int_{-\infty}^{i+1} f(y)\,\mathrm{d}y\right)^q \int_n^{n+1} w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}\\ &= \left(\sum_{n\in\mathbb{Z}}\int_n^{n+1}\left(\sup_{s\ge t} u(s)\int_{-\infty}^s f(y)\,\mathrm{d}y\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}\\ &= \left(\int_{\mathbb{R}}\left(\sup_{s\ge t} u(s)\int_{-\infty}^s f(y)\,\mathrm{d}y\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}, \qquad (4.2)\ \texttt{bla2}\end{aligned}$$

and (1.10) follows.

Conversely, assume that (1.10) is satisfied. Let $\mathbf{a} \in \mathbb{R}_+^{\mathbb{Z}}$ be arbitrary. Define

$$f = \sum_{n\in\mathbb{Z}} a_n \chi_{[n,n+1)}.$$

Then we get (4.2) as above, and (4.1) holds now with identity in place of inequality. Hence, (1.1) follows.

The equivalence between (1.2) and (1.11) can be obtained analogously. □

Now we can complete the proofs of the main results.

Proof of Theorem 2.1. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be as in Theorem 1.1. Let C_1 be the least constant (including the possibility $C_1 = \infty$) such that (1.1) holds for all $\mathbf{a} \in \mathbb{R}_+^{\mathbb{Z}}$.

Assume that $1 < p \le q$. From Theorem 1.1 and [GOP06, Theorem 4.1] it follows that

$$\begin{aligned}C_1 &\approx \sup_{t\in\mathbb{R}}\sup_{x\ge t} u(x)\left(\int_{-\infty}^t w(s)\,\mathrm{d}s\right)^{\frac{1}{q}}\left(\int_{-\infty}^t v(s)^{\frac{1}{1-p}}\,\mathrm{d}s\right)^{\frac{p-1}{p}}\\ &\quad + \sup_{t\in\mathbb{R}}\left(\int_t^\infty \sup_{y\ge s} u(y)^q w(s)\,\mathrm{d}s\right)^{\frac{1}{q}}\left(\int_{-\infty}^t v(s)^{\frac{1}{1-p}}\,\mathrm{d}s\right)^{\frac{p-1}{p}}\\ &= B_1 + B_2.\end{aligned}$$

Notice that [GOP06, Theorem 4.1] is stated for inequality (1.10) in which the integration domain is replaced by $(0,\infty)$ and where the function u is continuous. Therefore, to get the result in the form we need, we have to use a change of variables and a monotone approximation of u by continuous functions. Anyway, we have

$$\begin{aligned} B_1 &= \sup_{n\in\mathbb{Z}} \sup_{t\in[n,n+1)} \sup_{x\geq t} u(x) \left(\int_{-\infty}^t w(s)\,\mathrm{d}s\right)^{\frac{1}{q}} \left(\int_{-\infty}^t v(s)^{\frac{1}{1-p}}\,\mathrm{d}s\right)^{\frac{p-1}{p}} \\ &= \sup_{n\in\mathbb{Z}} \overset{\downarrow}{u}_n \left(\sum_{i\leq n} w_i\right)^{\frac{1}{q}} \left(\sum_{k\leq n} v_k^{\frac{1}{1-p}}\right)^{\frac{p-1}{p}} \end{aligned}$$

and

$$\begin{aligned} B_2 &= \sup_{n\in\mathbb{Z}} \sup_{t\in[n,n+1)} \left(\int_t^\infty \sup_{y\geq s} u(y)^q w(s)\,\mathrm{d}s\right)^{\frac{1}{q}} \left(\int_{-\infty}^t v(s)^{\frac{1}{1-p}}\,\mathrm{d}s\right)^{\frac{p-1}{p}} \\ &= \sup_{n\in\mathbb{Z}} \sup_{t\in[n,n+1)} \left(\int_t^{n+1} \sup_{y\geq s} u(y)^q w(s)\,\mathrm{d}s \int_{n+1}^\infty \sup_{y\geq s} u(y)^q w(s)\,\mathrm{d}s\right)^{\frac{1}{q}} \\ &\qquad \times \left(\int_n^t v(s)^{\frac{1}{1-p}}\,\mathrm{d}s + \int_{-\infty}^n v(s)^{\frac{1}{1-p}}\,\mathrm{d}s\right)^{\frac{p-1}{p}} \\ &= \sup_{n\in\mathbb{Z}} \sup_{\lambda\in[0,1)} \left(\lambda\, \overset{\downarrow}{u}{}_n^q w_n + \sum_{i\geq n+1} \overset{\downarrow}{u}{}_i^q w_i\right)^{\frac{1}{q}} \left((1-\lambda) v_n^{\frac{1}{1-p}} + \sum_{k\leq n-1} v_k^{\frac{1}{1-p}}\right)^{\frac{p-1}{p}} \\ &\approx \sup_{n\in\mathbb{Z}} \left(\sum_{i\geq n} \overset{\downarrow}{u}{}_i^q w_i\right)^{\frac{1}{q}} \left(\sum_{k\leq n} v_k^{\frac{1}{1-p}}\right)^{\frac{p-1}{p}}. \end{aligned}$$

To verify the latter equivalence, observe that

$$\sup_{\lambda\in[0,1]} (X+\lambda x)^\alpha (Y+(1-\lambda)y)^\beta \;\leq\; (X+x)^\alpha (Y+y)^\beta \;\leq\; 2^{\alpha+\beta} \sup_{\lambda\in[0,1]} (X+\lambda x)^\alpha (Y+(1-\lambda)y)^\beta$$

holds for all $x, y, X, Y \in [0,\infty)$ and $\alpha, \beta \in (0,\infty)$. (In case of doubts set $\lambda = \frac{1}{2}$.) Combining the obtained estimates gives (i).

If $1 < p$ and $q > p$, then we use Theorem 1.1 and [GOP06, Theorem 4.4] and proceed similarly as above.

In the remaining cases where $p \leq 1$ we use Corollary 3.10, [GOP06, Theorems 4.1, 4.4] and proceed analogously again. □

Proof of Theorem 2.2. This proof is analogous to that of Theorem 2.1. We use Theorem 1.1 and Corollary 3.10 and the characterizations concerning inequalities for positive functions which are found in [Kře17a, Theorems 6 and 7]. Details are omitted. □

References

ACS17 [ACS17] Angela Alberico, Andrea Cianchi, and Carlo Sbordone. Continuity properties of solutions to the p-Laplace system. *Adv. Calc. Var.*, 10(1):1–24, 2017.

CGMP2 [CGMP08] María Carro, Amiran Gogatishvili, Joaquim Martín, and Luboš Pick. Weighted inequalities involving two Hardy operators with applications to embeddings of function spaces. *J. Operator Theory*, 59(2):309–332, 2008.

CM19 [CM19] Paola Cavaliere and Zdeněk Mihula. Compactness for Sobolev-type trace operators. *Nonlinear Anal.*, 183:42–69, 2019.

GE98 [GE98] Karl-Goswin Grosse-Erdmann. *The blocking technique, weighted mean operators and Hardy's inequality*, volume 1679 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1998.

GKPS [GKPS17] Amiran Gogatishvili, Martin Křepela, Luboš Pick, and Filip Soudský. Embeddings of Lorentz-type spaces involving weighted integral means. *J. Funct. Anal.*, 273(9):2939–2980, 2017.

GM2 [GM17a] Amiran Gogatishvili and Rza Ch. Mustafayev. Iterated Hardy-type inequalities involving suprema. *Math. Inequal. Appl.*, 20(4):901–927, 2017.

GM1 [GM17b] Amiran Gogatishvili and Rza Ch. Mustafayev. Weighted iterated Hardy-type inequalities. *Math. Inequal. Appl.*, 20(3):683–728, 2017.

opic2006weighted [GOP06] Amiran Gogatishvili, Bohumír Opic, and Luboš Pick. Weighted inequalities for Hardy-type operators involving suprema. *Collect. Math.*, 57(3):227–255, 2006.

GP-discr [GP03] Amiran Gogatishvili and Luboš Pick. Discretization and anti-discretization of rearrangement-invariant norms. *Publ. Mat.*, 47(2):311–358, 2003.

ishvili2007reduction [GP07] Amiran Gogatishvili and Luboš Pick. A reduction theorem for supremum operators. *J. Comput. Appl. Math.*, 208(1):270–279, 2007.

Kre3 [Kře17a] Martin Křepela. Integral conditions for Hardy-type operators involving suprema. *Collect. Math.*, 68(1):21–50, 2017.

Kre1 [Kře17b] Martin Křepela. Iterating bilinear Hardy inequalities. *Proc. Edinb. Math. Soc. (2)*, 60(4):955–971, 2017.

vej17 [Mus16] Vít Musil. Optimal Orlicz domains in Sobolev embeddings into Marcinkiewicz spaces. *J. Funct. Anal.*, 270(7):2653–2690, 2016.

vej19 [Mus19] Vít Musil. Fractional maximal operator in Orlicz spaces. *J. Math. Anal. Appl.*, 474(1):94–115, 2019.

Sinn94 [Sin94] Gord Sinnamon. Spaces defined by the level function and their duals. *Studia Math.*, 111(1):19–52, 1994.

Sinn03 [Sin03] Gord Sinnamon. Transferring monotonicity in weighted norm inequalities. *Collect. Math.*, 54(2):181–216, 2003.

len15 [Sla15] Lenka Slavíková. Compactness of higher-order Sobolev embeddings. *Publ. Mat.*, 59(2):373–448, 2015.

Si-St96 [SS96] Gord Sinnamon and Vladimir D. Stepanov. The weighted Hardy inequality: new proofs and the case $p = 1$. *J. London Math. Soc. (2)*, 54(1):89–101, 1996.

E-mail address, A. Gogatishvili: gogatish@math.cas.cz
ORCiD: 0000-0003-3459-0355
E-mail address, M. Křepela: martin.krepela@math.uni-freiburg.de
ORCiD: 0000-0003-0234-1645
E-mail address, R. Oľhava: olhava@karlin.mff.cuni.cz
ORCiD: 0000-0002-9930-0454
E-mail address, L. Pick: pick@karlin.mff.cuni.cz
ORCiD: 0000-0002-3584-1454

[1]Institute of Mathematics, Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic

[2]Department of Applied Mathematics, University of Freiburg, Ernst-Zermelo-Strasse 1, 791 04 Freiburg, Germany

[3]Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic

[4]Institute of Applied Mathematics and Information Technologies, Faculty of Science, Charles University, Albertov 6, 128 43 Praha 2, Czech Republic

[5]Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic