INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

Some new results related to Lorentz $G\Gamma$–spaces and interpolation

Alberto Fiorenza
Maria Rosaria Formica
Amiran Gogatishvili
Jean Michel Rakotoson

Preprint No. 11-2019

PRAHA 2019

SOME NEW RESULTS RELATED TO LORENTZ $G\Gamma$-SPACES AND INTERPOLATION

ALBERTO FIORENZA[1], **MARIA ROSARIA FORMICA**[2], **AMIRAN GOGATISHVILI**[3], and **JEAN MICHEL RAKOTOSON**[4*]

Abstract. We compute the K-functional related to some couple of spaces as small or classical Lebesgue space or Lorentz-Marcinkiewicz spaces completing the result of [11]. This computation allows to determine the interpolation space in the sense of Peetre for such couple. It happens that the result is always a $G\Gamma$-space, since this last space covers many spaces.
The motivations of such study are various, among them we wish to obtain a regularity estimate for the so called very weak solution of a linear equation in a domain Ω with data in the space of the integrable function with respect to the distance function to the boundary of Ω.

1. Introduction

The present work finds its motivation in the recent results in [10, 7, 18]. The original question comes from an unpublished manuscript by H. Brezis (see comments in [7]) and later presented in [5] (see also the mention made in [22]) concerning the following problem, Let f is given in $L^1(\Omega, \text{dist}\,(x, \partial\Omega))$ (Ω bounded smooth open set of $\mathbb{R}^N$), H. Brezis shows the existence and uniqueness of a function $v \in L^1(\Omega)$ satisfying

$$|v|_{L^1(\Omega)} \leqslant c|f|_{L^1(\Omega,\text{dist}\,(x,\partial\Omega))}$$

with

$$GD(\Omega) = \begin{cases} -\displaystyle\int_\Omega v\Delta\varphi\, dx = \int_\Omega f\varphi\, dx, \quad \forall\varphi \in V_1(\Omega), \\ \text{with } C_0^2(\overline{\Omega}) = \Big\{\varphi \in C^2(\overline{\Omega}),\ \varphi = 0 \text{ on } \partial\Omega\Big\}. \end{cases}$$

Therefore, the question of the integrability of the generalized derivative $v : \partial_i v = \dfrac{\partial v}{\partial x_i}$ arises in a natural way and was raised already in the note by H. Brezis and developed in [7], [18], [19]. More generally, the question of the regularity of u is arised, according to f.

Date: Received: xxxxxx; Revised: yyyyyy; Accepted: zzzzzz.

*Corresponding author.

2010 *Mathematics Subject Classification.* Primary 46E30,46B70, Secondary 35J65.

Key words and phrases. Grand and Small Lebesgue spaces, classical Lorentz-spaces, Interpolation, very weak solution.

In [9, 10], we have shown the following theorem:

Theorem 1.1.
Let Ω be a bounded open set of class C^2 of $\mathbb{R}^n$, $|\Omega| = 1$ and $\alpha \geqslant \dfrac{1}{n'}$ where $n' = \dfrac{n}{n-1}$, $f \in L^1(\Omega;\delta)$, with $\delta(x) = \operatorname{dist}(x;\partial\Omega)$.

Consider $u \in L^{n',\infty}(\Omega)$, the very weak solution (v.w.s.) of

$$-\int_\Omega u\Delta\varphi dx = \int_\Omega f\varphi dx \qquad \forall\, \varphi \in C^2(\overline{\Omega}),\ \varphi = 0 \text{ on } \partial\Omega. \tag{1.1}$$

Then,

(1) *if $f \in L^1\Big(\Omega;\delta(1+|\operatorname{Log}\delta|)^\alpha\Big)$ and $\alpha > \dfrac{1}{n'}$:*

$$u \in L^{(n',n\alpha-n+1}(\Omega) = G\Gamma(n',1;w_\alpha),\ \ w_\alpha(t) = t^{-1}(1-\operatorname{Log} t)^{\alpha-1-\frac{1}{n'}}$$

and

$$||u||_{G\Gamma(n',1;w_\alpha)} \leqslant K_0|f|_{L^1(\Omega;\delta(1+|\operatorname{Log}\delta|)^\alpha)} \tag{1.2}$$

(2) *if $f \in L^1\Big(\Omega;\delta\big(1+|\operatorname{Log}\delta|\big)^{\frac{1}{n'}}\Big)$ then*

$u \in L^{n'}(\Omega)$ and similar estimate as (1.2) holds.

Note that the assumption on the regularity of Ω needed in the proof of Theorem 1.1 is necessary for the development of the theory of very weak solutions; we stress that the estimates in this paper will be obtained following arguments valid regardless of the regularity of Ω, which will be definitively dropped in our statements.

The Lorentz $G\Gamma$-space is defined as follows :

Definition 1.2. of Generalized Gamma space with double weights (Lorentz-$G\Gamma$)
Let $w_1,\ w_2$ be two weights on $(0,1)$, $m \in [1,+\infty]$, $1 \leqslant p < +\infty$. We assume the following conditions:

- c1) *There exists $K_{12} > 0$ such that $w_2(2t) \leqslant K_{12}w_2(t)$ $\forall\, t \in (0,1/2)$. The space $L^p(0,1;w_2)$ is continuously embedded in $L^1(0,1)$.*
- c2) *The function $\displaystyle\int_0^t w_2(\sigma)d\sigma$ belongs to $L^{\frac{m}{p}}(0,1;w_1)$.*

A generalized Gamma space with double weights is the set :

$$G\Gamma(p,m;w_1,w_2) = \left\{ v : \Omega \to \mathbb{R} \text{ measurable } \int_0^t v_*^p(\sigma)w_2(\sigma)d\sigma \text{ is in } L^{\frac{m}{p}}(0,1;w_1) \right\}.$$

A similar definition has been considered in [13]. They were interested in the embeddings between $G\Gamma$-spaces.

Property 1.3.
Let $G\Gamma(p,m;w_1,w_2)$ be a Generalized Gamma space with double weights and let us define for $v \in G\Gamma(p,m;w_1,w_2)$

$$\rho(v) = \left[\int_0^1 w_1(t)\left(\int_0^t v_*^p(\sigma)w_2(\sigma)d\sigma\right)^{\frac{m}{p}} dt\right]^{\frac{1}{m}}$$

with the obvious change for $m = +\infty$.
Then,

1. *ρ is a quasinorm.*
2. *$G\Gamma(p,m;w_1,w_2)$ endowed with ρ is a quasi-Banach function space.*
3. *If $w_2 = 1$*

$$G\Gamma(p,m;w_1,1) = G\Gamma(p,m;w_1).$$

Example 1.1. of weights
Let $w_1(t) = (1-\operatorname{Log} t)^\gamma, \quad w_2(t) = (1-\operatorname{Log} t)^\beta$ with $(\gamma,\beta) \in \mathbb{R}^2$. Then

$$w_2 \text{ satisfies condition c1) and } w_1 \text{ and } w_2 \text{ are in } L_{exp}^{\max(\gamma;\beta)}\big(]0,1[\big).$$

Question 1 The natural question is *how to extend of Theorem 1.1 for $\alpha < \dfrac{1}{n'}$ and how to improve the estimate when $\alpha = \dfrac{1}{n'}$?*

Since the solution of (1.1) satisfies also

$$|u|_{L^{n',\infty}(\Omega)} \leqslant K_1 |f|_{L^1(\Omega;\delta)}, \tag{1.3}$$

the natural idea to obtain an estimate is to use the real interpolation method of Marcinkiewicz (see [3, 4, 6]) to derive

$$|u|_{(L^{n',\infty},L^{(n')})_{\alpha,1}} \leqslant K_2 |f|_{L^1(\Omega;\delta(1+|\operatorname{Log}\delta|)^\alpha)} \qquad \text{for } 0<\alpha\leqslant 1. \tag{1.4}$$

Note that $L^{(n',1} = L^{(n'}$ (see below for a full definition.)

Question 2 *How to characterize the space $\Big(L^{n',\infty}(\Omega), L^{(n'}(\Omega)\Big)_{\alpha,1}$?*

Here, we give a complete answer to those questions when $n = 2$ (see section 4.). In the general case, a particular answer from our work made in [11] leads us to : Since $L^{n',\infty}(\Omega) \subset L^{n')}$, then we have

$$\Big(L^{n',\infty}(\Omega), L^{(n'}(\Omega)\Big)_{\alpha,1} \subset \Big(L^{n')}(\Omega), L^{(n'}(\Omega)\Big)_{\alpha,1}$$

and we have shown in [11] the following

Theorem 1.4. (characterization of the interpolation between Grand and Small Lebesgue space)

$$\left(L^{n')}(\Omega), L^{(n'}(\Omega)\right)_{\alpha,1} = G\Gamma(n';1;w_1;w_2) \text{ with } w_1(t) = \frac{(1-\operatorname{Log} t)^{\alpha-1}}{t},\ w_2(t) = \frac{1}{1-\operatorname{Log} t}.$$

(see next section for the definition of $G\Gamma$).

Therefore, we have the following **non optimal result** but valid for all α.

Proposition 1.5.
Let u be the solution of (1.1). Then,

$$||u||_{G\Gamma(n';1;w_1;w_2)} = \int_0^1 (1-\operatorname{Log} t)^\alpha \left(\int_0^t \frac{u_*^{n'}(x)dx}{1-\operatorname{Log} x}\right)^{\frac{1}{n'}} \frac{dt}{(1-\operatorname{Log} t)t} \leqslant K_4 |f|_{L^1(\Omega;\delta(1+|\operatorname{Log}\delta|)^\alpha)}$$

whenever $0<\alpha<1$.

Here, we shall introduce different results on the following interpolation spaces between $(L^{n'}, L^{(n'})_{\theta,r}$ and $(L^{n',\infty}, L^{n'})_{\theta,r}$.

Theorem 1.6.
For $0<\theta<1,\ r\in[1,+\infty[$

$$(L^{n'}, L^{(n'})_{\theta,r} = G(n',r;w_1;1)$$

$w_1(t) = t^{-1}(1-\operatorname{Log} t)^{r\frac{\theta}{n}-1}$.

Corollary 1.7. of Theorem 1.6
For $0<\theta<1$, *one has*

$$(L^{n'}, L^{(n'})_{\theta,1} = L^{(n',\theta}.$$

As in [11, 9], the proofs of the above results rely on the computation of the K-functional. In particular, we will show the following

Theorem 1.8.
The K-functional for $(L^{n',\infty}, L^{n'})$ is given by, for $t\in]0,1[$, $f\geqslant 0$ in $L^{n',\infty}$

$$K_0(f;t) = t\sup\left\{\left(\int_E f_*^{n'}(\sigma)d\sigma\right)^{\frac{1}{n'}};\quad t^{-n'} = \int_E \frac{dx}{x}\right\}$$

Remark 1.9.
Setting $d\nu = \dfrac{dx}{x}$, $|E|_\nu = \displaystyle\int_E d\nu$, $f_{*,\nu}$ *the decreasing rearrangement of a nonnegative function* f *with respect to the measure* ν, *then we can write the preceding theorem as :*

Theorem 1.10.
The K*-functional for the couple* $(L^{p,\infty}, L^p)$ *is given, for* $f \geqslant 0$ *in* $L^p + L^{p,\infty}$, $t > 0$

$$K_0(f;t) = t\left(\int_0^{t^{-p}} \Big(\psi(s)\Big)^p_{*,\nu}(x)dx\right)^{\frac{1}{p}}.$$

Here $1 \leqslant p < +\infty$, $\psi(s) = s^{\frac{1}{p}} f_*(s)$, $s \in (0,1)$.

From this result, we can recover the following result due to Maligranda and Persson (see [17] :

Theorem 1.11.
Let $0 < \theta < 1$, $1 < p < +\infty$. *Then*

$$(L^{p,\infty}, L^p)_{\theta,\frac{p}{\theta}} = L^{p,\frac{p}{\theta}}.$$

Here $L^{p,\frac{p}{\theta}}$ *is the usual Lorentz space.*

Applying Theorem 1.8 with real interpolation method of Marcinkiewicz, we then deduce the following partial answer for very weak solution :

Proposition 1.12.
For $0 < \alpha \leqslant 1$, *let* u *be the solution of (1.1). Then one has a constant* $c > 0$ *such that*

$$\int_0^1 t^{-\alpha} \sup_{\{E:|E|_\nu = t^{-n'}\}} \left(\int_E u_*^{n'}(x)dx\right)^{\frac{1}{n'}} dt \leqslant c|f|_{L^1(\Omega;\delta(1+|\mathrm{Log}\,\delta|)^{\frac{\alpha}{n'}})}.$$

Other consequences of the above interpolation results are the interpolation inequalities, we state few of them.

Property 1.13. (Interpolation inequalities for small and grand Lebesgue spaces)

(1) *Let* $1 \geqslant \alpha > \dfrac{1}{n'}$ *then* $\forall\, v \in L^{(n'}$

$$||v||_{L^{n',\infty}} \leqslant c||v||^{1-\alpha}_{L^{n')}} ||v||^{\alpha}_{L^{(n'}}.$$

(2) *For any* $\alpha \in]0,1[$, *one has*

$$||v||_{(L^{n',\infty}, L^{(n'})_{\alpha,1}} \leqslant c||v||^{1-\alpha}_{L^{n'}} ||v||^{\alpha}_{L^{(n'}} \qquad \forall\, v \in L^{(n'}.$$

2. Notation and Primary results

For a measurable function $f : \Omega \to \mathbb{R}$, we set for $t \geqslant 0$

$$D_f(t) = \text{measure}\left\{x \in \Omega : |f(x)| > t\right\}$$

and f_* the decreasing rearrangement of $|f|$,

$$f_*(s) = \inf\left\{t : D_f(t) \leqslant s\right\} \text{ with } s \in \big(0, |\Omega|\big),\ |\Omega| \text{ is the measure of } \Omega,$$

that we shall assume to be equal to 1 for simplicity.

If A_1 and A_2 are two quantities depending on some parameters, we shall write

$A_1 \lesssim A_2$ if there exists $c > 0$ (independent of the parameters) such that $A_1 \leqslant cA_2$

$$A_1 \simeq A_2 \text{ if and only if } A_1 \lesssim A_2 \text{ and } A_2 \lesssim A_1.$$

We recall also the following definition of interpolation spaces.
Let $(X_0, ||\cdot||_0),\ (X_1, ||\cdot||_1)$ two Banach spaces contained continuously in a Hausdorff topological vector space (that is (X_0, X_1) is a compatible couple).
For $g \in X_0+X_1,\ t > 0$ one defines the so called K functional $K(g, t; X_0, X_1) \doteq K(g, t)$ by setting

$$K(g,t) = \inf_{g=g_0+g_1} \big(||g_0||_0 + t||g_1||_1\big). \tag{2.1}$$

For $0 \leqslant \theta \leqslant 1,\ 1 \leqslant p \leqslant +\infty,\ \alpha \in \mathbb{R}$ we shall consider

$$(X_0, X_1)_{\theta,p;\alpha} = \left\{g \in X_0+X_1,\ ||g||_{\theta,p;\alpha} = ||t^{-\theta-\frac{1}{p}}\big(1-\operatorname{Log} t\big)^{\alpha} K(g,t)||_{L^p(0,1)} \text{ is finite}\right\}.$$

Here $||\cdot||_V$ denotes the norm in a Banach space V. The weighted Lebesgue space $L^p(0, 1; \omega)$, $0 < p \leqslant +\infty$ is endowed with the usual norm or quasi norm, where ω is a weight function on $(0, 1)$.
Our definition of the interpolation space is different from the usual one (see [3, 21]) since we restrict the norms on the interval $(0, 1)$.
If we consider ordered couple, i.e. $X_1 \hookrightarrow X_0$ and $\alpha = 0$,

$$(X_0, X_1)_{\theta,p;0} = (X_0, X_1)_{\theta,p}$$

is the interpolation space as it is defined by J. Peetre (see [3, 21, 4]).

2.1. **Some remarkable $G\Gamma$-spaces.** In this paragraph, we want to prove among other that $G\Gamma$-spaces cover many well-known spaces.

Proposition 2.1.
Consider the classical Lorentz space $\Lambda^p(w_2)$. Then it is equal to the set $\Big\{f:\Omega\to\mathbb{R}\ \textit{measurable}:\ \left(\int_0^1 f_*^p(\sigma)w_2(\sigma)d\sigma\right)^{\frac{1}{p}}=||f||_{\Lambda^p(w_2)}<+\infty\Big\}.$
If w_1 and w_2 are integrable and w_2 satisfies c1) then

$$G\Gamma(p,m;w_1,w_2)=\Lambda^p(w_2).$$

Proof

If $v\in\Lambda^p(w_2)$ then $\rho(v)\leqslant||v||_{\Lambda^p(w_2)}\left[\int_0^1 w_1(t)dt\right]^{\frac{1}{m}}<+\infty.$

Conversely, let v be such that $\rho(v)<+\infty$. We have for some $a>0$, $\int_a^1 w_1(t)dt>0$. Then for all $t\geqslant a$

$$\left(\int_0^a f_*^p(\sigma)w_2(\sigma)d\sigma\right)^{\frac{m}{p}}\leqslant\left(\int_0^t f_*^p(\sigma)w_2(\sigma)d\sigma\right)^{\frac{m}{p}},$$

from which we derive after multiplying by $w_1(t)$ and integrating from a to 1,

$$\left(\int_0^a f_*^p(\sigma)w_2(\sigma)d\sigma\right)^{\frac{1}{p}}\leqslant\frac{\rho(v)}{\left[\int_a^1 w_1(t)dt\right]^{\frac{1}{m}}}\lesssim\rho(v)<+\infty. \tag{2.2}$$

Between $(a,1)$, we have :

$$\left(\int_a^1 f_*^p(\sigma)w_2(\sigma)d\sigma\right)^{\frac{1}{p}}\leqslant f_*(a)||w_2||_{L^1}^{\frac{1}{p}}\lesssim\int_0^a f_*(\sigma)d\sigma=\int_0^1 f_*(\sigma)\chi_{[0,a]}(\sigma)d\sigma. \tag{2.3}$$

The condition c1) implies

$$\int_0^a f_*(\sigma)d\sigma\lesssim\left(\int_0^1(f_*\chi_{[0,a]})^p(\sigma)w_2(\sigma)d\sigma\right)^{\frac{1}{p}}. \tag{2.4}$$

So that relations (2.2) to (2.4) imply

$$\left(\int_a^1 f_*^p(\sigma)w_2(\sigma)d\sigma\right)^{\frac{1}{p}}\lesssim\left(\int_0^a f_*^p(\sigma)w_2(\sigma)d\sigma\right)^{\frac{1}{p}}\lesssim\rho(v)<+\infty. \tag{2.5}$$

This shows

$$||f||_{\Lambda^p(w_2)}\lesssim\rho(v).$$

◊

Next we want to focus in a special case :

Proposition 2.2.
Assume that $w_1(t) = t^{-1}(1-\operatorname{Log} t)^{\gamma}$, $w_2(t) = (1-\operatorname{Log} t)^{\beta}$, $(\gamma,\beta) \in \mathbb{R}^2$, $m \in [1,+\infty[$, $p \in [1,+\infty[$.

(1) *If* $\gamma < -1$ *then* $G\Gamma(p,m;w_1,w_2) = \Lambda^p(w_2)$.
(2) *If* $\gamma > -1$, *and* $\gamma + \beta\dfrac{m}{p} + 1 < 0$ *then*

$$G\Gamma(p,m;w_1,w_2) = L^p.$$

(3) *If* $\gamma > -1$ *and* $\gamma + \beta\dfrac{m}{p} + 1 \geqslant 0$ *then*

$$G\Gamma(p,m;w_1,w_2) = G\Gamma(p,m;\overline{w_1},1),\ \overline{w_1}(t) = t^{-1}(1-\operatorname{Log} t)^{\gamma+\beta\frac{m}{p}}.$$

Proof
For the first statement, we observe that if $\gamma + 1 < 0$, $\displaystyle\int_0^1 (1-\operatorname{Log} t)^{\gamma}\frac{dt}{t}$ is finite. Then applying Proposition 2.1 we derive the first result.
For the case $\gamma + 1 > 0$, we shall need the following lemma whose proof is in [11]:

Lemma 2.3.
Let $t_k = 2^{1-2^k}$, $k \in \mathbb{N}$, $\lambda > 0$, $q > 0$, H *a nonnegative locally integrable function on* $(0,1)$ *satisfying*

$$\int_0^1 H(x)dx \lesssim \int_0^{\frac{1}{2}} H(x)dx.$$

Then

(1) $2^k \approx 1 - \operatorname{Log} x$, $x \in [t_{k+1}, t_k]$.
(2)

$$\begin{aligned}\int_0^1 \left[(1-\operatorname{Log} t)^{\lambda}\int_0^t H(x)dx\right]^q \frac{dt}{(1-\operatorname{Log} t)t} \ &\approx\ \sum_{k\in\mathbb{N}}\left(\int_0^{t_k} H(x)dx\right)^q 2^{\lambda k q}\\ &\approx\ \sum_{k\in\mathbb{N}}\left(2^{\lambda k}\int_{t_{k+1}}^{t_k} H(x)dx\right)^q.\end{aligned}$$

We shall apply this Lemma with $H(x) = f_*^p(x)(1-\operatorname{Log} x)^\beta$. We have $\int_0^1 H(x)dx \lesssim \int_0^{\frac{1}{2}} H(x)dx$ since f_*^p is decreasing and $\int_0^1 (1-\operatorname{Log} t)^\gamma dt < +\infty \quad \forall\, \gamma \in \mathbb{R}$. Indeed

$$\begin{aligned}
\left(\int_{\frac{1}{2}}^1 H(x)dx\right)^{\frac{1}{p}} &\lesssim f_*\left(\frac{1}{2}\right) \lesssim \int_0^{\frac{1}{2}} f_*(t)dt \\
&\leqslant \left(\int_0^{\frac{1}{2}} f_*^p(t)(1-\operatorname{Log} t)^\beta dt\right)^{\frac{1}{p}} \cdot \left(\int_0^{\frac{1}{2}} (1-\operatorname{Log} t)^{-\beta\frac{p'}{p}} dt\right)^{\frac{1}{p'}} \\
&\lesssim \left(\int_0^{\frac{1}{2}} H(x)dx\right)^{\frac{1}{p}}.
\end{aligned}$$

Applying statement 2. of this Lemma 2.3, we derive

$$\begin{aligned}
\rho^m(f) &= \int_0^1 (1-\operatorname{Log} t)^\gamma \left(\int_0^t H(x)dx\right)^{\frac{m}{p}} \frac{dt}{t} \\
&= \int_0^1 \left[(1-\operatorname{Log} t)^{(\gamma+1)\frac{p}{m}} \int_0^t H(x)dx\right]^{\frac{m}{p}} \frac{dt}{(1-\operatorname{Log} t)t} \qquad if\ \gamma+1>0 \\
&\approx \sum_{k\in\mathbb{N}} \left(2^{\lambda k} \int_{t_{k+1}}^{t_k} H(x)dx\right)^q \text{ with } \lambda = (\gamma+1)\frac{p}{m},\ q = \frac{m}{p}
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
&\approx \sum_{k\in\mathbb{N}} \left(2^{(\lambda+\beta)k} \int_{t_{k+1}}^{t_k} f_*^p(x)dx\right)^q \\
&\approx \int_0^1 \left[(1-\operatorname{Log} t)^{\lambda+\beta} \int_0^t f_*^p(x)dx\right]^q \frac{dt}{(1-\operatorname{Log} t)t} \\
&\approx \int_0^1 (1-\operatorname{Log} t)^{(\lambda+\beta)q-1} \left(\int_0^t f_*^p(x)dx\right)^q \frac{dt}{t}.
\end{aligned} \tag{2.7}$$

Lemma 2.4.
If $(\lambda+\beta)q = \gamma+1+\beta\frac{m}{p} < 0$ *then*

$$R(f) = \int_0^1 (1-\operatorname{Log} t)^{(\lambda+\beta)q-1} \left(\int_0^t f_*^p(x)dx\right)^q \frac{dt}{t} \approx \left(\int_0^1 f_*^p(x)dx\right)^q.$$

Proof of Lemma 2.4
Since $K_0 = \int_0^1 (1-\operatorname{Log} t)^{(\lambda+\beta)q-1}\frac{dt}{t} < +\infty$ if $(\lambda+\beta)q<0$, then

$$R(f) \leqslant \left(\int_0^1 f_*^p(x)dx\right)^q K_0. \tag{2.8}$$

Let $a>0$. Then

$$\left(\int_0^a f_*^p(x)dx\right)^q \leqslant \frac{R(f)}{\displaystyle\int_a^1 (1-\operatorname{Log} t)^{(\lambda+\beta)q-1}\frac{dt}{t}} \lesssim R(f), \tag{2.9}$$

and by the monotonicity of f_* :

$$\left(\int_a^1 f_*^p(x)dx\right)^q \lesssim \left(\int_0^a f_*^p(x)dx\right)^q \lesssim R(f). \tag{2.10}$$

From (2.9) and (2.10), we derive the result. This ends the proof of Lemma 2.4♢

With the help of Lemma 2.4, we have shown that if $\gamma>-1$ and $\gamma+\beta\dfrac{m}{p}+1<0$ then

$$G\Gamma(p,m;w_1,w_2)=L^p$$

If $\gamma>-1$, $\gamma+\beta\dfrac{m}{p}+1\geqslant 0$, then the equality comes from the definition of $G\Gamma(p,m;\overline{w_1})$.

This ends of the proof of Proposition 2.2 ♢

We shall need in particular the following Corollary, consequence of relation (2.7) and the following

Definition 2.5. of the small Lebesgue space [16, 8]

The small Lebesgue space associated to the parameter $p\in]1,+\infty[$ *and* $\theta>0$ *is the set*

$L^{(p,\theta}(\Omega)=$

$$\left\{f:\Omega\to\mathbb{R}\ \textit{measurable}:\ ||f||_{(p,\theta}=\int_0^1(1-\operatorname{Log} t)^{-\frac{\theta}{p}+\theta-1}\left(\int_0^t f_*^p(\sigma)d\sigma\right)^{1/p}\frac{dt}{t}<+\infty\right\}.$$

Corollary 2.6. of Proposition 2.2

If $m=1,\gamma+1+\dfrac{\beta}{p}>0$, $\gamma>-1$ *and* $\beta\in\mathbb{R}$ *then*

$$G\Gamma(p,1;w_1,w_2)=L^{(p,\theta},\ \theta=p'\left(\gamma+1+\frac{\beta}{p}\right).$$

3. K-functional computation

3.1. The case of the couple $(L^{n'},L^{(n'})$.

Theorem 3.1.
Let $\varphi(t) = e^{1-\frac{1}{t^n}},\ 0 < t \leqslant 1$. *Then*

$$K(f,t;L^{n'},L^{(n'}) \approx t\int_{\varphi(t)}^{1}(1-\operatorname{Log}\sigma)^{-\frac{1}{n'}}\left(\int_0^\sigma f_*^{n'}(x)dx\right)^{\frac{1}{n'}}\frac{d\sigma}{\sigma}\doteq K^2(t)$$

for all $f \in L^{n'} + L^{(n'}$.

Proof:
First , let us show:

$$K^2(t) \lesssim K(f,t;L^{n'},L^{(n'}).$$

Let $f = g + h \in L^{n'} + L^{(n'}$. Then, for all x, $f_*(x) \leqslant g_*\left(\dfrac{x}{2}\right) + h_*\left(\dfrac{x}{2}\right)$. Therefore, we have

$$\begin{aligned} K^2(t) &\leqslant ||g||_{L^{n'}}t\int_{\varphi(t)}^{1}(1-\operatorname{Log}\sigma)^{-\frac{1}{n'}}\frac{d\sigma}{\sigma} + t||h||_{L^{(n'}} \\ &\lesssim ||g||_{L^{n'}} + t||h||_{L^{(n'}} \end{aligned}$$

Taking the infinimum, one derives

$$K^2(t) \lesssim K(f,t;L^{n'},L^{(n'}). \tag{3.1}$$

For the converse, we adopt the same decomposition as in [11]

$$g = \big(|f| - f_*\big(\varphi(t)\big)\big)_+, \quad h = f - g$$

Then

$$f_* = h_* + g_*, \qquad g_* = \Big(f_* - f_*\big(\varphi(t)\big)\Big)_+,$$

$$h_* = f_*\big(\varphi(t)\big)\chi_{[0,\varphi(t)]} + f_*(s)\chi_{[\varphi(t),1]}$$

$$\begin{aligned} ||g||_{L^{n'}} &\lesssim \left[\int_0^{\frac{\varphi(t)}{2}}\Big(f_*(s) - f_*\big(\varphi(t)\big)_+^{n'}ds\right]^{\frac{1}{n'}} \lesssim \left[\int_0^{\frac{\varphi(t)}{2}} f_*^{n'}(s)ds\right]^{\frac{1}{n'}} \\ &= ||f||_{L^{n'}(0,\frac{\varphi(t)}{2})} \lesssim \frac{K^2(t)}{\displaystyle\int_{\varphi(t)/2}^{1}(1-\operatorname{Log}\sigma)^{-\frac{1}{n'}}\frac{d\sigma}{\sigma}} \lesssim K^2(t). \end{aligned} \tag{3.2}$$

As in [11], we have

$$\begin{aligned} t||h||_{L^{(n'}} &\leqslant t\left(\int_0^{\varphi(t)}(1-\operatorname{Log} s)^{-1/n'}s^{1/n'}\frac{ds}{s}\right)f_*\big(\varphi(t)\big) \\ &+t\left(\int_{\varphi(t)}^1(1-\operatorname{Log} s)^{-1/n'}\frac{ds}{s}\right)\varphi(t)^{1/n'}f_*\big(\varphi(t)\big) \\ &+t\int_{\varphi(t)}^1(1-\operatorname{Log} s)^{-1/n'}\left(\int_{\varphi(t)}^s f_*^{n'}(x)dx\right)^{1/n'}\frac{ds}{s} \qquad (3.3)\\ &= I_1+I_2+I_3. \end{aligned}$$

Since

$$\int_0^{\varphi(t)} s^{1/n'}(1-\operatorname{Log} s)^{-1/n'}\frac{ds}{s}\lesssim \varphi(t)^{1/n'}\Big(1-\operatorname{Log}\varphi(t)\Big)^{-1/n'},$$

we obtain for the first term I_1

$$I_1 \lesssim t\Big(1-\operatorname{Log}\varphi(t)\Big)^{-1/n'}\varphi(t)^{1/n'}f_*\big(\varphi(t)\big)\lesssim K^2(t) \qquad (3.4)$$

$$I_2\lesssim t^{1/n'}\varphi(t)^{1/n'}f_*\Big(\varphi(t)\Big)\lesssim t^{1/n'}\sup_{0<s<\varphi(t)} s^{1/n'}f_*(s)\lesssim K^2(t) \qquad (3.5)$$

and

$$I_3\lesssim K^2(t) \qquad (3.6)$$

with relations (3.4) to (3.6), we derive

$$t||h||_{L^{(n'}}\leqslant I_1+I_2+I_3\lesssim K^2(t). \qquad (3.7)$$

Thus relations (3.2) and (3.7) infer :

$$||g||_{L^{n'}}+t||h||_{L^{(n'}}\lesssim K^2(t). \qquad (3.8)$$

Thus

$$K(f,t;L^{n'},L^{(n')})\lesssim K^2(t). \qquad (3.9)$$

The combination of the above relations (3.9), (3.1) gives Theorem 3.1. ♢

Corollary 3.2. Theorem 3.1

One has, for $r\in[1,+\infty[,\ \ 0<\theta<1$,

$$||f||^r_{(L^{n'},L^{(n')})_{\theta,r}}\approx\int_0^1(1-\operatorname{Log} x)^{\frac{\theta r}{n}}\left(\int_0^x f_*^{n'}(s)ds\right)^{\frac{r}{n'}}\frac{dx}{x(1-\operatorname{Log} x)}.$$

Proof:

One has for $f\in L^{n'}+L^{(n'},\ 1\leqslant r+\infty$

$$||f||^r_{(L^{n'},L^{(n')})_{\theta,r}}=\int_0^1[t^{-\theta}K(f,t)]^r\frac{dt}{t} \qquad (3.10)$$

Using Theorem 3.1 and making a change of variable $x=\varphi(t)$ that is $t=(1-\operatorname{Log}x)^{-\frac{1}{n}}$, one derives from relation (3.10)

$$||f||^r_{(L^{n'},L^{(n')})_{\theta,r}}\approx J_f$$

$$J_f=\int_0^1\left[(1-\operatorname{Log}x)^{\frac{\theta-1}{n}}\int_x^1(1-\operatorname{Log}\sigma)^{-\frac{1}{n'}}\left(\int_0^\sigma f_*^{n'}(x)dx\right)^{\frac{1}{n'}}\frac{d\sigma}{\sigma}\right]^r\frac{dx}{x(1-\operatorname{Log}x)}.$$

Applying Hardy's inequality (taking into account that $\theta<1$), we have

$$J_f\lesssim\int_0^1\left[(1-\operatorname{Log}x)^{\frac{\theta}{n}}\left(\int_0^x f_*^{n'}(\sigma)d\sigma\right)^{\frac{1}{n'}}\right]^r\frac{dx}{x(1-\operatorname{Log}x)}=\widetilde{J}_f.$$

For the converse, since we have for all $x>0$

$$\int_x^1(1-\operatorname{Log}\sigma)^{-\frac{1}{n'}}\left(\int_0^\sigma f_*^{n'}(s)ds\right)^{-\frac{1}{n'}}\frac{d\sigma}{\sigma}\geqslant\left(\int_0^x f_*^{n'}\right)^{\frac{1}{n'}}(1-\operatorname{Log}x)^{-\frac{1}{n'}}|\operatorname{Log}x|,$$

we then have

$$J_f\geqslant\int_0^1\left[(1-\operatorname{Log}x)^{\frac{\theta}{n}-1-\frac{1}{r}}|\operatorname{Log}x|\left(\int_0^x f_*^{n'}(s)ds\right)^{\frac{1}{n'}}\right]^r\frac{dx}{x}. \tag{3.11}$$

From this relation we deduce

$$\widetilde{J}_f\lesssim J_f+\int_0^1\left[(1-\operatorname{Log}x)^{\frac{\theta}{n}-1-\frac{1}{r}}\left(\int_0^x f_*^{n'}(s)ds\right)^{\frac{1}{n'}}\right]^r\frac{dx}{x}\doteq J_f+I_r, \tag{3.12}$$

while to estimate the last integral, one has

$$I_r\leqslant||f||^r_{L^{n'}}\int_0^1(1-\operatorname{Log}x)^{(\frac{\theta}{n})r-1}\frac{dx}{x}\leqslant c||f||^r_{L^{n'}}.$$

Since $(L^{n'},L^{(n')})_{\theta,r}$ is continuously embedded in $L^{n'}$, we then have

$$I_r\leqslant c||f||^r_{(L^{n'},L^{(n')})_{\theta,r}}. \tag{3.13}$$

Thus, we derive

$$\widetilde{J}_f\lesssim J_f+I_r\lesssim||f||^r_{(L^{n'},L^{(n')})_{\theta,r}}.$$

$\diamondsuit$

Proof of Theorem 1.6

We derive it from Corollary 3.2 of Theorem 3.1. $\diamondsuit$

4. K- functional of the couple $(L^{2,\infty}, L^{(2)})$

4.1. Computation of K- functional of the couple $(L^{2,\infty}, L^{(2)})$.

Theorem 4.1. *Let Ω an open bounded set of $\mathbb{R}^2$, $|\Omega| = 1$. Then for all $t \in (0,1)$, $f \in L^{2,\infty} + L^{(2}$, we have*

$$K(f,t;L^{2,\infty},L^{(2)}) \approx \sup_{0<s<\varphi(t)} s^{\frac{1}{2}} f_*(s) + t\int_{\varphi(t)}^1 (1-\operatorname{Log} s)^{-\frac{1}{2}} \left(\int_{\varphi(t)}^s f_*^2(x)\,dx\right)^{\frac{1}{2}} \frac{ds}{s} \tag{4.1}$$

where $\varphi(t) = e^{1-1/t^2}$.

We will use the following results.

Proposition 4.2. (see [11, Proposition 2.2])
Let $\beta \in \mathcal{R}$, $-\infty < \alpha < 1$. Then, there exists $c_{\alpha\beta} > 0$:

$$\int_0^a t^{-\alpha}(1-\operatorname{Log} t)^{\beta} dt \leqslant c_{\alpha\beta} a^{1-\alpha}(1-\operatorname{Log} a)^{\beta}, \qquad \forall\, a \in [\,0,1\,].$$

Lemma 4.3. see [2, Lemma 6.3]
Let $-\infty < \alpha < \infty$, $0 < \beta < \infty$. Then there is a number $N = N(\alpha,\beta) \geq 1$ such that on the interval $(0,1)$ we have

(a): $t^{-\beta}(N - \operatorname{Log} t)^{\alpha}$ *is a decreasing function of t;*
(b): $t^{\beta}(N - \operatorname{Log} t)^{\alpha}$ *is an increasing function of t.*

It sufficies to take $N = 1 + \frac{|\alpha|}{\beta}$.

Lemma 4.4. (see [1, Lemma 2.2])
Let A_0 and A_1 be two r.i. Banach function spaces over $[0,1]$ and let Φ_0, Φ_1 be their fundamental functions, satisfying the following conditions:

i): *there exists a constant C such that, for $i = 0;1$ and for all $t > 0$*

$$\int_0^t \frac{ds}{\Phi_i(s)} \leqslant \frac{Ct}{\Phi_i(t)}, \tag{C.0}$$

ii): *there exists a constant C such that, for all $t > 0$*

$$\frac{\Phi_1(t)}{\Phi_0(t)} \left\| \frac{\chi_{[0,t]}}{\Phi_1} \right\|_{A_0} \lambda \leqslant C, \tag{C.1}$$

$$\frac{\Phi_0(t)}{\Phi_1(t)} \left\| \frac{\chi_{[t,1]}}{\Phi_0} \right\|_{A_1} \leqslant C. \tag{C.2}$$

Then

$$K\left(f, \frac{\Phi_0(t)}{\Phi_1(t)}; A_0, A_1\right) \approx \left\| f_*\chi_{[0,t]} \right\|_{A_0} + \frac{\Phi_0(t)}{\Phi_1(t)} \left\| f_*\chi_{[t,1]} \right\|_{A_1}, \quad 0 < t < 1.$$

Proof of Theorem 4.1.

For the moment we consider the more general case $n > 1$.
Let $A_0 = L^{n',\infty}$ e $A_1 = L^{(n'}$. Then

$$\Phi_0(t) \approx t^{\frac{1}{n'}}, \qquad \Phi_1(t) \approx t^{\frac{1}{n'}}(1 - \operatorname{Log} t)^{\frac{1}{n}}.$$

We check if condition (C.0) holds.

$$\begin{aligned} \int_0^t \frac{ds}{\Phi_0(s)} &= \int_0^t s^{-\frac{1}{n'}}\, ds = \frac{n'}{n'-1}\left[s^{1-\frac{1}{n'}}\right]_0^t = \frac{n'}{n'-1}\frac{t}{t^{\frac{1}{n'}}} \\ &\Rightarrow \\ & \int_0^t \frac{ds}{\Phi_0(s)} \leqslant \frac{ct}{\Phi_0(t)} \end{aligned}$$

By Proposition 4.2

$$\begin{aligned} \int_0^t \frac{ds}{\Phi_1(s)} &= \int_0^t s^{-\frac{1}{n'}}(1-\operatorname{Log} s)^{-\frac{1}{n}}\, ds \\ &\leqslant ct^{1-\frac{1}{n'}}(1-\operatorname{Log} t)^{-\frac{1}{n}} \\ &= \frac{ct}{t^{\frac{1}{n'}}(1-\operatorname{Log} t)^{\frac{1}{n}}} \approx \frac{ct}{\Phi_1(t)}. \end{aligned}$$

We check if condition (C.1) holds. Since

$$\chi_{[0,t]}(s) = \begin{cases} 1, & \text{if } 0 < s < t < 1 \\ 0, & \text{if } 0 < t < s < 1 \end{cases}$$

we have

$$\begin{aligned} \frac{\Phi_1(t)}{\Phi_0(t)}\left\|\frac{\chi_{[0,t]}}{\Phi_1}\right\|_{A_0} &\approx (1-\operatorname{Log} t)^{\frac{1}{n}} \sup_{0<s<1} s^{\frac{1}{n'}}\left(\frac{\chi_{[0,t]}(s)}{s^{\frac{1}{n'}}(1-\operatorname{Log} s)^{\frac{1}{n}}}\right)_* \\ &\approx (1-\operatorname{Log} t)^{\frac{1}{n}} \sup_{0<s<t} s^{\frac{1}{n'}}(s^{-\frac{1}{n'}}(1-\operatorname{Log} s)^{-\frac{1}{n}})_* \end{aligned}$$

By Lemma 4.3, with $\alpha = -\frac{1}{n}$ and $\beta = \frac{1}{n'}$, there exists $N = N(n) \geqslant 1$ such that $s^{-\frac{1}{n'}}(N - \operatorname{Log} s)^{-\frac{1}{n}}$ is decreasing, $s \in (0,1)$. It suffices to choose

$$N = 1 + \frac{|-\frac{1}{n}|}{\frac{1}{n'}} = \frac{n}{n-1} = n'.$$

Since $1 - \operatorname{Log} s \approx N - \operatorname{Log} s,\ \ 0 < t \leqslant 1$ (see [2, p. 27)], we have

$$(1-\operatorname{Log} t)^{\frac{1}{n}} \sup_{0<s<t} s^{\frac{1}{n'}}(s^{-\frac{1}{n'}}(N-\operatorname{Log} s)^{-\frac{1}{n}})_* \approx$$

$$(1-\operatorname{Log} t)^{\frac{1}{n}} \sup_{0<s<t} s^{\frac{1}{n'}} s^{-\frac{1}{n'}}(N-\operatorname{Log} s)^{-\frac{1}{n}} \approx 1.$$

Therefore

$$\frac{\Phi_1(t)}{\Phi_0(t)}\left\|\frac{\chi_{[0,t]}}{\Phi_1}\right\|_{A_0}\leqslant C.$$

We check if condition (C.2) holds.

$$\frac{\Phi_0(t)}{\Phi_1(t)}\left\|\frac{\chi_{[t,1]}}{\Phi_0}\right\|_{A_1}\approx$$

$$(1-\operatorname{Log} t)^{-\frac{1}{n}}\int_0^1(1-\operatorname{Log} s)^{-\frac{1}{n'}}\left(\int_0^s\left(\frac{\chi_{[t,1](\sigma)}}{\sigma^{\frac{1}{n'}}(1-\operatorname{Log}\sigma)^{\frac{1}{n}}}\right)_*^{n'}d\sigma\right)^{\frac{1}{n'}}\frac{ds}{s}.$$

We observe that

- if $0<s<t<1 \quad\Rightarrow\quad \chi_{[t,1]}(\sigma)=0$ since $0<\sigma<s$,
- if $0<t<s<1 \quad\Rightarrow\quad \chi_{[t,1]}(\sigma)=\begin{cases}1 & \text{if } t<\sigma<s\\ 0 & \text{if } 0<\sigma<t\end{cases}.$

Therefore, arguing as above, from Lemma 4.3 we have

$$\begin{aligned}
\frac{\Phi_0(t)}{\Phi_1(t)}\left\|\frac{\chi_{[t,1]}}{\Phi_0}\right\|_{A_1} &\approx (1-\operatorname{Log} t)^{-\frac{1}{n}}\int_t^1(1-\operatorname{Log} s)^{-\frac{1}{n'}}\left(\int_t^s\left(\sigma^{-\frac{1}{n'}}(1-\operatorname{Log}\sigma)^{-\frac{1}{n}}\right)_*^{n'}d\sigma\right)^{\frac{1}{n'}}\frac{ds}{s}\\
&\approx (1-\operatorname{Log} t)^{-\frac{1}{n}}\int_t^1(1-\operatorname{Log} s)^{-\frac{1}{n'}}\left(\int_t^s\sigma^{-1}(1-\operatorname{Log}\sigma)^{-\frac{1}{n-1}}d\sigma\right)^{\frac{1}{n'}}\frac{ds}{s}\\
&\approx (1-\operatorname{Log} t)^{-\frac{1}{n}}\int_t^1(1-\operatorname{Log} s)^{-\frac{1}{n'}}\left[-(1-\operatorname{Log} s)^{\frac{n-2}{n-1}}+(1-\operatorname{Log} t)^{\frac{n-2}{n-1}}\right]^{\frac{1}{n'}}\frac{ds}{s}\\
&\approx (1-\operatorname{Log} t)^{-\frac{1}{n}}\int_t^1(1-\operatorname{Log} s)^{-\frac{1}{n'}}(1-\operatorname{Log} t)^{\frac{n-2}{n}}\frac{ds}{s}\\
&= (1-\operatorname{Log} t)^{\frac{n-3}{n}}\int_t^1(1-\operatorname{Log} s)^{-\frac{1}{n'}}\frac{ds}{s}\\
&= (1-\operatorname{Log} t)^{\frac{n-3}{n}}\left[\frac{-(1-\operatorname{Log} s)^{1-\frac{1}{n'}}}{1-\frac{1}{n'}}\right]_t^1\\
&= n(1-\operatorname{Log} t)^{\frac{n-3}{n}}[-1+(1-\operatorname{Log} t)^{\frac{1}{n}}]\\
&\approx (1-\operatorname{Log} t)^{\frac{n-2}{n}}
\end{aligned}$$

and the last quantity is bounded only for $n=2$.

By Lemma 4.4 we get

$$\begin{aligned}
K\left(f,\frac{\Phi_0(t)}{\Phi_1(t)};L^{2,\infty},L^{(2}\right) &\approx \left\|f_*\chi_{[0,t]}\right\|_{L^{2,\infty}}+\frac{\Phi_0(t)}{\Phi_1(t)}\left\|f_*\chi_{[t,1]}\right\|_{L^{(2}}\\
&\approx \sup_{0<s<1} s^{\frac{1}{2}}(f_*\chi_{[0,t]}(s))_*\\
&\quad +(1-\operatorname{Log} t)^{-\frac{1}{2}}\int_0^1(1-\operatorname{Log} s)^{-\frac{1}{2}}\left(\int_0^s (f_*\chi_{[t,1]}(\sigma))_*^2\,d\sigma\right)^{\frac{1}{2}}\frac{ds}{s}\\
&\approx \sup_{0<s<t} s^{\frac{1}{2}}f_*(s)\\
&\quad +(1-\operatorname{Log} t)^{-\frac{1}{2}}\int_t^1(1-\operatorname{Log} s)^{-\frac{1}{2}}\left(\int_t^s f_*(\sigma)^2\,d\sigma\right)^{\frac{1}{2}}\frac{ds}{s}.
\end{aligned}$$

If we set $\tau=\frac{\Phi_0(t)}{\Phi_1(t)}=(1-\operatorname{Log} t)^{-\frac{1}{2}}$, that is $t=e^{1-\frac{1}{\tau^2}}=\varphi(\tau)$, we get

$$K(f,\tau,L^{2,\infty},L^{(2})\approx \sup_{0<s<\varphi(\tau)} s^{\frac{1}{2}}f_*(s)+\tau\int_{\varphi(\tau)}^1(1-\operatorname{Log} s)^{-\frac{1}{2}}\left(\int_{\varphi(\tau)}^s f_*(\sigma)^2\,d\sigma\right)^{\frac{1}{2}}\frac{ds}{s}$$

as desired.

4.2. Computation of the norm in $\left(L^{2,\infty},L^{(2}\right)_{\theta,r}=V$, $0<\theta<1$, $r\in[1,+\infty[$.

Let $f\in V$, then we write

$$\begin{aligned}
||f||_V^r &= J_1+J_2,\ \text{with}\\
J_1 &= \int_0^1\left[t^{-\theta}\sup_{0<s<\varphi(t)} s^{\frac{1}{2}}f_*(s)\right]^r\frac{dt}{t} &(4.2)\\
J_2 &= \int_0^1\left[t^{1-\theta}\int_{\varphi(t)}^1(1-\operatorname{Log} s)^{-\frac{1}{2}}\left(\int_{\varphi(t)}^s f_*^2(\tau)d\tau\right)^{\frac{1}{2}}\frac{ds}{s}\right]^r\frac{dt}{t} &(4.3)
\end{aligned}$$

Making change of variable $x=\varphi(t)=e^{1-\frac{1}{t^2}}$, we deduce that

$$\begin{aligned}
J_1 &\approx \int_0^1\left[\Big(1-\operatorname{Log} x\Big)^{\frac{\theta}{2}}\sup_{0<s<x} s^{\frac{1}{2}}f_*(s)\right]^r\frac{dx}{(1-\operatorname{Log} x)x} &(4.4)\\
J_2 &\approx \int_0^1\left[\Big(1-\operatorname{Log} x\Big)^{\frac{\theta-1}{2}}\int_x^1\Big(1-\operatorname{Log} s\Big)^{-\frac{1}{2}}\left(\int_x^s f_*^2\right)^{\frac{1}{2}}\frac{ds}{s}\right]^r\frac{dx}{(1-\operatorname{Log} x)x} &(4.5)
\end{aligned}$$

Let us temporarily $V_0=G\Gamma(2,r;w_1,1)$ with $w_1=x^{-1}(1-\operatorname{Log} x)^{\frac{r\theta}{2}-1}$. We know that the norm is

$$||f||_{V_0}^r=\int_0^1\left[\Big(1-\operatorname{Log} x\Big)^{\frac{\theta}{2}}\left(\int_0^x f_*^2d\tau\right)^{\frac{1}{2}}\right]^r\frac{dx}{(1-\operatorname{Log} x)x}.$$

The main theorem in that section is the following :

Theorem 4.5.
Let $0 < \theta < 1,\ r \in [\,1, +\infty[$. *Then,*

(1) $(L^{2,\infty}, L^{(2)})_{\theta,r} = G\Gamma(2, r; w_1, 1)$ *with equivalent norms.*
Moreover, one has for all $f \in (L^{2,\infty}, L^{(2)})_{\theta,r}$ *:*

(2) $J_1 \lesssim ||f||^r_{V_0} \approx \displaystyle\sum_{k=0}^{+\infty} 2^{\frac{k\theta}{2}r} \left(\int_{t_{k+1}}^{t_k} f_*^2(\sigma) d\sigma \right)^{\frac{r}{2}}$ *with* $t_k = 2^{1-2^k}$;

(3) $J_2 \approx ||f||^r_{V_0}$;

(4) $J_2 \lesssim ||f||^r_{V} \approx J_1 + J_2 \lesssim ||f||^r_{V_0}$.

Remark 4.6.
From the above results, we then deduce that in the case of the dimension 2,

$$(L^{2,\infty}, L^{(2)})_{\theta,r} = (L^2, L^{(2)})_{\theta,r}.$$

In particular if $r = 1$, one has

$$(L^{2,\infty}, L^{(2)})_{\theta,1} = L^{(2,\theta} \quad \text{for all } 0 < \theta \leqslant 1. \tag{4.6}$$

As a byproduct of this relation (4.6) we have an improvement of Theorem 1.1 :

Corollary 4.7. of Theorem 4.5
Assume that $n = 2$ *and the unique solution of (1.1) satisfies, for all* $0 < \theta \leqslant 1$

$$||u||_{L^{(2,\theta}} \leqslant K_\theta ||f||_{L^1(\Omega;\delta(1+|\mathrm{Log}\,\delta|)^\theta)}.$$

Proof of Corollary 4.7
Since $L^{(2,1} = L^{(2}$, from Theorem 1.1, we have :

$$||u||_{L^{(2}} \leqslant K_1 ||f||_{L^1(\Omega,\delta(1+|\mathrm{Log}\,\delta|))} \tag{4.7}$$

and

$$||u||_{L^{2,\infty}} \leqslant K ||f||_{L^1(\Omega;\delta)}. \tag{4.8}$$

Therefore we deduce :

$$||u||_{L^{(2,\theta}} = ||u||_{(L^{2,\infty}, L^{(2)})_{\theta,1}} \leqslant K ||f||_{L^1(\Omega;\delta(1+|\mathrm{Log}\,\delta|)^\theta)}.$$

♢

The improvement comes from the fact that the relations (4.7) and (4.8) are almost optimal, therefore we obtain a good estimate for the interpolated spaces. We can see that fact in the case $\theta = \dfrac{1}{2}$ compared with the statement (2) of Theorem 1.1.

Proof of Theorem 4.5

Statement (1) is a consequence of statements (3) and (4). So we first show statement (3). For this we have for $f \in L^0(\Omega)$, for all $x \in]\,0,1\,]$

$$\sup_{0<s<x} s^{\frac{1}{2}} f_*(s) \leqslant \left(\int_0^x f_*^2(\sigma)d\sigma\right)^{\frac{1}{2}}. \tag{4.9}$$

Thus, using the expression J_1 in relation (4.4), one has $J_1 \lesssim ||f||_{V_0}^r$

To show the remainder part of this statement, we use Lemma 2.3 with $\lambda = \theta$, $q = \dfrac{r}{2}$, $H(x) = f_*^2(x)$.
Since H is decreasing, we have :

$$\int_0^1 H(x)dx \leqslant 2\int_0^{\frac{1}{2}} H(x)dx. \tag{4.10}$$

Thus, we may apply directly the Lemma to get the equivalence.
To prove statement (3), we follow the same argument as in [11]. We set for convenience

$$G(x) = \int_x^1 (1-\operatorname{Log} t)^{-\frac{1}{2}} \left(\int_x^t f_*^2(\sigma)d\sigma\right)^{\frac{1}{2}} \frac{dt}{t}$$

From relation (4.5), we then have

$$J_2 \approx \sum_{k=0}^{+\infty} \int_{t_{k+1}}^{t_k} (1-\operatorname{Log} x)^{\frac{\theta-1}{2}r-1} G(x)^r \frac{dx}{x}. \tag{4.11}$$

Since $2^k \approx 1-\operatorname{Log} x$ for $x \in [\,t_{k+1}, t_k\,]$, we deduce from relation (4.11) that

$$J_2 \approx \sum_{k=0}^{+\infty} 2^{kr\frac{\theta-1}{2}} G(t_{k+1})^r. \tag{4.12}$$

Let us write as in [11]

$$G(t_{k+1}) = \sum_{i=0}^{k} \int_{t_{i+1}}^{t_i} (1-\operatorname{Log} s)^{-\frac{1}{2}} \left(\int_{t_{k+1}}^{s} f_*^2(\sigma)d\sigma\right)^{\frac{1}{2}} \frac{ds}{s}. \tag{4.13}$$

Applying again Lemma 2.3 statement (1), we have

$$G(t_{k+1}) \approx \sum_{i=0}^{k} 2^{\frac{i}{2}} \left(\int_{t_{k+1}}^{t_i} f_*^2(t)dt\right)^{\frac{1}{2}} = \sum_{i=0}^{k} 2^{\frac{i}{2}} \left(\sum_{j=i}^{k} \int_{t_{j+1}}^{t_j} f_*^2(t)dt\right)^{\frac{1}{2}}. \tag{4.14}$$

Using Lemma 6.1 of [11] (see also [15]), we deduce from (4.14)

$$G(t_{k+1}) \approx \sum_{i=0}^{k} 2^{\frac{i}{2}} \left(\int_{t_{i+1}}^{t_i} f_*^2(t)dt \right)^{\frac{1}{2}}. \tag{4.15}$$

Combining relations (4.13) to (4.15) we have :

$$J_2 \approx \sum_{k=0}^{+\infty} 2^{kr\frac{\theta-1}{2}} \left[\sum_{i=0}^{k} 2^{\frac{i}{2}} \left(\int_{t_{i+1}}^{t_i} f_*^2(t)dt \right)^{\frac{1}{2}} \right]^r, \tag{4.16}$$

thus

$$J_2 \approx \sum_{k=0}^{+\infty} \left[2^{\frac{k\theta}{2}} \left(\int_{t_{k+1}}^{t_k} f_*^2(t)dt \right)^{\frac{1}{2}} \right]^r. \tag{4.17}$$

(we have used the Gol'dman-Heinig- Stepanov's Lemma (see [15, 14]), to derive this last relation.)
This shows statement (3).
Since $||f||_V^r \approx J_1 + J_2$, we deduce that

$$||f||_{V_0}^r \approx J_2 \leqslant ||f||_V^r \lesssim ||f||_{V_0}^r. \tag{4.18}$$

This ends the proof of Theorem 4.5. ♢

5. The K-functional for the couple $(L^{p,\infty}, L^p)$, $1 < p < +\infty$

Theorem 5.1.
For a measurable set $E \subset [0,1]$, we denote $|E|_\nu = \displaystyle\int_E \frac{dx}{x}$ and for $f \in L^{p,\infty} + L^p$, $1 < p < +\infty$, we define

$$K_p(f,t) = t \sup \left\{ \left(\int_E f_*^p(\sigma)d\sigma \right)^{\frac{1}{p}} : |E|_\nu = t^{-p} \right\} \qquad t \in]0,1].$$

Then

$$K(f,t; L^{p,\infty}, L^p) \approx K_p(f,t)$$

and

$$K_p(f,t) = t \left[\int_0^{t^{-p}} \psi_{*,\nu}(x)^p dx \right]^{\frac{1}{p}}$$

where $\psi(s) = s^{\frac{1}{s}} f_*(s)$, $\psi_{*,\nu}$ its decreasing rearrangement with respect to the measure ν.

Proof:

Let $f = g + h \in L^{p,\infty} + L^p$. Then, $f_*(s) \leqslant g_*\left(\frac{s}{2}\right) + h_*\left(\frac{s}{2}\right)$, for $s \in]0,1]$

$$K_p(f,t) \leqslant t \sup_{|E|_\nu = t^{-p}} \left(\int_E g_*\left(\frac{s}{2}\right)^p\right)^{\frac{1}{p}} + t \sup_{E:=|E|_\nu = t^{-p}} \left(\int_E h_*\left(\frac{s}{2}\right)^p ds\right)^{\frac{1}{p}}. \tag{5.1}$$

The first term can be bound as follows :

$$t \sup_{|E|_\nu = t^{-p}} \left(\int_E \left[s^{\frac{1}{p}} g_*\left(\frac{s}{2}\right)\right]^p \frac{ds}{s}\right) \lesssim t||g||_{L^{p,\infty}} \sup_{|E|_\nu = t^{-p}} |E|_\nu^{\frac{1}{p}} = ||g||_{L^{p,\infty}} \tag{5.2}$$

While the second term satisfies

$$t \sup_{|E|_\nu = t^{-p}} \left(\int_E h_*\left(\frac{s}{2}\right)^p ds\right)^{\frac{1}{p}} \leqslant t||h||_{L^p}. \tag{5.3}$$

From the three last relations, we have

$$K_p(f,t) \lesssim ||g||_{L^{p,\infty}} + t||h||_{L^p}. \tag{5.4}$$

From which we derive

$$K_p(f,t) \lesssim K(f,t;L^{p,\infty},L^p). \tag{5.5}$$

For the converse , let t be fixed and set $\psi(s) = s^{\frac{1}{p}} f_*(s)$, $s \in [0,1]$, $\psi_{*,\nu}$ will denote its decreasing rearrangement with respect to ν, $A_t = \{s : \psi(s) > \psi_{*,\nu}(t^{-p})\}$.
By equimesurability, we have

$$|A_t|_\nu = t^{-p}.$$

Let us consider the measure preserving mapping $\sigma : \mathbb{R} \to (0,+\infty)$ such that $f = f_* \circ \sigma$ and set $f_i = g_i \circ \sigma$, $i = 1,2$ where, for $s \in (0,1)$

$$\begin{aligned} g_1(s) &= s^{-\frac{1}{p}} \psi_{*,\nu}(t^{-p}) \chi_{A_t}(s) + f_*(s) \chi_{A_t^c}(s) \\ g_2(s) &= s^{-\frac{1}{p}} \Big(\psi(s) - \psi_{*,\nu}(t^{-p})\Big) \chi_{A_t} \end{aligned}$$

and A_t^c is the complement of A_t in $(0,1)$, say $A_t^c = \Big\{s : \psi(s) \leqslant \psi_{*,\nu}(t^{-p})\Big\}$.

Since σ is measure preserving we have

$$||f_2||_{L^p}^p = ||g_2||_{L^p}^p = \int_0^{|A_t|_\nu} \Big(\psi_{*,\nu}(x) - \psi_{*,\nu}(t^{-p})\Big)^p dx.$$

From which we derive

$$||f_2||_{L^p}^p \leqslant \int_0^{t^{-p}} \psi_{*,\nu}(x)^p dx. \tag{5.6}$$

While for f_1, we have

$$\begin{aligned} ||f_1||_{L^{p,\infty}} &= ||g_1||_{L^{p,\infty}} \leqslant \sup_s \left[\psi_{*,\nu}(t^{-p})\chi_{A_t}(s) + s^{\frac{1}{p}} f_*(s)\chi_{A_t^c}(s)\right] \\ &\leqslant \sup_s \left[\psi_{*,\nu}(t^{-p})\chi_{A_t}(s) + \psi(s)\chi_{A_t^c}(s)\right] \\ &\leqslant \psi_{*,\nu}(t^{-p}) \text{ (by definition of } A_t^c). \end{aligned} \tag{5.7}$$

Since $f = f_1 + f_2 \in L^{p,\infty} + L^p$, we derive from relation (5.6) and (5.7) that

$$K(f,t;L^{p,\infty},L^p) \leqslant ||f_1||_{L^{p,\infty}} + t||f_2||_{L^p} \leqslant \psi_{*,\nu}(t^{-p}) + t\left[\int_0^{t^{-p}} \psi^p_{*,\nu}(x)dx\right]^{\frac{1}{p}}. \tag{5.8}$$

Since the function $x \to \psi_{*,\nu}(x)$ is decreasing one has

$$\psi_{*,\nu}(t^{-p}) \leqslant t\left[\int_0^{t^{-p}} \psi^p_{*,\nu}(x)dx\right]^{\frac{1}{p}}. \tag{5.9}$$

Thus, we derive from (5.8) and (5.9)

$$K(f,t;L^{p,\infty},L^p) \lesssim t\left[\int_0^{t^{-p}} \psi^p_{*,\nu}(x)dx\right]^{\frac{1}{p}}. \tag{5.10}$$

Making use of the Hardy Littlewood (see[20]), we have

$$\int_0^{t^{-p}} \psi^p_{*,\nu}(x)dx = \mathop{\mathrm{Max}}_{|E|_\nu = t^{-p}} \int_E \psi^p(s)\frac{ds}{s} = \mathop{\mathrm{Max}}_{|E|_\nu = t^{-p}} \int_E f_*(s)^p ds. \tag{5.11}$$

Thus

$$K_p(f,t) = t\left[\int_0^{t^{-p}} \psi_{*,\nu}(x)^p dx\right]^{\frac{1}{p}}. \tag{5.12}$$

This equality with relation (5.10) leads to

$$K(f,t;L^{p,\infty},L^p) \lesssim K_p(f,t).$$

◇

As we noticed at the beginning, we recover the Maligranda-Persson's results stating that

$$(L^{p,\infty},L^p)_{\theta,\frac{p}{\theta}} = L^{p,\frac{p}{\theta}}.$$

Proof of Maligranda-Persson's result

One has, from the above result

$$||f||^{\frac{p}{\theta}}_{(L^{p,\infty},L^p)_{\theta,\frac{p}{\theta}}} = \int_0^{+\infty}\left[t^{1-\theta}\left(\int_0^{t^{-p}} \Phi(x)dx\right)^{\frac{1}{p}}\right]^{\frac{p}{\theta}} \frac{dt}{t} \doteq I_0$$

where we set temporarily $\Phi(x) = \psi_{*,\nu}(x)^p$.

Making the following change of variable $\sigma = t^{-p}$, we derive that

$$I_0 \approx \int_0^{+\infty}\left[\sigma^{\theta-1}\left(\int_0^\sigma \Phi(x)dx\right)\right]^{\frac{1}{\theta}}\frac{d\sigma}{\sigma} = \int_0^{+\infty}\left[\frac{1}{\sigma}\int_0^\sigma \Phi(x)dx\right]^{\frac{1}{\theta}} d\sigma,$$

but by the Hardy inequality this integral is equivalent to $\displaystyle\int_0^{+\infty}\Phi(x)^{\frac{1}{\theta}}dx$.
Therefore, we have

$$I_0^{\frac{\theta}{p}} \approx \left[\int_0^{+\infty}\psi_{*,\nu}(x)^{\frac{p}{\theta}}dx\right]^{\frac{\theta}{p}} = \left[\int_0^1\left[s^{\frac{1}{p}}f_*(s)\right]^{\frac{p}{\theta}}\frac{ds}{s}\right]^{\frac{\theta}{p}} \quad \text{(by equimesurability).}$$

This last quantity is equivalent to the norm of f in $L^{p,\frac{p}{\theta}}$. ♢

6. Some interpolation inequalities for Small and Grand Lebesgue spaces

One may combine the above results with some standard results on interpolation spaces to deduce few inequalities as Property 1.13.
We recall the following result that can be found in [21].

Theorem 6.1.
Let E_0 and E_1 be two Banach spaces continuously embedded into some topological vector space.
For $0 \leqslant \theta \leqslant 1$, one has

$$(E_0, E_1)_{\theta,1} \subset E \subset E_0 + E_1$$

if and only if

$$\text{there exists } c > 0 : \forall\, a \in E_0 \cap E_1 \qquad ||a||_E \leqslant c||a||_0^{1-\theta}||a||_1^\theta$$

where $||\cdot||_i$ denotes the norm in E_i, $i = 0, 1$.

Proof of Property 1.13
We apply the above Theorem 6.1 with $E_0 = L^{n')}$, $E_1 = L^{(n'}$.
Then from Theorem 1.4 and Corollary 2.6 of Proposition 2.2 one has

$$\left(L^{n')}, L^{(n'}\right)_{\alpha,1} = L^{(n',\beta}$$

with $\beta = n\alpha - n + 1$ and $\alpha > \dfrac{1}{n'}$.
Since $L^{(n',\beta} \subset L^{n',\infty} \subset L^{n')}$, we deduce the result from Theorem 6.1 with $E =$

$L^{n',\infty}, \ \theta = \alpha.$

The same argument holds for the second inequality, since

$$\left(L^{n'}, L^{(n'}\right)_{\alpha,1} = L^{(n',\alpha} \subset E = \left(L^{n',\infty}, L^{(n'}\right)_{\alpha,1}.$$

$\diamondsuit$

Acknowledgment :
The second author has been partially supported by Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and by Università degli Studi di Napoli Parthenope through the project "sostegno alla Ricerca individuale".
This work of the third author has been partially supported by Shota Rustaveli National Science Foundation of Georgia (SRNSFG) [grand number FR17-589], by Grant 18-00580S of the Czech Science Foundation and RVO:67985840.

References

[1] **J. Bastero, F. Ruiz**, Elementary reverse Hölder type inequalities with application to operator interpolation theory, *Proc. A.M.S.* **124 (10)** (1996), 3183-3192.

[2] **C. Bennett, K. Rudnick**, *On Lorentz-Zygmund spaces* Dissertationes Math. (Rozrrawy Mat.) 175 (1980) 67 pp.

[3] **C. Bennett, R. Sharpley**, *Interpolation of Operators*, Academic Press, 1988.

[4] **J. Bergh, J. Löfstrom**, *Interpolation spaces. An introduction,* Grundlehren der Mathematischen Wissenschaften, N° 223, Springer-Verlag, Berlin-New-York, 1976.

[5] **H. Brezis, T. Cazenave, Y. Martel, A. Ramiandrisoa**, Blow up for $u_t - \mathbb{D}u = g(u)$ revisited. *Advance in Diff. Eq.* **1**, (1996) 73-90.

[6] **J. Cerdà, H. Coll**, Interpolation of classical Lorentz Spaces, *Positivity* **7**, 3, (2003) 225-234.

[7] **J.I. Díaz, J.M. Rakotoson,** On the differentiability of the very weak solution with right-hand side data integrable with respect to the distance to the boundary *J. Functional Analysis* **257** (2009) 807-831.

[8] **G. Di Fratta, A. Fiorenza,** A direct approach in the duality of grand and small Lebesgue spaces, *Nonlinear Anal. TMA* **70** (2009) 2582-2592.

[9] **J. I. Díaz, D. Gómez-Castro, J. M. Rakotoson, R. Temam**, Linear diffusion with singular absorption potential and/or unbounded convective flow: the weighted space approach, *Discrete and Continuous Dynamical Systems* **38**, 2, (2018) 509-546

[10] **A. Fiorenza, M.R. Formica, J.M. Rakotoson**, Pointwise estimates for $G\Gamma$-functions and applications, *Differential Integral Equations* **30**, 11-12 (2017) 809-824.

[11] **A. Fiorenza, M.R. Formica, A. Gogatishvili, T. Kopaliani, J.M. Rakotoson**, Characterization of interpolation between Grand, small or classical Lebesgue spaces, *Non Linear Analysis* **177** (2018) 422-453. DOI https://doi.org/10.1016/j.na.2017.09.005

[12] **A. Fiorenza, M. Krbec,** On an optimal decomposition in Zygmund spaces, *Georgian Math. J.* **9**, (2) (2002), 271–286.

[13] **A. Gogatishvili, M. Krepela, L. Pick, F. Soudsky**, Embeddings of Lorentz-type spaces involving weighted integral means, *J.F.A.* **273**, 9 (2017) 2939-2980.

[14] **A. Gogatishvili, L. Pick**, Discretization and anti-discretization of rearrangement-invariant norms, *Publ. Mat.* **47** (2003) 311-358.

[15] **M.L. Gol'dman, H.P. Heinig, V.D. Stepanov**, On the principle of duality in Lorentz spaces, *Canad. J. Math.* **48** 5, (1996) 959-979.

[16] **T. Iwaniec, C. Sbordone**, On the integrability of the jacobian under minimal, *Arch. Rat. Mech. Anal.* **119** (1992), 129-143.

[17] **L. Maligranda, L.E. Persson,** Real interpolation between weighted L^p and Lorentz spaces, *Bull. Polish Acad. Sci. Math.*, 35 (1987), 765–778.

[18] **J.M. Rakotoson,** New hardy inequalities and behaviour of linear elliptic equations *J. Functional Analysis* **263** 9 (2012) 2893-2920.

[19] **J.M. Rakotoson**, A sufficient condition for a blow-up on the space. Absolutely conditions functions for the very weak solution *AMO* **73**(2016) 153-163.

[20] **J.M. Rakotoson,** *Rearrangement Relatif Un instrument d'estimation pour les problèmes aux limites,* Springer (2008).

[21] **L. Tartar**, *An introduction to Sobolev spaces and Interpolation spaces*, Springer-Verlag Berlin (2007)

[22] **L. Veron** *Singularities of solutions of second order quasilinear equations*, Longman, Edinburgh Gate, Harlow,(1995) p 176-180.

[1] Università di Napoli "Federico II",via Monteoliveto, 3, I-80134 Napoli, **Italy**and Istituto per le Applicazioni del Calcolo "Mauro Picone" Consiglio Nazionale delle Ricerche via Pietro Castellino, 111 I-80131Napoli, **Italy**

Email address: fiorenza@unina.it

[2]Università degli Studi di Napoli "Parthenope", via Generale Parisi 13, 80132, Napoli, **Italy**

Email address: mara.formica@uniparthenope.it

[3]Institute of Mathematics of the Czech Academy of Sciences - Žitná, 115 67 Prague 1, **Czech Republic**

Email address: gogatish@math.cas.cz

[4] Laboratoire de Mathématiques et Applications - Université de Poitiers,Avenue Marie et Pierre Curie,Téléport 2,BP 30179,86692 Futuroscope Chasseneuil Cedex, **France**

Email address: rako@math.univ-poitiers.fr, jean.michel.rakotoson@univ-poitiers.fr