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Compactifiable classes of compacta

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Dedicated to the memory of Petr Simon,
member of Seminar on Topology at Charles University.

1 Introduction

Let us consider two classes $\mathcal{C}$ and $\mathcal{D}$ of topological spaces. We say that these classes are *equivalent* (and we write $\mathcal{C} \cong \mathcal{D}$) if every space in $\mathcal{C}$ is homeomorphic to a space in $\mathcal{D}$ and vice versa.

Given a class of metrizable compacta $\mathcal{C}$, we are interested whether $\mathcal{C}$ (up to the equivalence) can be disjointly composed into one metrizable compactum such that the corresponding quotient space is also a metrizable compactum. If $\mathcal{C}$ is a class of continua, this is equivalent to finding a metrizable compactum whose set of connected components is equivalent to $\mathcal{C}$ (see Observation 1.12).

The idea of disjointly composing topological spaces is captured by the following notion.

Definition 1.1. A *composition* $\mathcal{A}$ consists of a continuous map $q\colon A \to B$ between topological spaces. In this context, A is called the *composition space*, B is called the *indexing space*, and q is called the *composition map*. We write $\mathcal{A}(q\colon A \to B)$ to introduce all names at once. The idea is that the composition map q captures how its fibers are composed in the composition space A.

The following language gives us some flexibility when working with compositions.

- $\mathcal{A}$ is a *composition of an indexed family of topological spaces* $(A_b)_{b\in B}$ if $q^{-1}(b) = A_b$ for every $b \in B$. Of course the family $(A_b)_{b\in B}$ is a decomposition of A (i.e. $A_b \cap A_{b'} = \emptyset$ for every $b \neq b' \in B$ and $\bigcup_{b\in B} A_b = A$) and is determined by $\mathcal{A}$. On the other hand, every decomposition $(A_b)_{b\in B}$ of a topological space A induces the unique map $q\colon A \to B$ with fibers $(A_b)_{b\in B}$ and the composition $\mathcal{A}(q\colon A \to B)$ if the map q is continuous.

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- $\mathcal{A}$ is a *composition of an indexed family of embeddings* $(e_b\colon A_b \hookrightarrow A)_{b\in B}$ if $q^{-1}(b) = \operatorname{rng}(e_b)$ for every $b \in B$. Again, $(\operatorname{rng}(e_b))_{b\in B}$ is necessarily a decomposition of A.
- $\mathcal{A}$ is a *composition of a class of topological spaces* $\mathcal{C}$ if the family $\{q^{-1}(b) : b \in B\}$ is equivalent to $\mathcal{C}$.

We are interested in the following special types of compositions.

- $\mathcal{A}$ is a *compact composition* if both A and B are metrizable compacta.
- $\mathcal{A}$ is a *Polish composition* if both A and B are Polish spaces.

Remark 1.2. In [9] P. Minc constructed a compact 2^ω-indexed composition of a family of pairwise non-homeomorphic compactifications of a ray with remainders being copies of an arbitrary fixed nondegenerate metrizable continuum.

Remark 1.3. Given a composition $\mathcal{A}(q\colon A \to B)$ of a family $(A_b)_{b\in B}$, the spaces A_b are all nonempty if and only if the composition map q is surjective.

Definition 1.4. A class of spaces $\mathcal{C}$ is called *compactifiable* (or *Polishable*) if there is a compact (or Polish) composition of $\mathcal{C}$, i.e. if there is a continuous map $q\colon A \to B$ between metrizable compacta (or Polish spaces) such that $\{q^{-1}(b) : b \in B\} \cong \mathcal{C}$. Note that the spaces $q^{-1}(b)$ are necessarily metrizable compacta (or Polish spaces).

Construction 1.5 (rectangular compositions)**.** Let A, B be topological spaces and let $F \subseteq A \times B$. By F^b we denote the subset of A corresponding to the slice of F through b, i.e. $F^b = \{a \in A\colon (a, b) \in F\}$. For every $b \in B$ let e_b denote the canonical embedding $F^b \to F^b \times \{b\} \subseteq F$. The set F induces the composition $\mathcal{A}_F(\pi_B{\restriction}_F\colon F \to B)$ of the family $(e_b)_{b\in B}$. If the spaces A, B are metrizable compacta (or Polish spaces) and the set F is closed (or G_δ) in $A \times B$, then the composition $\mathcal{A}_F$ is compact (or Polish).

Moreover, every composition can essentially be obtained this way. For a composition $\mathcal{A}(q\colon A \to B)$ we consider the graph of q, $G = \{(a, q(a)) : a \in A\} \subseteq A \times B$, which is closed if B is Hausdorff. Since A is homeomorphic to G and $G^b = q^{-1}(b)$ for every $b \in B$, the compositions $\mathcal{A}$ and $\mathcal{A}_G$ are essentially the same.

Construction 1.6 (pullback compositions)**.** Let $\mathcal{A}(q\colon A \to B)$ be a composition and let $f\colon B' \to B$ be a continuous map. The *pullback of* $\mathcal{A}$ *along* f is the composition $\mathcal{A}'(q'\colon A' \to B')$ where $A' := \{(a, b') \in A \times B' : q(a) = f(b')\}$ and $q' := \pi_B{\restriction}_{A'}$, so $\mathcal{A}'$ is the rectangular composition induced by $A' \subseteq A \times B'$.

If $\mathcal{A}$ is a composition of spaces $(A_b)_{b\in B}$, then $\mathcal{A}'$ is essentially a composition of $(A_{f(b')})_{b'\in B'}$ since for every $b' \in B'$ we have the canonical embedding $e_{b'} : A_{f(b')} \to A_{f(b')} \times \{b'\} \subseteq A'$ and so $\mathcal{A}'$ is formally a composition of $(e_{b'})_{b'\in B'}$. This way we change the indexing space so that each space A_b has $f^{-1}(b)$-many copies in A'.

Moreover, A' is a closed subset $A \times B'$ if B is Hausdorff. Hence, if $\mathcal{A}$ is a compact (or Polish) composition and B' is a metrizable compactum (or a Polish space), then $\mathcal{A}'$ is a compact (or Polish) composition as well.

Corollary 1.7 (subcompositions)**.** If $\mathcal{A}(q\colon A \to B)$ is a compact (or Polish) composition of spaces $(A_b)_{b\in B}$ and $C \subseteq B$ is F_σ (or analytic), then the class $\{A_c : c \in C\}$ is compactifiable (or Polishable).

Proof. In the compact case with closed $C \subseteq B$, it is enough to consider the *induced subcomposition* $\mathcal{A}_C(q\colon q^{-1}[C] \to C)$, which may be viewed as a special case of the pullback construction. If $C = \bigcup_{n\in\omega} C_n$ for some closed sets $C_n \subseteq B$, then $\{A_c : c \in C\}$ is a countable union of compactifiable classes, which is compactifiable as we will show later (Observation 1.14). In the Polish case, there is a Polish space B' and a continuous surjection $f\colon B' \twoheadrightarrow C$, so the pullback of $\mathcal{A}$ along f is a Polish composition of $\{A_{f(b')} : b' \in B'\} = \{A_c : c \in C\}$. □

Remark 1.8. By an *analytic* set or space we mean just a continuous image of a Polish space, it does not have to be a subspace of a Polish space or even be metrizable.

Lemma 1.9. Let A be a Polish space, let B be an analytic space, let $F \subseteq A \times B$ be a G_δ subset, and let $\mathcal{A}_F(q\colon F \to B)$ be the corresponding rectangular composition. Moreover, let B' be a Polish space and let $f\colon B' \to B$ be a continuous map. The pullback $\mathcal{A}'(q'\colon F' \to B')$ of $\mathcal{A}_F$ along f is a Polish composition.

Proof. We need to show that the composition space F' is Polish. We have $F' = \{((a,b),b') \in (A \times B) \times B' : (a,b) \in F \text{ and } b = f(b')\}$, which is canonically homeomorphic to $G := \{(a,b') \in A \times B' : (a, f(b')) \in F\} = g^{-1}[F]$ where $g := \mathrm{id}_A \times f\colon A \times B' \to A \times B$. Since F is G_δ in $A \times B$, G is G_δ in the Polish space $A \times B'$. □

By combining the previous observations we obtain the following characterizations.

Theorem 1.10. The following conditions are equivalent for a class of spaces $\mathcal{C}$.

(i) $\mathcal{C}$ is compactifiable.

(ii) There is a metrizable compactum A and a closed equivalence relation $E \subseteq A \times A$ such that $\{E^a : a \in A\} \cong \mathcal{C}$.

(iii) There is a metrizable compactum A, a metrizable σ-compact space B, and a closed set $F \subseteq A \times B$ such that $\{F^b\colon b \in B\} \cong \mathcal{C}$.

(iv) There is a closed set $F \subseteq [0,1]^\omega \times 2^\omega$ such that $\{F^b : b \in 2^\omega\} \cong \mathcal{C}$.

Theorem 1.11. The following conditions are equivalent for a class of spaces $\mathcal{C}$.

(i) $\mathcal{C}$ is Polishable.

(ii) There is a Polish space A and a closed equivalence relation $E \subseteq A \times A$ such that $\{E^a : a \in A\} \cong \mathcal{C}$.

(iii) There is a Polish space A, an analytic space B, and a G_δ set $F \subseteq A \times B$ such that $\{F^b \colon b \in B\} \cong \mathcal{C}$.

(iv) There is a G_δ set $F \subseteq [0,1]^\omega \times \omega^\omega$ such that $\{F^b : b \in \omega^\omega\} \cong \mathcal{C}$.

(v) There is a closed set $F \subseteq (0,1)^\omega \times \omega^\omega$ such that $\{F^b : b \in \omega^\omega\} \cong \mathcal{C}$.

Proof. (i) $\implies$ (ii). For a composition $\mathcal{A}(q\colon A \to B)$ of $\mathcal{C}$ it is enough to consider the equivalence $E := \{(a, a') \in A \times A : q(a) = q(a')\}$ induced by q.

(ii) $\implies$ (iii) is trivial.

(iii) $\implies$ (i). We consider the induced composition $\mathcal{A}_F(q\colon F \to B)$ (Construction 1.5). In the compact case with B compact we are done. If $B = \bigcup_{n\in\omega} B_n$ for some compacta B_n, then each $F \cap (A \times B_n)$ induces a compact composition of $\{F^b : b \in B_n\}$, and $\mathcal{C}$ is equivalent to a countable union of compactifiable classes, which is compactifiable by Observation 1.14. In the Polish case, there is a Polish space B' and a continuous surjection $f\colon B' \twoheadrightarrow B$. Let $\mathcal{A}'$ be the pullback of $\mathcal{A}_F$ along f (Construction 1.6). As in Corollary 1.7, $\mathcal{A}'$ is a composition of $\{F^b : b \in B\} \cong \mathcal{C}$, and it is Polish by Lemma 1.9.

(i) $\implies$ (iv), (v). Let $\mathcal{A}(q\colon A \to B)$ be a compact (or Polish) composition of $\mathcal{C}$. We may suppose that B is nonempty. Otherwise, $\mathcal{C}$ is empty as well, and we put $F := \emptyset$. Recall that every nonempty metrizable compactum is a continuous image of the Cantor space 2^ω and that every nonempty Polish space is a continuous image of the Baire space ω^ω, so we may suppose that $B = 2^\omega$ (or ω^ω) by Construction 1.6. Recall that every separable metrizable space may be embedded into the Hilbert cube $[0,1]^\omega$, so we may suppose that $A \subseteq [0,1]^\omega$. Let F be the graph of q. By the second part of Construction 1.5, $\{F^b : b \in B\} \cong \mathcal{C}$ and F is closed in $A \times B$. Since A is compact (or Polish), $A \times B$ and so F is closed (or G_δ) in $[0,1]^\omega$. This proves (iv). The proof of (v) is analogous and uses the fact that every Polish space may be embedded into $(0,1)^\omega$ as a closed subspace [6, 4.17].

The implications (iv), (v) $\implies$ (iii) are trivial. □

Observation 1.12. A class of nonempty metrizable continua $\mathcal{C}$ is compactifiable if and only if there exists a metrizable compactum A whose set of components is equivalent to $\mathcal{C}$.

Proof. Let $\mathcal{A}(q\colon A \to B)$ be a compact composition of $\mathcal{C}$. By Theorem 1.10 the indexing space B may be taken zero-dimensional (e.g. the Cantor space), and hence the spaces $q^{-1}(b)$ are precisely the components of A.

On the other hand, let A be a metrizable compactum whose set of components is equivalent to $\mathcal{C}$. Let $q\colon A \to B$ be the quotient map induced by the decomposition of A

into its components. Since A is a metrizable compactum, the components are equal to the quasi-components, and hence B is totally separated, in particular Hausdorff. Therefore, B is a metrizable compactum and q induces the desired compact composition. □

Let us conclude this section with basic observations about (non)existence of compactifiable or Polishable classes.

Remark 1.13. If a class $\mathcal{C}$ is compactifiable (or Polish), then so are the classes $\mathcal{C} \setminus \{\emptyset\}$ and $\mathcal{C} \cup \{\emptyset\}$. This is because if a map $q\colon A \to B$ induces a compact composition, then the maps $q\colon A \to q[A]$ and $q\colon A \to B \oplus \{\infty\}$ induces compact compositions as well. For Polishable $\mathcal{C}$ the case "$\mathcal{C} \cup \{\emptyset\}$" is the same, but the case "$\mathcal{C} \setminus \{\emptyset\}$" needs a comment. The map $q\colon A \to q[A]$ may not directly induce a Polish composition since $q[A]$ may not be G_δ in B. Nevertheless, it is analytic, so we use Corollary 1.7. In fact, this gives us the composition $\mathcal{A}_E$ for $E = \{(a, a') \in A \times A : q(a) = q(a')\}$.

Observation 1.14. Every countable union of compactifiable (or Polishable) classes is compactifiable (or Polishable).

Proof. Let I be a set and for every $i \in I$ let $\mathcal{A}_i(q_i\colon A_i \to B_i)$ be a composition of a class $\mathcal{C}_i$. We consider the *sum composition* $\mathcal{A}(q\colon A \to B) := \sum_{i\in I} \mathcal{A}_i$, i.e. $A := \sum_{i\in I} A_i$, $B := \sum_{i\in I} B_i$, and $q := \sum_{i\in I} q_i\colon A \to B$. Clearly, $\mathcal{A}$ is a composition of $\bigcup_{i\in I} \mathcal{C}$. If I is finite (or countable) and the compositions $\mathcal{A}_i$ are compact (or Polish), then $\mathcal{A}$ is also compact (or Polish).

It remains to consider a countable sum of compact compositions that is not compact. Without loss of generality, $\emptyset \notin \mathcal{C}_i \neq \emptyset$ for every $i \in I$ (Remark 1.13), and so A and B are separable metrizable locally compact non-compact spaces. We consider their one-point compactifications A^+ and B^+, which are metrizable, and the corresponding extension $q^+\colon A^+ \to B^+$ of the map q. The map q^+ is continuous since q is perfect (i.e. closed with compact fibers), and it induces a composition of $\bigcup_{i\in I} \mathcal{C}_i \cup \{\{\infty\}\}$, so if the given classes contain a one-point space, we are done. Otherwise, we take any space $C \in \bigcup_{i\in I} \mathcal{C}_i$, attach it to the point $\infty \in A^+$, and modify the definition of q^+ accordingly. □

Corollary 1.15. Every countable family of metrizable compacta is compactifiable. Every countable family of Polish spaces is Polishable.

Question 1.16 (by Jan Starý)**.** Is the class of all countable compact ordinals compactifiable?

Remark 1.17. We require metrizability (or equivalently existence of a countable base) in the definition of compact composition not only to obtain a notion stronger than Polish composition, but because otherwise the corresponding compactifiability would be trivial. Using the one-point compactification as in the previous proof, we may easily construct a composition with compact composition space and compact indexing space for any family of compacta.

Observation 1.18. By Theorem 1.11 there are at most $\mathfrak{c}$-many nonequivalent Polishable classes since there are only $\mathfrak{c}$-many G_δ subsets of $[0,1]^\omega \times \omega^\omega$. On the other hand, there are $\mathfrak{c}$-many nonhomeomorphic metrizable compact spaces – even in the real line. Hence, there are exactly $2^\mathfrak{c}$-many nonequivalent classes of metrizable compacta and also exactly $2^\mathfrak{c}$-many nonequivalent classes of Polish spaces. This cardinal argument gives us that many classes of metrizable compacta are not Polishable.

2 Compactifiability and hyperspaces

We often consider a class of spaces that is equivalent to a family of subspaces of some fixed ambient space. Therefore, it is natural to consider how compactifiability of such family is related to its properties when viewed as a subset of a hyperspace.

For a topological space X we shall consider the hyperspaces of all subsets $\mathcal{P}(X)$, of all closed subsets $\mathcal{Cl}(X)$, of all compact subsets $\mathcal{K}(X)$, and of all subcontinua $\mathcal{C}(X)$ endowed with the *Vietoris topology*. We include the empty set in the families. Recall that the *lower Vietoris topology* τ_V^- is generated by the sets $U^- = \{A : A \cap U \neq \emptyset\}$ for $U \subseteq X$ open, and the *upper Vietoris topology* τ_V^+ is generated by the sets $U^+ = \{A : A \subseteq U\}$ for $U \subseteq X$ open. The Vietoris topology τ_V is their join.

Also recall that if X is metrizable by a metric d, the corresponding *Hausdorff metric* d_H on $\mathcal{Cl}(X)$ is defined by $d_H(A,B) = \max(\delta(A,B), \delta(B,A))$ where $\delta(A,B) = \sup_{x\in A} d(x,B) = \sup_{x\in A} \inf_{y\in B} d(x,y) = \inf\{\varepsilon : A \subseteq N_\varepsilon(B)\}$. We have $\delta(\emptyset, B) = 0$ for every B, and $\delta(A,\emptyset) = \infty$ for every $A \neq \emptyset$, and also $\delta(A,B) = \infty$ for every A unbounded and B bounded. Hence, strictly speaking, d_H is an extended metric, but we may always cap it at 1 or suppose that $d \leq 1$ and interpret the infima in $[0,1]$, so $\inf \emptyset = 1$. In any case, the singleton $\{\emptyset\}$ is clopen in $\mathcal{Cl}(X)$ with both Vietoris topology and Hausdorff metric topology.

The Vietoris topology and the topology induced by the Hausdorff metric are not comparable on $\mathcal{Cl}(X)$ in general, but they coincide on $\mathcal{K}(X)$. If X is compact or Polish, so is $\mathcal{K}(X)$. Also, $\mathcal{C}(X)$ is a closed subspace of $\mathcal{K}(X)$ if X is Hausdorff. For reference on the mentioned properties see [6, 4.F].

Construction 2.1 (from hyperspace to composition)**.** Let X be a topological space and let $\mathcal{F} \subseteq \mathcal{P}(X)$. We consider the set $A_\mathcal{F} := \{(x,F) : x \in F \in \mathcal{F}\} \subseteq X \times \mathcal{F}$. Let us denote the corresponding composition (Construction 1.5) by $\mathcal{A}_\mathcal{F}$. Since $(A_\mathcal{F})^F = F$ for every $F \in \mathcal{F}$, we have that $\mathcal{A}_\mathcal{F}$ is a composition of the family $\mathcal{F}$ with composition space $A_\mathcal{F}$ and indexing space $\mathcal{F}$. The composition map is just the projection $\pi_\mathcal{F}{\restriction}_{A_\mathcal{F}}$. Also,

$A_\mathcal{F} = \mathcal{R}_\in \cap (X \times \mathcal{F})$ where $\mathcal{R}_\in := \{(x, F) \in X \times \mathcal{P}(X) : x \in F\}$ is the membership relation.

Observation 2.2. If X is a regular space, then the membership relation of closed sets is closed, i.e. $\mathcal{R}_\in \cap (X \times \mathcal{C}l(X))$ is closed in $X \times \mathcal{C}l(X)$ (even with respect to τ_V^+).

Proof. If $F \in \mathcal{C}l(X)$ and $x \in X \setminus F$, then there are disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $F \subseteq V$. We have that $U \times V^+$ is a neighborhood of (x, F) disjoint with $\mathcal{R}_\in$. □

Proposition 2.3.

(i) If X is a metrizable compactum and $\mathcal{F}$ is an F_σ subset of $\mathcal{K}(X)$ (or $\mathcal{C}(X)$), then $\mathcal{F}$ is a compactifiable class of compacta (or continua).

(ii) If X is a Polish space and $\mathcal{F}$ is an analytic subset of $\mathcal{C}l(X)$ (or $\mathcal{K}(X)$ or $\mathcal{C}(X)$), then $\mathcal{F}$ is a Polishable class of Polish spaces (or compacta or continua).

Proof. It is enough to use the set $A_\mathcal{F} \subseteq X \times \mathcal{F}$ from Construction 2.1 and Theorem 1.10 and 1.11. Note that it does not matter if $\mathcal{C}l(X)$ is not Polish. □

Next, we shall introduce a construction in the opposite direction, i.e. turning a composition into a subset of a hyperspace. But first, let us recall some further properties of hyperspaces and their induced maps.

Observation 2.4. If a space X is identified with the family of its singletons $[X]^1$, then it becomes a subspace of $\mathcal{P}(X)$ with respect to all τ_V^-, τ_V^+, and τ_V since for every open $U \subseteq X$ we have $U^- \cap [X]^1 = U^+ \cap [X]^1 = [U]^1$.

Notation 2.5. Let $f\colon X \to Y$ be a map between sets. We define

- $f^*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ by $f^*(A) = f[A]$,
- $f^{-1*}\colon Y \to \mathcal{P}(X)$ by $f^{-1*}(y) = f^{-1}(y)$,
- $f^{-1**}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ by $f^{-1**}(B) = f^{-1}[B]$.

This notation comes from [8, 5.9].

Proposition 2.6. Let $f\colon X \to Y$ be a map between topological spaces. Recall the properties of the induced maps defined above (see [8, 5.10]).

(i) f is continuous $\iff$ f^* is τ_V^--cont. $\iff$ f^* is τ_V^+-cont. $\iff$ f^* is τ_V-cont.

(ii) f is an embedding $\iff$ f^* is τ_V^--emb. $\iff$ f^* is τ_V^+-emb. $\iff$ f^* is τ_V-emb.

(iii) f is an open embedding $\iff$ f^* is τ_V^+-open emb. $\iff$ f^* is τ_V-open emb.

(iv) f is a closed embedding $\iff$ f^* is τ_V^--closed emb. $\iff$ f^* is τ_V-closed emb.

(v) f is open $\iff$ f^{-1*} is τ_V^--continuous $\iff$ f^{-1**} is τ_V^--continuous.

(vi) f is closed $\iff$ f^{-1*} is τ_V^+-continuous $\iff$ f^{-1**} is τ_V^+-continuous.

(vii) f is closed and open $\iff$ f^{-1*} is τ_V-continuous $\iff$ f^{-1**} is τ_V-continuous.

(viii) f is continuous $\implies$ f^{-1*} is τ_V-(closed and open) onto its image.

Proof. This was mainly proved by Michael [8, 5.10]. We just add that f does not have to be onto, the claims are often true also for lower and upper Vietoris topology separately, and even when the empty set is allowed. Compact proofs may be obtained by considering the following equalities.

$$
\begin{aligned}
(f^*)^{-1}[B^-] &= f^{-1}[B]^- & (f^*)^{-1}[B^+] &= f^{-1}[B]^+ \\
(f^*)[A^-] &= f[A]^- \cap \operatorname{rng}(f^*) & (f^*)[A^+] &= f[A]^+ \subseteq \operatorname{rng}(f^*) \quad \text{for } f \text{ injective} \\
(f^{-1*})^{-1}[A^-] &= f[A] & (f^{-1*})^{-1}[A^+] &= f^\forall[A] := Y \setminus f[X \setminus A] \\
(f^{-1**})^{-1}[A^-] &= f[A]^- & (f^{-1**})^{-1}[A^+] &= f^\forall[A]^+
\end{aligned}
$$

$$
(f^{-1*})[B] = \begin{cases} f^{-1}[B]^- \cap \operatorname{rng}(f^{-1*}) & \text{if } B \subseteq \operatorname{rng}(f) \\ f^{-1}[B]^+ \cap \operatorname{rng}(f^{-1*}) & \text{if } B \nsubseteq \operatorname{rng}(f) \text{ or } f \text{ is onto} \end{cases}
$$

Regarding the embeddings, if f is an embedding and $U \subseteq X$ is open, then $f[U] = V \cap \operatorname{rng}(f)$ for some open $V \subseteq Y$. Therefore, we have

$$
\begin{aligned}
f^*[U^-] &= f[U]^- \cap \operatorname{rng}(f) = (V \cap \operatorname{rng}(f))^- \cap \operatorname{rng}(f^*) = V^- \cap \operatorname{rng}(f^*), \\
f^*[U^+] &= f[U]^+ = (V \cap \operatorname{rng}(f))^+ = V^+ \cap \operatorname{rng}(f^*),
\end{aligned}
$$

and so f^* is a τ_V^--, τ_V^+- and hence a τ_V-embedding. Regarding the closedness and openness, observe that $\operatorname{rng}(f^*) = \operatorname{rng}(f)^+$, so if $\operatorname{rng}(f)$ is open, then $\operatorname{rng}(f^*)$ is τ_V^+-open, and if $\operatorname{rng}(f)$ is closed, then $\operatorname{rng}(f^*)$ is τ_V^--closed. For the backward implications we may use Observation 2.4 since f may be viewed as a restriction $[X]^1 \to [Y]^1$ of f^*, and $\operatorname{rng}(f)$ is essentially $\operatorname{rng}(f^*) \cap [Y]^1$. □

Definition 2.7. A composition $\mathcal{A}(q\colon A \to B)$ is called a *strong composition* if the composition map q is closed and open and $|B \setminus \operatorname{rng}(q)| \leq 1$. A class of spaces $\mathcal{C}$ is called *strongly compactifiable* (or *strongly Polishable*) if there is a strong compact (or strong Polish) composition of $\mathcal{C}$.

The strongness of a composition means that the corresponding decomposition of A is continuous (closedness correspond to upper semi-continuity and openness to lower semi-continuity). Note that the rather technical condition $|B \setminus \operatorname{rng}(q)| \leq 1$ and also clopenness of $\operatorname{rng}(q)$ can be obtained for every composition by removing $B \setminus \operatorname{rng}(q)$ and then eventually adding a clopen point (Remark 1.13). Also, the closedness of q is trivial for compact compositions.

Construction 2.8 (from composition to hyperspace)**.** To every composition $\mathcal{A}(q\colon A \to B)$ we assign the disjoint family $\mathcal{F}_{\mathcal{A}} := \{q^{-1}(b) : b \in B\} \subseteq \mathcal{P}(A)$.

We have $q^{-1*}\colon B \twoheadrightarrow \mathcal{F}_{\mathcal{A}} \subseteq \mathcal{P}(A)$, so we have two natural topologies on $\mathcal{F}_{\mathcal{A}}$ – the quotient topology induced by q^{-1*} from B, and the subspace topology induced from the hyperspace $\mathcal{P}(A)$. By Proposition 2.6 the Vietoris topology is finer than the quotient topology. The converse holds if and only if q is both closed and open. The map q^{-1*} is a homeomorphism with respect to the quotient topology if and only if it is a bijection, which happens if and only if $|B \setminus \operatorname{rng}(q)| \leq 1$. Therefore, $\mathcal{F}_{\mathcal{A}}$ is homeomorphic to B via q^{-1*} if and only if the composition $\mathcal{A}$ is strong.

In this case, if $\mathcal{A}$ is a compact (or Polish) composition of compacta, then $\mathcal{F}_{\mathcal{A}}$ is compact (or Polish), and so it is a closed (or G_δ) subset of the compact (or Polish) hyperspace $\mathcal{K}(A)$.

Observation 2.9. If $\mathcal{A}(q\colon A \to B)$ is a strong composition, then the family $\mathcal{F}_{\mathcal{A}}$ is closed in every Hausdorff space $\mathcal{H} \subseteq \mathcal{P}(A)$ containing it.

Proof. Let us consider the family $\mathcal{F}^{\bigcup} := \{F \in \mathcal{H} : q^{-1}[q[F]] = F\}$, which is closed since $q^{-1**} \circ q^*$ is continuous and $\mathcal{H}$ is Hausdorff, and the family $\mathcal{F}^{\downarrow} := (q^*)^{-1}[[B]^{\leq 1}]$, which is also closed since $B \cong \mathcal{F}_{\mathcal{A}}$ is Hausdorff, and so $[B]^{\leq 1}$ is closed in $\mathcal{P}(B)$. To conclude, it is enough to observe that $\mathcal{F}_{\mathcal{A}} \subseteq \mathcal{F}^{\bigcup} \cap \mathcal{F}^{\downarrow} \subseteq \mathcal{F}_{\mathcal{A}} \cup \{\emptyset\}$. □

Lemma 2.10. Let X, Y be topological spaces, and let $R \subseteq X \times Y$. Let us consider the map $\rho\colon Y \to \mathcal{P}(X)$ defined by $\rho(y) := R^y$.

(i) The map $\pi_Y{\restriction}_R\colon R \to Y$ is open if and only if the map ρ is τ_V^--continuous.

(ii) The map $\pi_Y{\restriction}_R\colon R \to Y$ is closed if and only if the map ρ is τ_V^+-continuous and every set $R^y \times \{y\}$ has a basis of rectangular neighborhoods (r.n.b.), i.e. every its neighborhood in R contains a neighborhood of form $R \cap (U \times V)$ for some open sets U and V. The r.n.b. condition is satisfied if $\operatorname{rng}(\rho) \subseteq \mathcal{K}(X)$.

Proof. The necessity of $\tau_V^{+/-}$-continuity follows from equality $\rho = \pi_X^* \circ (\pi_Y{\restriction}_R)^{-1*}$ and from Proposition 2.6. The open case follows from equality $\pi_Y[R \cap (U \times V)] = \{y \in V : R^y \cap U \neq \emptyset\} = \rho^{-1}[U^-] \cap V$. The map $\pi_Y{\restriction}_R$ is closed if and only if for every closed $F \subseteq R$ and every $y \in Y \setminus \pi_Y[F]$ there is an open neighborhood W of y disjoint with $\pi_Y[F]$. Considering $R \cap (X \times W)$ gives us necessity of the r.n.b. condition. On the other hand, if $U \times V$ is an open neighborhood of $R^y \times \{y\} \neq \emptyset$ disjoint with F, then we put $W := \rho^{-1}[U^+] \cap V$. Note that $z \in \rho^{-1}[U^+]$ if and only if $R^z \subseteq U$. Hence, if $(x, z) \in R$ and $z \in W$, then $(x, z) \in U \times V$ and so it cannot be in F. If $R^y = \emptyset$, then we put $W := Y \setminus \pi_Y[R]$, which is open since $\pi_Y[R] = \rho^{-1}[X^-]$ and X^- is τ_V^+-closed. The r.n.b. condition holds if every $R^y \times \{y\}$ is compact by the tube lemma [4, 3.1.15]. □

Corollary 2.11. Let $\mathcal{A}_\mathcal{F}(q\colon A_\mathcal{F} \to \mathcal{F})$ be the composition obtained by Construction 2.1 from a family $\mathcal{F} \subseteq \mathcal{P}(X)$. We have that the map q is open and $|\mathcal{F} \setminus \operatorname{rng}(q)| \leq 1$. If $\mathcal{F} \subseteq \mathcal{K}(X)$, then q is also closed, and hence the composition is strong.

Proof. The map q is the projection $\mathcal{R}_\in \cap (X \times \mathcal{F}) \to \mathcal{F}$, so we may use Lemma 2.10. The corresponding map ρ is $\mathrm{id}\colon \mathcal{F} \to \mathcal{P}(X)$, which is both τ_V^-- and τ_V^+-continuous. The fact that $|\mathcal{F} \setminus \operatorname{rng}(q)| \leq 1$ is clear since there is only one empty set. □

Corollary 2.12. Let $\mathcal{A}(q\colon A \to B)$ be a composition of spaces $(A_b)_{b\in B}$, let $f\colon B' \to B$ be a continuous map, and let $\mathcal{A}'(q'\colon A' \to B')$ be the pullback of $\mathcal{A}$ along f (Construction 1.6). If q is open, so is q'. If q is closed and every space A_b is compact, then q is also closed. It follows that strong compositions of compact spaces are preserved by pullbacks (such that $|f^{-1}[B \setminus \operatorname{rng}(f)]| \leq 1$).

Proof. We apply Lemma 2.10 to $A' \subseteq A \times B'$. The corresponding map ρ is $q^{-1*} \circ f$, which is τ_V^-- (or τ_V^+-)continuous if q is open (or closed) by Proposition 2.6. □

By putting all the previous claims and propositions together, we obtain the following characterizations. These may be compared with Theorem 1.10 and 1.11.

Theorem 2.13. The following conditions are equivalent for a class of spaces $\mathcal{C}$.

(i) $\mathcal{C}$ is strongly compactifiable.

(ii) There is a metrizable compactum X and a closed family $\mathcal{F} \subseteq \mathcal{K}(X)$ such that $\mathcal{F} \cong \mathcal{C}$.

(iii) There is a closed zero-dimensional disjoint family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ such that $\mathcal{F} \cong \mathcal{C}$.

Theorem 2.14. The following conditions are equivalent for a class of spaces $\mathcal{C}$.

(i) $\mathcal{C}$ is a strongly Polishable class of compacta.

(ii) There is a Polish space X and an analytic family $\mathcal{F} \subseteq \mathcal{K}(X)$ such that $\mathcal{F} \cong \mathcal{C}$.

(iii) There is a G_δ zero-dimensional disjoint family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ such that $\mathcal{F} \cong \mathcal{C}$.

(iv) There is a closed zero-dimensional disjoint family $\mathcal{F} \subseteq \mathcal{K}((0,1)^\omega)$ such that $\mathcal{F} \cong \mathcal{C}$.

Proof. Let $\mathcal{F} \subseteq \mathcal{K}(X)$. Construction 2.1 gives us the corresponding composition $\mathcal{A}_\mathcal{F}$, which is strong by Corollary 2.11. If X is a metrizable compactum and $\mathcal{F}$ is closed, then the composition $\mathcal{A}_\mathcal{F}$ is compact. If X is Polish and $\mathcal{F}$ is analytic, then there is a continuous surjection $f\colon Y \twoheadrightarrow \mathcal{F}$ from a Polish space Y such that $|f^{-1}(\emptyset)| \leq 1$. The pullback of $\mathcal{A}_\mathcal{F}$ along f (Construction 1.6) is a composition of $\mathcal{F}$ that is Polish by Lemma 1.9 and strong by Corollary 2.12.

On the other hand, let $\mathcal{A}(q\colon A \to B)$ be a strong compact (or Polish) composition of $\mathcal{C}$. Without loss of generality, B is zero-dimensional (we use Construction 1.6 as in

Theorem 1.10 and 1.11 together with Corollary 2.12). Construction 2.8 gives us the corresponding zero-dimensional disjoint family $\mathcal{F}_\mathcal{A} \subseteq \mathcal{K}(A)$, which is closed by Observation 2.9. The space A embeds into $[0,1]^\omega$, and so $\mathcal{K}(A)$ embeds into $\mathcal{K}([0,1]^\omega)$ by Proposition 2.6. In the compact case, $\mathcal{F}_\mathcal{A}$ is compact and so closed in $\mathcal{K}([0,1]^\omega)$. In the Polish case, $\mathcal{F}_\mathcal{A}$ is Polish and so G_δ in $\mathcal{K}([0,1]^\omega)$. Moreover, A embeds into $(0,1)^\omega$ as a closed subspace [6, 4.17], and so $\mathcal{K}(A)$ embeds into $\mathcal{K}((0,1)^\omega)$ as a closed subspace by Proposition 2.6. Hence, $\mathcal{F}_\mathcal{A}$ becomes a closed subset of $\mathcal{K}((0,1)^\omega)$.

The remaining implications are trivial. □

Proposition 2.15. Let $\mathcal{A}(q\colon A \to B)$ be a Polish composition of compacta such that the composition map q is closed. The family $\mathcal{F} \subseteq \mathcal{K}(A)$ obtained via Construction 2.8 is G_δ.

Proof. Let us start with the closed family $\mathcal{H} := (q^*)^{-1}[\mathcal{B}] \cap \mathcal{K}(A)$ where $\mathcal{B} = [B]^{\leq 1}$ or $[B]^1$ depending on whether $\emptyset \in \mathcal{F}$. For every $F \subseteq A$ let us put $\widehat{F} := q^{-1}[q[F]]$. For $F \in \mathcal{H}$ we have that $\widehat{F}$ is the only member of $\mathcal{F}$ such that $F \subseteq \widehat{F}$. Let d be a compatible metric on A and let $\mathcal{G}_n := \{F \in \mathcal{K}(A) : d_H(F, \widehat{F}) < \frac{1}{n}\}$. Clearly, $\mathcal{F} = \mathcal{H} \cap \bigcap_{n\in\mathbb{N}} \mathcal{G}_n$, so it is enough to show that each $\mathcal{G}_n$ is open.

Let $F \in \mathcal{G}_n$. There is $\varepsilon > 0$ such that $d_H(F, \widehat{F}) < \frac{1}{n} - 4\varepsilon$. Note that for every H the inequality $d_H(H, \widehat{H}) < r$ is equivalent to $N_{r-\delta}(H) \supseteq \widehat{H}$ for some $\delta > 0$. Since F is compact, there are points $\{x_i : i < k\} \subseteq F$ such that $F \subseteq \bigcup_{i<k} B_d(x_i, \varepsilon)$. Since the maps q^{-1**} and q^* are τ_V^+-continuous, there is an open set $U \subseteq A$ containing F such that $\widehat{H} \in N_\varepsilon(\widehat{F})^+$ for every $H \in U^+$.

Let $\mathcal{U} := \bigcap_{i<k} B_d(x_i, \varepsilon)^- \cap U^+ \cap \mathcal{K}(A)$ and let $H \in \mathcal{U}$. We have $N_\varepsilon(H) \supseteq \{x_i : i < k\}$, so $N_{2\varepsilon}(H) \supseteq F$. Since $N_{\frac{1}{n}-4\varepsilon}(F) \supseteq \widehat{F}$, we have $N_{\frac{1}{n}-2\varepsilon}(H) \supseteq \widehat{F}$, and since $N_\varepsilon(\widehat{F}) \supseteq \widehat{H}$, we have $N_{\frac{1}{n}-\varepsilon}(H) \supseteq \widehat{H}$. Therefore, $d_H(H, \widehat{H}) < \frac{1}{n}$, and $\mathcal{U}$ is an open neighborhood of F that is contained in $\mathcal{G}_n$. □

Corollary 2.16. Every countable sum of compact compositions is a strong Polish composition, and hence every compactifiable class is strongly Polishable. Also, in the definition of strong Polishability it is enough that the witnessing composition map is closed.

We may formulate a characterization of compactifiability using families in hyperspaces.

Theorem 2.17. The following conditions are equivalent for a class of spaces $\mathcal{C}$.

(i) $\mathcal{C}$ is compactifiable.

(ii) There is a metrizable compactum X and a family $\mathcal{F} \subseteq \mathcal{K}(X)$ such that $\mathcal{F} \cong \mathcal{C}$ and $(\mathcal{F}, \tau)$ is a metrizable compactum for a topology $\tau \supseteq \tau_V^+$.

(iii) There is a G_δ disjoint family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ such that $\mathcal{F} \cong \mathcal{C}$ and $(\mathcal{F}, \tau_V^+)$ is a zero-dimensional metrizable compactum.

Proof. For (ii) $\implies$ (i) we use Construction 2.1 on $(\mathcal{F}, \tau)$. We obtain a composition of $\mathcal{C}$ that is compact by Observation 2.2.

(i) $\implies$ (iii). Let $\mathcal{A}(q\colon A \to B)$ be a compact composition of $\mathcal{C}$. We may suppose that B is zero-dimensional by Theorem 1.10, that $|B \setminus \operatorname{rng}(q)| \leq 1$ by Observation 1.13, and that $A \subseteq [0,1]^\omega$. The family $\mathcal{F}_\mathcal{A}$ obtained by Construction 2.8 is disjoint and by Proposition 2.15 G_δ in $\mathcal{K}(A) \subseteq \mathcal{K}([0,1]^\omega)$. Since $|B \setminus \operatorname{rng}(q)| \leq 1$ and the map q is closed, $q^{-1*}\colon B \to (\mathcal{F}, \tau_V^+)$ is a homeomorphism.

(iii) $\implies$ (ii) is trivial. $\square$

Question 2.18. Is there a similar characterization for Polishable classes?

Question 2.19. Figure 1 summarizes the implications between composition-related properties and descriptive complexity of the corresponding subsets of the space of all metrizable compacta $\mathcal{K}([0,1]^\omega)$. We do not know which implications can be reversed. Namely, we have the following questions.

(i) Is there a compactifiable class that is not strongly compactifiable?

(ii) Is there a strongly Polishable class that is not compactifiable?

(iii) Is there a Polishable class that is not strongly Polishable?

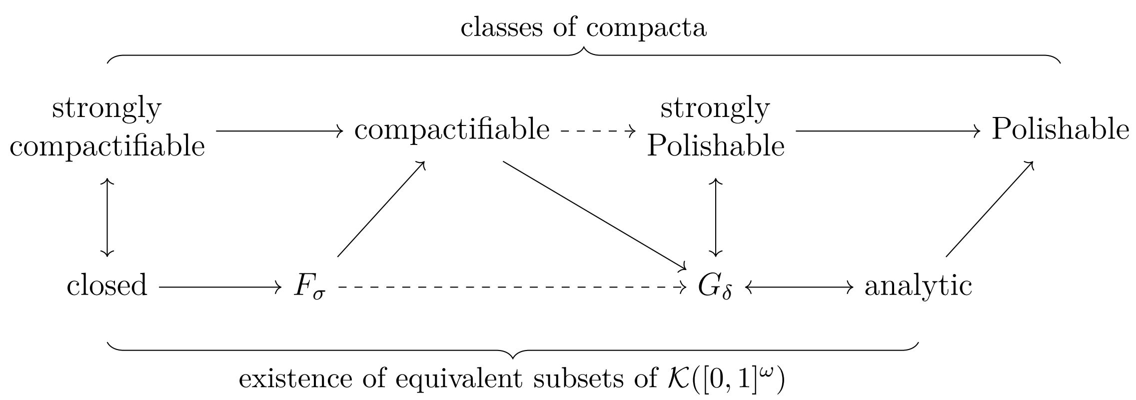

Figure 1: Implications between the classes considered.

To summarize, this chapter relates (strong) compactifiability or Polishability of a class of metrizable compacta to the lowest complexity of its realizations in the hyperspace $\mathcal{K}([0,1]^\omega)$. We study this complexity in $\mathcal{K}([0,1]^\omega)$ up to the equivalence in [1].

2.1 The Wijsman hypertopologies

So far we have considered mostly the hyperspace of all compact subsets $\mathcal{K}(X)$ endowed with the Vietoris topology (or equivalently Hausdorff metric topology for metrizable X).

There we have the $1:1$ correspondence between subsets of the hyperspace and strong compositions (Construction 2.1 and 2.8). On the other hand, we are limited to Polishable classes of compact rather than Polish spaces.

For a Polish space X we would like $\mathcal{C}l(X)$ to be Polish as well, but the Vietoris topology on $\mathcal{C}l(X)$ is not metrizable unless X is compact, and the Hausdorff metric topology is not separable unless X is compact. That is why we consider so-called *Wijsman topology*. The Wijsman topology induced by the metric d is the one projectively generated by the family $\{d(x,\cdot)\colon \mathcal{C}l(X) \to \mathbb{R}\}_{x\in X}$. It was shown in [2] that $\mathcal{C}l(X)$ with a Wijsman topology is a Polish space for a Polish space X. Usually the Wijsman topology is defined only on $\mathcal{C}l(X)\setminus \{\emptyset\}$, and is then extended to $\mathcal{C}l(X)$ in a way related to the one-point compactification. For our purposes we may use the projectively generating definition directly to $\mathcal{C}l(X)$, which results in $\{\emptyset\}$ being clopen.

The Wijsman topology is coarser than both Vietoris and Hausdorff metric topology, but in general they are not equal even on $\mathcal{K}(X)$. In general, $\mathcal{K}(X)$ is an $F_{\sigma\delta}$-subspace of $\mathcal{C}l(X)$ with respect to the Wijsman topology, but it is not necessarily G_δ [2]. Given a metric d on X we may identify a set $A \in \mathcal{C}l(X)$ with the function $d(\cdot, A)\colon X \to \mathbb{R}$. Therefore, the Wijsman topology is inherited from the space of all continuous functions $C(X,\mathbb{R})$ with the topology of pointwise convergence. On the other hand, $d_H(A,B) = \sup_{x\in X}|d(x,A) - d(x,B)|$, so the Hausdorff metric topology is inherited from $C(X,\mathbb{R})$ with the topology of uniform convergence.

The Observation 2.2 holds also for the Wijsman topologies.

Observation 2.20. If X is a metrizable space and $\mathcal{C}l(X)$ is endowed with a Wijsman topology, then $\mathcal{R}_\in \cap (X \times \mathcal{C}l(X))$ is closed in $X \times \mathcal{C}l(X)$.

Proof. If $F \in \mathcal{C}l(X)$ and $x \in X \setminus F$, then $r := d(x,F) > 0$. We put $U = \{y \in X : d(x,y) < \frac{r}{2}\}$ and $\mathcal{V} = \{H \in \mathcal{C}l(X) : d(x,H) > \frac{r}{2}\}$, so $U \times \mathcal{V}$ is a neighborhood of (x,F) disjoint with $\mathcal{R}_\in$. □

It follows that we may use Construction 2.1 also for Wijsman hyperspaces to obtain Polish compositions. Note that since the Wijsman topologies are coarser than the Vietoris topology, they have more analytic subsets.

Proposition 2.21. If X is a Polish space and $\mathcal{C}l(X)$ is endowed with a Wijsman topology, then every analytic subset of $\mathcal{C}l(X)$ is a Polishable class of Polish spaces.

Remark 2.22. Since every Polish space can be embedded as a closed subspace to $(0,1)^\omega$, the hyperspace $\mathcal{C}l((0,1)^\omega)$ endowed with a Wijsman topology may be viewed as a Polish space of all Polish spaces.

Question 2.23. Can more of the compactifiability theory from Vietoris hyperspaces be done in the setting of $\mathcal{C}l(X)$ or $\mathcal{C}l((0,1)^\omega)$ with a Wijsman topology?

3 Induced classes

In this section we shall analyze how the properties of being compactifiable and Polishable are preserved under various modifications and constructions of induced classes.

Proposition 3.1. Strongly compactifiable, compactifiable, strongly Polishable, and Polishable classes are stable under countable unions.

Proof. For compactifiable and Polishable classes this is Observation 1.14. Let $\mathcal{C}_n$, $n \in \omega$, be strongly Polishable classes. By Theorem 2.14 each of them is equivalent to an analytic family $\mathcal{F}_n \subseteq \mathcal{K}([0,1]^\omega)$. We have $\bigcup_{n\in\omega} \mathcal{C}_n \cong \bigcup_{n\in\omega} \mathcal{F}_n$, which is also analytic and hence strongly Polishable. In the strongly compactifiable case we proceed analogously, but end up with an F_σ family $\bigcup_{n\in\omega} \mathcal{F}_n$. The conclusion follows from the non-trivial fact, that every F_σ family in $\mathcal{K}([0,1]^\omega)$ is equivalent to a closed family, and hence is strongly compactifiable [1, Theorem 3.6]. □

Remark 3.2. In the previous proof we have used the fact that $\bigcup_{i\in I} \mathcal{C}_i \cong \bigcup_{i\in I} \mathcal{D}_i$ for every collection of equivalent classes $\mathcal{C}_i \cong \mathcal{D}_i$, $i \in I$. However, it is not necessary that even $\mathcal{C}_i \cap \mathcal{C}_j \cong \mathcal{D}_i \cap \mathcal{D}_j$, so we cannot use the same argument for proving preservation under intersections, but see Proposition 3.31.

Observation 3.3. Let X be a metric space. The map $\operatorname{diam}\colon \mathcal{P}(X) \to [0,\infty)$ is both (τ_V^+, τ_U)- and (τ_V^-, τ_L)-continuous, where τ_U and τ_L are the upper and lower semicontinous topologies on $[0,\infty)$. It follows that diam is continuous.

Proof. If $\operatorname{diam}(A) < r$, then there is $\varepsilon > 0$ such that $\operatorname{diam}(N_\varepsilon(A)) < r$. Hence, $\operatorname{diam}(A') < r$ for every $A' \in N_\varepsilon(A)^+$. If $\operatorname{diam}(A) > r$, then there are points $x, y \in A$ and $\varepsilon > 0$ such that $d(x,y) \geq r + 2\varepsilon$. Hence, $\operatorname{diam}(A') > r$ for every $A' \in B(x,\varepsilon)^- \cap B(y,\varepsilon)^-$. □

Corollary 3.4. Let $\mathcal{A}(q\colon A \to B)$ be a compact composition of a family $(A_b)_{b\in B}$. For every $\varepsilon > 0$ the set $B_\varepsilon := \{b \in B : \operatorname{diam}(A_b) \geq \varepsilon\}$ is closed, and the set $B_0 := \{b \in B : \operatorname{diam}(A_b) > 0\}$ is F_σ. It follows that the corresponding families of spaces are also compactifiable.

Proof. The map $(\operatorname{diam} \circ q^{-1*})\colon B \to [0,\infty)$ is upper semi-continuous since q^{-1*} is τ_V^+-continuous and diam is (τ_V^+, τ_U)-continuous by Observation 3.3. Note that the intervals $[\varepsilon,\infty)$ are τ_U-closed, and so the interval $(0,\infty)$ is τ_U-F_σ. □

In definitions of many natural classes of compacta, sometimes degenerate spaces are included, sometimes they are excluded. The following proposition shows that with respect to compactifiability, it does not matter.

Proposition 3.5. If a class of metrizable compacta $\mathcal{C}$ is strongly compactifiable, compactifiable, strongly Polishable, or Polishable, then so are the classes $\mathcal{C} \cup \{\emptyset\}$, $\mathcal{C} \setminus \{\emptyset\}$, $\mathcal{C} \cup \{1\}$, and $\mathcal{C}_{>1}$, where 1 denotes a one-point space and $\mathcal{C}_{>1}$ denotes the class of all nondegenerate members of $\mathcal{C}$.

Proof. The additive cases $\mathcal{C} \cup \{\emptyset\}$ and $\mathcal{C} \cup \{1\}$ follow directly from Proposition 3.1. The case $\mathcal{C} \setminus \{\emptyset\}$ for compactifiable and Polishable classes is covered by Observation 1.13. For strongly compactifiable and Polishable classes, it is easy since $\{\emptyset\}$ is clopen in $\mathcal{K}([0,1]^\omega)$, and so removing it from a realization of $\mathcal{C}$ does not change its complexity. Similarly, we obtain the $\mathcal{C}_{>1}$ case since the degenerate sets form a closed subset of the hyperspace. Hence, removing degenerate spaces from a realization of $\mathcal{C}$ preserves the G_δ complexity and turns a closed family to an F_σ family (since the hyperspace is metrizable), which is enough for $\mathcal{C}_{>1}$ to be strongly compactifiable by Proposition 3.1.

It remains to cover the $\mathcal{C}_{>1}$ case for compactifiable and Polishable $\mathcal{C}$. Let $\mathcal{A}(q\colon A \to B)$ be a composition of $\mathcal{C}$ and let $C := \{b \in B : |q^{-1}(b)| > 1\}$. On one hand, if A is a metric space, then C is the preimage $(q^{-1*})^{-1}[\mathcal{G}]$ of the family $\mathcal{G} := \{K \in \mathcal{K}(A) : \operatorname{diam}(K) > 0\}$, which is τ_V^+-open by Observation 3.3. Hence, if $\mathcal{A}$ is a compact composition, then q is closed, q^{-1*} is τ_V^+-continuous, and C is open and, in particular, F_σ, and so $\mathcal{C}_{>1}$ is compactifiable. On the other hand, C is the projection of the set $\{(a, a', b) \in A \times A \times B : q(a) = b = q(a'),\ a \neq a'\}$, which an intersection of a closed set and an open set. Hence, if $\mathcal{A}$ is a Polish composition, then C is analytic, and so $\mathcal{C}_{>1}$ is Polishable. □

Notation 3.6. Let $\mathcal{C}$ be a class of topological spaces.

- $\mathcal{C}^{\downarrow}$ denotes the class of all subspaces of members of $\mathcal{C}$.
- $\mathcal{C}^{\uparrow}$ denotes the class of all superspaces of members of $\mathcal{C}$.
- $\mathcal{C}^{\cong}$ denotes the class of all homeomorphic copies of members of $\mathcal{C}$.
- $\mathcal{C}^{\twoheadrightarrow}$ denotes the class of all continuous images of members of $\mathcal{C}$.
- $\mathcal{C}^{\twoheadleftarrow}$ denotes the class of all continuous preimages of members of $\mathcal{C}$, i.e. the class of all spaces than can be continuously mapped onto a member of $\mathcal{C}$.

We also denote the classes of all metrizable compacta and all continua by $\mathbf{K}$ and $\mathbf{C}$, respectively, so we can denote e.g. the class of all subcontinua of members of $\mathcal{C}$ by $\mathcal{C}^{\downarrow} \cap \mathbf{C}$. For a topological space X and a family $\mathcal{F} \subseteq \mathcal{P}(X)$, the notation $\mathcal{F}^{\uparrow} \cap \mathcal{P}(X)$ means "all supersets of members of $\mathcal{F}$ that are subsets of X, all endowed with the subspace topology". This is consistent with the definition of $\mathcal{C}^{\uparrow}$ above when $\mathcal{P}(X)$ is viewed as a set of topological spaces.

Observation 3.7. If $\mathcal{C}$ is a strongly compactifiable or strongly Polishable class of compacta, then so is the class $\mathcal{C} \cap \mathbf{C}$ of all continua from $\mathcal{C}$.

Proof. There is a closed (or G_δ) family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ such that $\mathcal{F} \cong \mathcal{C}$. We have $\mathcal{C} \cap \mathbf{C} \cong \mathcal{F} \cap \mathcal{C}([0,1]^\omega)$, which is closed (or G_δ) not only in $\mathcal{C}([0,1]^\omega)$, but also in $\mathcal{K}([0,1]^\omega)$ since $\mathcal{C}(X)$ is closed in $\mathcal{K}(X)$ for every Hausdorff space X. □

Question 3.8. Is the previous observation true also for compactifiable and Polishable classes?

Proposition 3.9. If $\mathcal{C}$ is a compactifiable (or Polishable) class, then $\mathcal{C}^{\downarrow} \cap \mathbf{K}$ is a strongly compactifiable (or strongly Polishable) class.

Proof. Let $\mathcal{A}(q\colon A \to B)$ be a witnessing composition. It is enough to observe that $\mathcal{C}^{\downarrow} \cap \mathbf{K} \cong (q^*)^{-1}[[B]^{\leq 1}] \cap \mathcal{K}(A)$, which is a closed subset of $\mathcal{K}(A)$ since the family of all degenerate subspaces of B, $[B]^{\leq 1}$, is τ_V^--closed in $\mathcal{P}(B)$. □

Corollary 3.10. Every hereditary class of metrizable compacta or continua with a universal element is strongly compactifiable. This includes the classes of all metrizable: compacta, totally disconnected compacta, continua, continua with dimension at most n, chainable continua, tree-like continua, dendrites, and others.

In order to obtain a similar result for the induced class $\mathcal{C}^{\uparrow} \cap \mathbf{K}$, we shall analyze the set $\mathcal{F}^{\uparrow} \cap \mathcal{K}(X)$ for a family $\mathcal{F} \subseteq \mathcal{K}(X)$. First, we shall need the following refinement of Observation 2.2.

Observation 3.11. If X is a Hausdorff space, then the inclusion relation of compacts sets is closed, i.e. $\mathcal{R}_{\subseteq} \cap \mathcal{K}(X)^2$ is closed in $\mathcal{K}(X)^2$ where $\mathcal{R}_{\subseteq} := \{(A,B) \in \mathcal{P}(X)^2 : A \subseteq B\}$.

Proof. If $x \in A \setminus B$ for some $A, B \in \mathcal{K}(X)$, then there are disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $B \subseteq V$, and hence $U^- \times V^+$ is an open neighborhood of (A,B) disjoint with $\mathcal{R}_{\subseteq}$. □

Lemma 3.12. Let X be a topological space.

(i) The map $\mathcal{K}\colon \mathcal{K}(X) \to \mathcal{K}(\mathcal{K}(X))$ that maps every compact set $A \subseteq X$ to its compact hyperspace $\mathcal{K}(A)$ is continuous.

(ii) The projection $\pi_2\colon \mathcal{R}_{\subseteq} \cap \mathcal{K}(X)^2 \to \mathcal{K}(X)$ is closed and open.

Proof. Let R denote the relation $\mathcal{R}_{\subseteq} \cap \mathcal{K}(X)^2$. Observe that for every $A \in \mathcal{K}(X)$ we have $R^A = A^+ \cap \mathcal{K}(X) = \mathcal{K}(A)$, which is compact. Hence, (i) $\iff$ (ii) by Lemma 2.10 since $\mathcal{K}$ is the map ρ for R. We shall prove (i). In fact, $\mathcal{K}$ is both $(\tau_V^-, \tau_V^-(\tau_V))$-continuous and $(\tau_V^+, \tau_V^+(\tau_V))$-continuous.

Let $A \in \mathcal{K}(X)$ and $\mathcal{V} \subseteq \mathcal{K}(X)$ open such that $\mathcal{K}(A) \in \mathcal{V}^-$ (or $\mathcal{V}^+$). To prove that $\mathcal{K}$ is τ_V^--continuous (or τ_V^+-continuous) it is enough to find $\mathcal{U}$ a τ_V^--open (or τ_V^+-open) neighborhood of A in $\mathcal{K}(X)$ such that $\mathcal{K}[\mathcal{U}] \subseteq \mathcal{V}^-$ (or $\mathcal{V}^+$). The set $\mathcal{V}$ is of the from

$\bigcup_{i\in I}\bigcap_{j\in J_i}\mathcal{V}_{i,j}$ where J_i are finite sets and every $\mathcal{V}_{i,j}$ is V^- or V^+ for some open set $V \subseteq X$.

Let us start with the τ_V^--continuity. Since $(\bigcup_{i\in I}\bigcap_{j\in J_i}\mathcal{V}_{i,j})^- = \bigcup_{i\in I}(\bigcap_{j\in J_i}\mathcal{V}_{i,j})^-$, we may suppose without loss of generality that $\mathcal{V} = \bigcap_{j<m} U_j^+ \cap \bigcap_{i<n} V_i^-$ for some open sets $U_j, V_i \subseteq X$. Also, $\bigcap_{j<m} U_j^+ = (\bigcap_{j<m} U_j)^+ =: U^+$. We put $\mathcal{U} := \bigcap_{i<n}(U \cap V_i)^-$. Since $\mathcal{K}(A) \in \mathcal{V}^-$, there is $B \in \mathcal{K}(A) \cap U^+ \cap \bigcap_{i<n} V_i^-$, so $B \cap (U \cap V_i) \neq \emptyset$ for every $i < n$, and since $A \supseteq B$, we have $A \in \mathcal{U}$. On the other hand, for every $B \in \mathcal{U}$ we may choose points $x_i \in B \cap U \cap V_i$ for $i < n$, and hence $\{x_i : i < n\} \in \mathcal{K}(B) \cap \mathcal{V}$, so $\mathcal{K}(B) \in \mathcal{V}^-$.

Now let us prove the τ_V^+-continuity. We have

$$\mathcal{K}(A) \subseteq \mathcal{V} = \bigcup_{i\in I}\bigcap_{j\in J_i}\mathcal{V}_{i,j} = \bigcap_{f\in\prod_{i\in I} J_i}\bigcup_{i\in I}\mathcal{V}_{i,f(i)} = \bigcap_{f\in F}\mathcal{V}_f$$

where $F := \prod_{i\in I} J_i$ and $\mathcal{V}_f := \bigcup_{i\in I}\mathcal{V}_{i,f(i)}$ for $f \in F$. Since $\mathcal{K}(A)$ is compact, we may suppose the sets I and F are finite. Since $(\bigcap_{f\in F}\mathcal{V}_f)^+ = \bigcap_{f\in F}\mathcal{V}_f^+$, it is enough to find for every $f \in F$ an open neighborhood $\mathcal{U}_f$ of A such that $\mathcal{K}[\mathcal{U}_f] \subseteq \mathcal{V}_f^+$. Therefore, we may suppose without loss of generality that $\mathcal{V} = \bigcup_{i<n} U_i^+ \cup \bigcup_{j<m} V_j^-$ for some open sets $U_i, V_j \subseteq X$. Also, $\bigcup_{j<m} V_j^- = (\bigcup_{j<m} V_j)^- =: V^-$.

We have $A \setminus V \in \mathcal{K}(A) \subseteq \mathcal{V} = \bigcup_{i<n} U_i^+ \cup V^-$, and $n > 0$ since $\emptyset \in \mathcal{K}(A) \setminus V^-$. Hence, there is some $i < n$ such that $A \setminus V \subseteq U_i$. We put $\mathcal{U} := (U_i \cup V)^+$. We have $A = (A \setminus V) \cup (A \cap V) \subseteq U_i \cup V$, so $A \in \mathcal{U}$. Let $B \in \mathcal{U}$. For every $C \in \mathcal{K}(B)$ we have $C \subseteq B \subseteq U_i \cup V$. Therefore, $\mathcal{K}(B) \subseteq (U_i \cup V)^+ \subseteq U_i^+ \cup V^- \subseteq \mathcal{V}$, and so $\mathcal{K}(B) \in \mathcal{V}^+$. □

Corollary 3.13. Let X be a topological space and $\mathcal{F} \subseteq \mathcal{K}(X)$.

(i) If $\mathcal{F}$ is closed, then $\mathcal{F}^\uparrow \cap \mathcal{K}(X)$ is closed.

(ii) If X is Polish and $\mathcal{F}$ is analytic, then $\mathcal{F}^\uparrow \cap \mathcal{K}(X)$ is analytic.

Proof. Observe that $\mathcal{F}^\uparrow \cap \mathcal{K}(X)$ is the π_2-image of the set $\mathcal{H} := \mathcal{R}_\subseteq \cap (\mathcal{F} \times \mathcal{K}(X))$. If $\mathcal{F}$ is closed, then $\mathcal{H}$ is closed in $\mathcal{R}_\subseteq \cap \mathcal{K}(X)^2$, and the claim follows since the map $\pi_2\restriction_{\mathcal{R}_\subseteq\cap\mathcal{K}(X)^2}$ is closed by Lemma 3.12. If $\mathcal{F}$ is analytic, then $\mathcal{H}$ is analytic since $\mathcal{K}(X)$ is Polish and $\mathcal{R}_\subseteq$ is closed in $\mathcal{K}(X)^2$ by Observation 3.11. The claim follows since the map π_2 is continuous. □

Proposition 3.14. If $\mathcal{C}$ is a strongly compactifiable or a strongly Polishable class of compacta, then so is the corresponding class of all metrizable compact superspaces $\mathcal{C}^\uparrow \cap \mathbf{K}$.

Proof. Let us denote the Hilbert cube by Q and let Z be a Z-set in Q that is homeomorphic to Q (it exists by [10, Lemma 5.1.3]). Our class $\mathcal{C}$ is equivalent to a closed or an analytic family $\mathcal{F} \subseteq \mathcal{K}(Z)$. We show that $\mathcal{C}^\uparrow \cap \mathbf{K}$ is equivalent to $\mathcal{F}^\uparrow \cap \mathcal{K}(Q)$, which is closed or analytic by Corollary 3.13. Clearly, every member of $\mathcal{F}^\uparrow \cap \mathcal{K}(Q)$ is homeomorphic to a member of $\mathcal{C}^\uparrow \cap \mathbf{K}$. On the other hand, let $K \in \mathcal{C}^\uparrow \cap \mathbf{K}$. We may suppose that

$K \in \mathcal{K}(Z)$. Since K has a subspace $C \in \mathcal{C}$, there is a homeomorphism $h\colon C \to F \in \mathcal{F}$. By [10, Theorem 5.3.7] h can be extended to a homeomorphism $\bar{h}\colon Q \to Q$. We have $K \cong \bar{h}[K] \in \mathcal{F}^{\uparrow} \cap \mathcal{K}(Q)$. □

Example 3.15. The class of all uncountable metrizable compacta is strongly compactifiable. Since every uncountable metrizable compactum contains a copy of the Cantor space, the class is equivalent to $\{2^\omega\}^{\uparrow} \cap \mathbf{K}$.

Proposition 3.16. If $\mathcal{C}$ is a strongly compactifiable or a strongly Polishable class of compacta, then so is the corresponding class of all metrizable compact continuous preimages $\mathcal{C}^{\twoheadleftarrow} \cap \mathbf{K}$.

Proof. Let Q denote the Hilbert cube $[0,1]^\omega$ and let $\mathcal{F} \subseteq \mathcal{K}(Q)$ be equivalent to $\mathcal{C}$. We will show that $\mathcal{C}^{\twoheadleftarrow} \cap \mathbf{K} \cong \mathcal{H} := \{K \in \mathcal{K}(Q \times Q) : \pi_2[K] \in \mathcal{F}\}$. Clearly, $\mathcal{H} \subseteq \mathcal{F}^{\twoheadleftarrow} \cap \mathbf{K}$. On the other hand, let $K \in \mathcal{F}^{\twoheadleftarrow} \cap \mathbf{K}$. There is an embedding $e\colon K \hookrightarrow Q$, and there is a continuous map $f\colon K \twoheadrightarrow Y \subseteq Q$ for some $Y \in \mathcal{F}$. The map $(e \triangle f)\colon K \to Q \times Q$ defined by $x \mapsto (e(x), f(x))$ is an embedding because of the embedding e, so $K \cong \operatorname{rng}(e \triangle f) \subseteq Q \times Q$. At the same time $\pi_2[\operatorname{rng}(e \triangle f)] = \operatorname{rng}(f) = Y$, and so $\operatorname{rng}(e \triangle f) \in \mathcal{H}$. Altogether, we have $\mathcal{C}^{\twoheadleftarrow} \cap \mathbf{K} \cong \mathcal{F}^{\twoheadleftarrow} \cap \mathbf{K} \cong \mathcal{H}$. Since $\mathcal{H} = (\pi_2^*)^{-1}[\mathcal{F}]$, if $\mathcal{F}$ is closed or analytic, so is $\mathcal{H}$. □

Example 3.17. Both the class of all connected metrizable compacta $\mathbf{C}$ and the class of all disconnected metrizable compacta $\mathbf{K} \setminus \mathbf{C}$ are strongly compactifiable – $\mathbf{C}$ by Corollary 3.10 and $\mathbf{K} \setminus \mathbf{C}$ since it is exactly $\{2\}^{\twoheadleftarrow} \cap \mathbf{K}$, where 2 denotes the two-point discrete space.

Example 3.18. The class of all metrizable compact spaces with infinitely many components is strongly compactifiable since it is exactly $\{\omega + 1\}^{\twoheadleftarrow} \cap \mathbf{K}$, where $\omega + 1$ denotes the convergent sequence.

Proof. For every metrizable compactum X we consider the equivalence $\sim$ induced by its components. $X/{\sim}$ may be viewed as a subspace of the Cantor space 2^ω. If X has infinitely many components, then $X/{\sim}$ contains a nontrivial converging sequence. The conclusion follows from the fact that every closed subspace of 2^ω is its retract. □

Example 3.19. Let $\mathcal{N}$ denote the class of all topological spaces that are *not* locally connected. The class of all non-Peano metrizable continua $\mathcal{N} \cap \mathbf{C}$ is strongly compactifiable since it is exactly $\{H\}^{\twoheadleftarrow} \cap \mathbf{C}$, where H denotes the harmonic fan. The class of all non-locally connected metrizable compacta $\mathcal{N} \cap \mathbf{K}$ is strongly compactifiable since it is exactly $\{\omega + 1, H\}^{\twoheadleftarrow} \cap \mathbf{K}$.

Proof. Since Peano continua are exactly continuous images of the unit interval, every continuum that maps continuously onto H (which is clearly not locally connected) is not Peano, so $\{H\}^{\twoheadleftarrow} \cap \mathbf{C} \subseteq \mathcal{N} \cap \mathbf{C}$. On the other hand, it is known that each member of $\mathcal{N} \cap \mathbf{C}$ maps continuously onto H [3].

Let $K \in \mathbf{K}$. By Example 3.18, K has infinitely many components if and only if K continuously maps onto $\omega+1$, and in this case K is not locally connected since it contains a convergent sequence such each its member and the limit are in different components. So we may suppose that K has finitely many components. If $K \in \mathcal{N}$, then one of the components is a non-Peano continuum, and so $K \in \{H\}^{\twoheadleftarrow}$ as before. On the other hand if $K \in \{H\}^{\twoheadleftarrow}$, then one of its components maps onto a subfan $H' \subseteq H$ that contains infinitely many endpoints of H. It follows that $H' \in \mathcal{N}$, and so $K \in \mathcal{N}$. □

Question 3.20. Is the class of all Peano continua compactifiable? We will show in Corollary 3.25 that it is strongly Polishable.

Example 3.21. Let $\mathcal{D}$ denote the class of all dendrites, $\mathcal{N}$ the class of all non-locally connected spaces, and S^1 the unit circle. Both $\mathcal{D}$ and $\mathbf{C} \setminus \mathcal{D}$ are strongly compactifiable classes – $\mathcal{D}$ by Corollary 3.10, and $\mathbf{C} \setminus \mathcal{D}$ since dendrites are exactly Peano continua not containing a simple closed curve, so $\mathbf{C} \setminus \mathcal{D} \cong (\{S^1\}^{\uparrow} \cup \mathcal{N}) \cap \mathbf{C}$, which is strongly compactifiable by Proposition 3.14 and Example 3.19.

In the following paragraphs we shall prove a preservation theorem for $\mathcal{C}^{\twoheadrightarrow} \cap \mathbf{K}$ and a necessary condition for being a strongly Polishable class.

Lemma 3.22. Let X, Y be metrizable. The following sets are G_δ.

- $\mathcal{G}_{\cong} := \{G \in \mathcal{K}(X \times Y) : G$ is a graph of a partial homeomorphism$\}$,
- $\mathcal{G}_{\twoheadrightarrow} := \{G \in \mathcal{K}(X \times Y) : G$ is a graph of a partial continuous surjection$\}$.

Proof. A set $G \in \mathcal{K}(X \times Y)$ is a member of $\mathcal{G}_{\cong}$ if and only if the maps $\pi_X{\restriction}_G$ and $\pi_Y{\restriction}_G$ are injective. The necessity is clear. On the other hand, if they are injective, then they are homeomorphisms onto their images since G is compact. It follows that G is the graph of the homeomorphism $\pi_Y{\restriction}_G \circ (\pi_X{\restriction}_G)^{-1} \colon \pi_X[G] \to \pi_Y[G]$. Analogously, $G \in \mathcal{G}_{\twoheadrightarrow}$ if and only if $\pi_X{\restriction}_G$ is injective.

For every $n \in \mathbb{N}$ let $\mathcal{F}_n := \{F \in \mathcal{K}(X \times Y) : |\pi_Y[F]| = 1$ and $\operatorname{diam}(F) \geq \frac{1}{n}\}$, which is a closed set since π_Y^* is continuous, $[Y]^1$ is closed in $\mathcal{K}(Y)$, and $\operatorname{diam} \colon \mathcal{K}(X \times Y) \to [0, \infty)$ is continuous. The map $\pi_Y{\restriction}_G$ is *not* injective if and only if there are $x_1 \neq x_2 \in X$ and $y \in Y$ such that $\{(x_1, y), (x_2, y)\} \subseteq G$ if and only if there is $n \in \mathbb{N}$ and a set $F \in \mathcal{F}_n$ such that $F \subseteq G$, i.e. if and only if $G \in \bigcup_{n \in \mathbb{N}} \mathcal{F}_n^{\uparrow} \cap \mathcal{K}(X \times Y)$, which is an F_σ set by Corollary 3.13. Analogously for $\pi_X{\restriction}_G$. □

It is known that the homeomorphic classification for compact metric spaces is analytic [5, Proposition 14.4.3]. We shall use the following formulation of the result.

Corollary 3.23. Let X, Y be Polish spaces. The following relations are analytic.

- $\mathcal{R}_{\cong} := \{(A, B) \in \mathcal{K}(X) \times \mathcal{K}(Y) : B$ is homeomorphic to $A\}$,

- $\mathcal{R}_{\twoheadrightarrow} := \{(A, B) \in \mathcal{K}(X) \times \mathcal{K}(Y) : B \text{ is a continuous image of } A\}$.

Proof. We have $\mathcal{R}_{\cong} = \{(\pi_X[G], \pi_Y[G]) : G \in \mathcal{G}_{\cong}\} = (\pi_X^* \vartriangle \pi_Y^*)[\mathcal{G}_{\cong}]$, which is a continuous image of a G_δ set by Lemma 3.22. Analogously for $\mathcal{R}_{\twoheadrightarrow}$. □

Proposition 3.24. If $\mathcal{C}$ is a strongly Polishable class of compacta, then the corresponding class of all metrizable compact continuous images $\mathcal{C}^{\twoheadrightarrow} \cap \mathbf{K}$ is also strongly Polishable.

Proof. There is an analytic family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ such that $\mathcal{C} \cong \mathcal{F}$. We have $\mathcal{C}^{\twoheadrightarrow} \cap \mathbf{K} \cong \mathcal{F}^{\twoheadrightarrow} \cap \mathcal{K}([0,1]^\omega) = \mathcal{R}_{\twoheadrightarrow}[\mathcal{F}] = \pi_2[\mathcal{H}]$ where $\mathcal{H} = \mathcal{R}_{\twoheadrightarrow} \cap (\mathcal{F} \times \mathcal{K}([0,1]^\omega))$, which is an analytic set by Corollary 3.23. □

We obtain a corollary dual to Corollary 3.10.

Corollary 3.25. Every class of metrizable compacta or continua closed under continuous images with a common model is strongly Polishable. This includes the class of all Peano continua (images of $[0,1]$) and the class of all weakly chainable continua (images of the pseudoarc).

We finally give our only necessary condition.

Theorem 3.26. If $\mathcal{C}$ is a strongly Polishable class of compacta, then $\mathcal{C}^{\cong} \cap \mathcal{K}(X)$ is analytic for every Polish space X.

Proof. There is an analytic set $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ such that $\mathcal{F} \cong \mathcal{C}$. We have $\mathcal{C}^{\cong} \cap \mathcal{K}(X) = \mathcal{R}_{\cong}[\mathcal{F}] = \pi_2[\mathcal{H}]$ where $\mathcal{R}_{\cong}$ is the relation of being homeomorphic on $\mathcal{K}([0,1]^\omega) \times \mathcal{K}(X)$ and $\mathcal{H} = \mathcal{R}_{\cong} \cap (\mathcal{F} \times \mathcal{K}(X))$, which is an analytic set by Corollary 3.23. □

Remark 3.27. Of course, if $\mathcal{C}$ is a class of metrizable compacta, X is a Polish space such that each member of $\mathcal{C}$ embeds into X, and $\mathcal{C}^{\cong} \cap \mathcal{K}(X)$ is analytic, then $\mathcal{C}$ is strongly Polishable.

Remark 3.28. For a strongly compactifiable class $\mathcal{C}$, the family $\mathcal{C}^{\cong} \cap \mathcal{K}([0,1]^\omega)$ is almost never closed. In fact, this happens if and only if $\mathcal{C}^{\cong}$ is one of countably many classes [1, Observation 4.3].

Example 3.29. By [6, Theorem 27.5] the class of all uncountable compacta in $\mathcal{K}([0,1]^\omega)$ is analytically complete. Together with Example 3.15 this shows that there is a strongly compactifiable class $\mathcal{C}$ such that $\mathcal{C}^{\cong} \cap \mathcal{K}([0,1]^\omega)$ is not Borel. It also follows that the class of all countable compacta is coanalytically complete, and hence is not strongly Polishable.

Example 3.30. By [7] the following classes are also coanalytically complete, and hence not strongly Polishable: hereditarily decomposable continua, dendroids, λ-dendroids, arcwise connected continua, uniquely arcwise connected continua, hereditarily locally connected continua.

Let us conclude with a result on preservation under intersections.

Proposition 3.31. Let $\{\mathcal{C}_n : n \in \omega\}$, $\mathcal{C}$, $\mathcal{D}$ be classes of metrizable compacta.

(i) If the classes $\mathcal{C}_n$ are strongly Polishable (or Polishable), then so is the class $\bigcap_{n\in\omega} \mathcal{C}_n^{\cong}$.

(ii) If the classes $\mathcal{C}$ and $\mathcal{D}$ are strongly Polishable (or Polishable), then so is the class $\mathcal{C} \cap \mathcal{D}^{\cong}$.

Proof. In the strongly Polishable case we have $\bigcap_{n\in\omega} \mathcal{C}_n^{\cong} \cong \bigcap_{n\in\omega} \mathcal{C}_n^{\cong} \cap \mathcal{K}([0,1]^\omega)$, which is an analytic set by Theorem 3.26.

In the Polishable case, by Theorem 1.11 for every $n \in \omega$ there is a G_δ subset $F_n \subseteq [0,1]^\omega \times \omega^\omega$ such that $\{F_n^x : x \in \omega^\omega\} \cong \mathcal{C}_n$. By [6, Theorem 28.8] the maps $\rho_n\colon \omega^\omega \to \mathcal{K}([0,1]^\omega)$ defined by $x \mapsto F_n^x$ are Borel. Let $i, j \in \omega$. We put $A_{i,j} := \{(x,y) \in \omega^\omega \times \omega^\omega : F_i^x \cong F_j^y\} = (\rho_i \times \rho_j)^{-1}[\mathcal{R}_{\cong}]$. Since the relation $\mathcal{R}_{\cong}$ is analytic and the map $\rho_i \times \rho_j$ is Borel, the set $A_{i,j}$ is analytic. Hence, also the set $A := \{(x_n)_{n\in\omega} \in (\omega^\omega)^\omega : (x_i, x_j) \in A_{i,j}$ for every $i, j \in \omega\}$ and its projection $\pi_0[A] \subseteq \omega^\omega$ are analytic. Observe that $\bigcap_{n\in\omega} \mathcal{C}_n^{\cong} \cong \{F_0^x : x \in \pi_0[A]\}$, and so the intersection is Polishable by Corollary 1.7.

Unlike $\mathcal{C} \cap \mathcal{D}$, the class $\mathcal{C} \cap \mathcal{D}^{\cong}$ is equivalent to $\mathcal{C}^{\cong} \cap \mathcal{D}^{\cong}$, which is (strongly) Polishable by the previous claim. □

Remark 3.32. A similar argument would give us that if $\mathcal{C}$ is strongly compactifiable and $\mathcal{D}^{\cong} \cap \mathcal{K}([0,1]^\omega)$ is closed, then $\mathcal{C} \cap \mathcal{D}^{\cong}$ is strongly compactifiable, but by Remark 3.28, $\mathcal{D}^{\cong}$ would have to be one of countably many special classes. One of these classes is the class of all metrizable continua $\mathbf{C}$, so Observation 3.7 is a special case.

Question 3.33. Is there a better theorem on preservation of (strong) compactifiability under intersections?

Example 3.34. We shall extend Example 3.21. Let $\mathcal{P}$ be the class of all Peano continua. The class $\mathcal{P} \setminus \mathcal{D}$ is strongly Polishable by Corollary 3.25 and Proposition 3.31 since it is equivalent to $\mathcal{P} \cap \{S^1\}^{\uparrow}$.

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