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**Free locally convex spaces  
with a small base**

*Saak Gabrielyan*

*Jerzy Kąkol*

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## Free locally convex spaces with a small base

Saak Gabrielyan · Jerzy Kąkol

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**Abstract** The paper studies the free locally convex space  $L(X)$  over a Tychonoff space  $X$ . Since for infinite  $X$  the space  $L(X)$  is never metrizable (even not Fréchet-Urysohn), a possible applicable generalized metric property for  $L(X)$  is welcome. We propose a concept (essentially weaker than first-countability) which is known under the name a  $\mathfrak{G}$ -base. A space  $X$  has a  $\mathfrak{G}$ -base if for every  $x \in X$  there is a base  $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  of neighborhoods at  $x$  such that  $U_\beta \subseteq U_\alpha$  whenever  $\alpha \leq \beta$  for all  $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ , where  $\alpha = (\alpha(n))_{n \in \mathbb{N}} \leq \beta = (\beta(n))_{n \in \mathbb{N}}$  if  $\alpha(n) \leq \beta(n)$  for all  $n \in \mathbb{N}$ . We show that if  $X$  is an Ascoli  $\sigma$ -compact space, then  $L(X)$  has a  $\mathfrak{G}$ -base if and only if  $X$  admits an Ascoli uniformity  $\mathcal{U}$  with a  $\mathfrak{G}$ -base. We prove that if  $X$  is a  $\sigma$ -compact Ascoli space of  $\mathbb{N}^{\mathbb{N}}$ -uniformly compact type, then  $L(X)$  has a  $\mathfrak{G}$ -base. As an application we show: (1) if  $X$  is a metrizable space, then  $L(X)$  has a  $\mathfrak{G}$ -base if and only if  $X$  is  $\sigma$ -compact, and (2) if  $X$  is a countable Ascoli space, then  $L(X)$  has a  $\mathfrak{G}$ -base if and only if  $X$  has a  $\mathfrak{G}$ -base.

**Keywords** free locally convex space ·  $\mathfrak{G}$ -base ·  $C_k(X)$  · compact resolution

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S. Gabrielyan

Department of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva, P.O. 653, Israel

E-mail: saak@math.bgu.ac.il

J. Kąkol

A. Mickiewicz University 61 – 614 Poznań, Poland and Institute of Mathematics, Czech Academy of Sciences, Czech Republic

E-mail: kakol@amu.edu.pl

## 1 Introduction

The class of free locally convex spaces  $L(X)$  over a (Tychonoff) space  $X$  is one of the most important classes in the category of locally convex spaces and continuous operators. This class was introduced by Markov [20] and intensively studied over the last half-century, see for example [1, 11, 14, 23, 25]. Recall that the *free locally convex space*  $L(X)$  over a space  $X$  is a pair consisting of a locally convex space  $L(X)$  and a continuous mapping  $i : X \rightarrow L(X)$  such that every continuous mapping  $f$  from  $X$  to a locally convex space  $E$  gives rise to a unique continuous linear operator  $\bar{f} : L(X) \rightarrow E$  with  $f = \bar{f} \circ i$ . The free locally convex space  $L(X)$  always exists and is unique.

It is well-known that  $L(X)$  is metrizable if and only if  $X$  is finite. Moreover,  $L(X)$  is a  $k$ -space if and only if  $X$  is a countable discrete space, see [13]. Therefore, seeking for concrete objects  $L(X)$  carrying some *small base* at zero might be interesting for specialist both from topology and functional analysis.

One of such possible concepts extending metrizability is related with locally convex spaces having a  $\mathfrak{G}$ -base. Following [18], a topological space  $X$  has a  $\mathfrak{G}$ -base at a point  $x \in X$  if it has a base  $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  of neighborhoods at  $x$  such that  $U_\beta \subseteq U_\alpha$  whenever  $\alpha \leq \beta$  for all  $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ , where  $\alpha = (\alpha(n))_{n \in \mathbb{N}} \leq \beta = (\beta(n))_{n \in \mathbb{N}}$  if  $\alpha(n) \leq \beta(n)$  for all  $n \in \mathbb{N}$ ;  $X$  has a  $\mathfrak{G}$ -base if it has a  $\mathfrak{G}$ -base at each point  $x \in X$ .

Originally, the concept of a  $\mathfrak{G}$ -base has been formally introduced in [10] in the realm of locally convex spaces for studying  $(DF)$ -spaces,  $C(X)$ -spaces and spaces in the class  $\mathfrak{G}$  in the sense of Cascales and Orihuela, see [19]. Every quasibarrelled locally convex space with a  $\mathfrak{G}$ -base has countable tightness both in the original and the weak topology, respectively; each precompact set in a locally convex space with a  $\mathfrak{G}$ -base is metrizable, see again [19]. It is easy to see that every metrizable group has a  $\mathfrak{G}$ -base at the identity. Topological groups with a  $\mathfrak{G}$ -base are thoroughly studied in [18], see also [15, 17].

Being motivated by several results of the above type (see [19] also for a long list of references), the authors in [18] posed the following general problem:

**Problem 1.1** ([18]) *For which spaces  $X$  the free locally convex space  $L(X)$  has a  $\mathfrak{G}$ -base?*

For a space  $X$  we denote by  $C_p(X)$  and  $C_k(X)$  the space  $C(X)$  of all continuous real-valued functions on  $X$  endowed with the pointwise topology  $\tau_p$  and the compact-open topology  $\tau_k$ , respectively. Recall that a space  $X$  is called an *Ascoli space* if every compact subset  $\mathcal{K}$  of  $C_k(X)$  is evenly continuous [2]. In other words,  $X$  is Ascoli if and only if the compact-open topology of  $C_k(X)$  is Ascoli in the sense of [21, p.45].

Using a deep result of Uspenskiĭ [25], for a wide class of topological spaces  $X$  we show that Problem 1.1 can be reformulated in the term of function spaces  $C(X)$ .

**Theorem 1.2** *Let  $X$  be a Dieudonné complete Ascoli space (in particular,  $X$  is a paracompact  $k$ -space or a metrizable space). Then  $L(X)$  has a  $\mathfrak{G}$ -base if and only if  $C_k(X)$  has a compact resolution swallowing compact subsets.*

Recall that a family  $\mathcal{K} = \{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  of compact subsets of a space  $Z$  is called a *compact resolution* if  $\mathcal{K}$  covers  $Z$  and  $K_\alpha \subseteq K_\beta$  whenever  $\alpha \leq \beta$  for all  $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ . Following Christensen [4], we say that  $\mathcal{K}$  *swallows compact sets of  $Z$*  if for every compact subset  $K$  of  $Z$  there is an  $\alpha \in \mathbb{N}^{\mathbb{N}}$  such that  $K \subseteq K_\alpha$ . The importance of this concept follows from the following deep result of Christensen: *A metrizable and separable space  $Z$  is Polish if and only if  $Z$  has a compact resolution swallowing compact sets.* Consequently, since  $C_k(X)$  is Polish if  $X$  is locally compact metrizable and separable, by Theorem 1.2 the space  $L(X)$  has a  $\mathfrak{G}$ -base. These results and Theorem 1.2 motivate the following question:

**Problem 1.3** *For which spaces  $X$ , the space  $C_k(X)$  has a compact resolution (swallowing compact sets)?*

This problem is of independent interest because (see for example [19, Theorem 9.9])  $C_k(X)$  has a compact resolution if and only if  $C_k(X)$  is  $K$ -analytic, i.e.  $C_k(X)$  is the image under an upper semi-continuous compact-valued map defined in  $\mathbb{N}^{\mathbb{N}}$ ; the same result holds for  $C_p(X)$ , see [24]. Moreover, Tkachuk proved in [24] that  $C_p(X)$  has a compact resolution swallowing compact sets if and only if  $X$  is a countable discrete space.

Christensen had already proved the following result (see also Corollary 2.3 below): *If  $X$  is a separable metrizable space, then  $C_k(X)$  has a compact resolution if and only if  $X$  is  $\sigma$ -compact.* Below we strengthen this result by showing that under the same assumption on  $X$  the space  $C_k(X)$  has even a compact resolution swallowing compact sets, see Corollary 2.10 below. These results motivate the question: *For which  $\sigma$ -compact spaces  $X$  the space  $C_k(X)$  has a compact resolution (swallowing compact sets)?* The aforementioned results explain our study of functions spaces with compact resolutions, see Section 2.

In Section 3 we prove Theorem 1.2 and obtain the following partial answers to Problem 1.1.

**Theorem 1.4** *Let  $X$  be an Ascoli  $\sigma$ -compact space. Then  $L(X)$  has a  $\mathfrak{G}$ -base if and only if  $X$  admits an Ascoli uniformity  $\mathcal{U}$  with a  $\mathfrak{G}$ -base.*

Theorem 1.4 needs a new concept which is stronger than to be an Ascoli space.

**Definition 1.5** A uniformity  $\mathcal{U}$  on a space  $X$  is said to be *Ascoli* if  $\mathcal{U}$  is admissible and any compact subset  $K$  of  $C_k(X)$  is uniformly equicontinuous with respect to  $\mathcal{U}$ , i.e. for every  $\varepsilon > 0$  there is  $U \in \mathcal{U}$  such that  $|f(x) - f(y)| < \varepsilon$  for every  $f \in K$  and each  $(x, y) \in U$ . We say that  $X$  is a *uniformly Ascoli space* if  $X$  has an Ascoli uniformity.

We provide also a sufficient condition on a  $\sigma$ -compact space  $X$  for which  $L(X)$  has a  $\mathfrak{G}$ -base. This approach requires some additional concept.

**Definition 1.6** A topological space  $X$  is a space of  $\mathbb{N}^{\mathbb{N}}$ -*uniformly compact type* if for every compact subset  $K$  of  $X$  the set  $\Delta_K = \{(x, x) \in X \times X : x \in K\}$  has an  $\mathbb{N}^{\mathbb{N}}$ -decreasing base  $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  of open neighborhoods in  $X \times X$ , i.e. for every open neighborhood  $U$  of  $\Delta_K$  there is  $\alpha \in \mathbb{N}^{\mathbb{N}}$  such that  $\Delta_K \subseteq U_\alpha \subseteq U$ .

**Theorem 1.7** *Let  $X$  be a  $\sigma$ -compact Ascoli space. If  $X$  is of  $\mathbb{N}^{\mathbb{N}}$ -uniformly compact type, then  $L(X)$  has a  $\mathfrak{G}$ -base.*

It is easy to show, see Proposition 2.7 below, that if  $X$  is a metrizable space or a countable space such that every point  $x \in X$  has a  $\mathfrak{G}$ -base, then  $X$  is of  $\mathbb{N}^{\mathbb{N}}$ -uniformly compact type. So Theorem 1.7 with Corollary 2.10 imply

**Corollary 1.8** *If  $X$  is a metrizable space, then  $L(X)$  has a  $\mathfrak{G}$ -base if and only if  $X$  is  $\sigma$ -compact.*

In particular, the space  $L(\mathbb{Q})$  has a  $\mathfrak{G}$ -base.

**Corollary 1.9** *If  $X$  is a countable Ascoli space, then  $L(X)$  has a  $\mathfrak{G}$ -base if and only if  $X$  has a  $\mathfrak{G}$ -base.*

Note that Corollaries 1.8 and 1.9 are proved independently in [3] using totally different methods.

## 2 Compact resolutions in function spaces

Recall that a subset  $A$  of a topological space  $X$  is called *functionally bounded* if every continuous function  $f \in C(X)$  is bounded on  $A$ . Recall also that a *resolution*  $\mathcal{A} = \{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  in  $X$  is a cover of  $X$  such that  $A_\alpha \subseteq A_\beta$  whenever  $\alpha \leq \beta$  for all  $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ . If all  $A_\alpha$  are functionally bounded, the resolution  $\mathcal{A}$  is called *functionally bounded*. Note also that by a result of Calbrix, see [19, Theorem 9.7], if  $C_p(X)$  is analytic, then  $X$  is  $\sigma$ -compact. Recall that a space  $Z$  is called *analytic* if it is a continuous image of  $\mathbb{N}^{\mathbb{N}}$ .

We shall use the following fact, see Corollary 9.1 of [19]. Recall that a (Tychonoff) space  $X$  is *cosmic* if it is a continuous image of a separable metric space.

**Fact 2.1** *Let  $X$  be a cosmic space. Then  $C_p(X)$  has a functionally bounded resolution if and only if  $X$  is  $\sigma$ -compact. Consequently, if  $C_k(X)$  has a compact resolution, then  $X$  is  $\sigma$ -compact.*

**Proposition 2.2** *Let  $X$  be a paracompact first countable space such that  $C_k(X)$  is an angelic space. Then  $X$  is Lindelöf.*

*Proof* If  $X$  is not Lindelöf, the space  $C_k(X)$  contains a closed subset  $A$  homeomorphic to  $\mathbb{N}^{\omega_1}$  by Lemma 1 of [22]. But the space  $\mathbb{N}^{\omega_1}$  is not angelic, so  $C_k(X)$  is also not angelic. This contradiction shows that  $X$  must be Lindelöf.

This yields the following

**Corollary 2.3** *Let  $X$  be a metrizable space. If  $C_k(X)$  has a functionally bounded resolution, then  $X$  is  $\sigma$ -compact.*

*Proof* Proposition 9.6 of [19] implies that  $C_p(X)$  is angelic. By [12, Theorem, page 31], the space  $C_k(X)$  is also angelic. Now Proposition 2.2 implies that  $X$  is Lindelöf. So being metrizable, the space  $X$  is separable, and hence  $X$  is a cosmic space. Thus  $X$  is  $\sigma$ -compact by Fact 2.1.

The next proposition completes Proposition 2.2.

**Proposition 2.4** *Let  $X$  be a paracompact Čech-complete space. Then  $C_k(X)$  is an angelic space if and only if  $X$  is Lindelöf.*

*Proof* Assume that  $C_k(X)$  is angelic. By a result of Frolík [5, 5.5.9], there is a perfect map  $f$  from  $X$  onto a complete metrizable space  $Y$ . Suppose that  $X$  is not Lindelöf. Then, since  $f$  is perfect,  $Y$  is also not Lindelöf by [5, Theorem 3.8.9]. As  $f^{-1}(K)$  is compact for every compact set  $K \subseteq Y$  by [5, Theorem 3.7.2], the space  $C_k(Y)$  embeds into  $C_k(X)$ , and hence  $C_k(Y)$  is also angelic. Now Proposition 2.2 implies that  $Y$  is Lindelöf, a contradiction. Thus  $X$  is a Lindelöf space.

Conversely, let  $X$  be Lindelöf. Then  $X$  has a compact resolution swallowing compact sets, see the proof of Proposition 4.7 in [16]. So  $C_p(X)$  is angelic by Example 4.1 and Theorem 4.5 of [19]. Therefore  $C_k(X)$  is angelic by [12, Theorem, page 31].

Recall that, for a space  $X$ , the family of sets of the form

$$[K, \varepsilon] := \{f \in C(X) : f(K) \subset (-\varepsilon, \varepsilon)\}$$

where  $K$  is a compact subset of  $X$ , is a base of the compact-open topology  $\tau_k$  on  $C(X)$ . Denote by  $\delta : X \rightarrow C_k(C_k(X))$  the canonical map defined by

$$\delta(x)(f) := f(x), \quad \forall x \in X, \forall f \in C(X).$$

**Proposition 2.5** *Let  $X$  be an Ascoli space. If  $C_k(X)$  has a compact resolution swallowing compact sets, then  $X$  has an Ascoli uniformity  $\mathcal{U}$  with a  $\mathfrak{G}$ -base.*

*Proof* Let  $\mathcal{K} := \{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  be a compact resolution swallowing compact sets of  $C_k(X)$ . Then, by [8], the space  $C_k(C_k(X))$  has a  $\mathfrak{G}$ -base  $\{V_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ , where  $V_\alpha := [K_\alpha, \alpha(1)^{-1}]$ . Since  $X$  is Ascoli space, the canonical map  $\delta : X \rightarrow C_k(C_k(X))$  is an embedding by Corollary 5.8 of [2]. For every  $\alpha \in \mathbb{N}^{\mathbb{N}}$ , define

$$U_\alpha := \{(x, y) \in X \times X : \delta(x) - \delta(y) \in V_\alpha\},$$

and let  $\mathcal{U}$  be the uniformity on  $X$  induced from  $C_k(C_k(X))$ . Clearly,  $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  is a  $\mathfrak{G}$ -base for  $\mathcal{U}$  and  $\mathcal{U}$  is admissible. We show that  $\mathcal{U}$  is also Ascoli.

Fix a compact subset  $K$  of  $C_k(X)$  and  $\varepsilon > 0$ . Take  $\alpha \in \mathbb{N}^{\mathbb{N}}$  such that  $K \subseteq K_\alpha$  and  $\alpha(1) > 1/\varepsilon$ . Now for every  $(x, y) \in U_\alpha$  and each  $f \in K$ , we obtain

$$|f(x) - f(y)| = |(\delta(x) - \delta(y))(f)| < \frac{1}{\alpha(1)} < \varepsilon,$$

Thus  $\mathcal{U}$  is an Ascoli uniformity.

We shall use the following encoding operation of elements of  $\mathbb{N}^{\mathbb{N}}$ . We encode each  $\alpha \in \mathbb{N}^{\mathbb{N}}$  into a sequence  $\{\alpha_i\}_{i \in \omega}$  of elements of  $\mathbb{N}^{\mathbb{N}}$  as follows. Consider an arbitrary decomposition of  $\mathbb{N}$  onto a disjoint family  $\{N_i\}_{i \in \omega}$  of infinite sets, where  $N_i = \{n_{k,i}\}_{k \in \mathbb{N}}$ . Now for  $\alpha = (\alpha(n))_{n \in \mathbb{N}}$  and  $i \in \omega$ , we set  $\alpha_i =$

$(\alpha_i(k))_{k \in \mathbb{N}}$ , where  $\alpha_i(k) := \alpha(n_{k,i})$  for every  $k \in \mathbb{N}$ . Conversely, for every sequence  $\{\alpha_i\}_{i \in \omega}$  of elements of  $\mathbb{N}^{\mathbb{N}}$  we define  $\alpha = (\alpha(n))_{n \in \mathbb{N}}$  setting  $\alpha(n) := \alpha_i(k)$  if  $n = n_{k,i}$ .

For a subset  $A$  of a set  $S$ , a subset  $B$  of  $S \times S$  and  $(a, b) \in B$ , we define

$$\Delta_A := \{(a, a) \in S \times S : a \in A\} \text{ and } B(a) := \{s \in S : (a, s) \in B\}.$$

Next definition generalizes the classical notion of spaces of pointwise countable type (due to Arhangel'skii) and also Definition 1.6.

**Definition 2.6** Let  $I$  be an ordered set and  $X$  a topological space. The space  $X$  is a space of

- (i) *I-compact type* if every compact subset  $K$  of  $X$  has an decreasing  $I$ -base  $\{U_i : i \in I\}$  of open neighborhoods, i.e.  $U_i \subseteq U_j$  for all  $i \geq j$  and for every open neighborhood  $U$  of  $K$  there is  $i \in I$  such that  $K \subseteq U_i \subseteq U$ ;
- (ii) *I-pointwise countable type* if for every  $x$  in  $X$  there exists a compact set  $K$  which has a decreasing  $I$ -base of open sets;
- (iii) *I-uniformly compact type* if for every compact subset  $K$  of  $X$  the set  $\Delta_K$  has an  $I$ -decreasing base  $\{U_i : i \in I\}$  of open neighborhoods in  $X \times X$ , i.e. for every open neighborhood  $U$  of  $\Delta_K$  there is  $i \in I$  such that  $\Delta_K \subseteq U_i \subseteq U$ .

As usual a decreasing  $\mathbb{N}^{\mathbb{N}}$ -base of a subset  $A$  of  $X$  is called a  $\mathfrak{G}$ -base of  $A$ . Next proposition provides possible two cases when  $X$  is of  $\mathbb{N}^{\mathbb{N}}$ -uniformly compact type.

**Proposition 2.7** A Tychonoff space  $X$  is of  $\mathbb{N}^{\mathbb{N}}$ -uniformly compact type if one of the following conditions holds:

- (i)  $X$  is a metrizable space;
- (ii)  $X$  is a countable space such that every point  $x \in X$  has a  $\mathfrak{G}$ -base.

*Proof* (i) For every compact subset  $K$  of  $X$ , the compact subset  $\Delta_K$  of the metrizable space  $X \times X$  has a decreasing base  $\{V_n : n \in \mathbb{N}\}$ . Then the family  $\{U_\alpha = V_{\alpha(1)} : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  is a  $\mathfrak{G}$ -base of  $K$ .

(ii) Let  $\{x_n : n \in \mathbb{N}\}$  be an enumeration of  $X$  and  $\{U_{\alpha, x_n} : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  be a  $\mathfrak{G}$ -base at  $x_n$ . Fix a compact subset  $K$  of  $X$ . If  $K$  is finite, then clearly the family

$$\left\{ \bigcup_{x_n \in K} U_{\alpha, x_n} \times U_{\alpha, x_n} : \alpha \in \mathbb{N}^{\mathbb{N}} \right\}$$

is a  $\mathfrak{G}$ -base at  $\Delta_K$ .

Assume that  $K$  is infinite. For every  $\alpha \in \mathbb{N}^{\mathbb{N}}$  with the encoding  $(\alpha_n)$ , we define

$$U_\alpha := \bigcup \{U_{\alpha_n, x_n} \times U_{\alpha_n, x_n} : x_n \in K\}.$$

We claim that the family  $\mathcal{U} = \{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  is a  $\mathfrak{G}$ -base at  $\Delta_K$ . Indeed, fix an open neighborhood  $U$  of  $\Delta_K$ . For every  $x_n \in K$  take  $\alpha_n \in \mathbb{N}^{\mathbb{N}}$  such that  $U_{\alpha_n, x_n} \times U_{\alpha_n, x_n} \subseteq U$ . Now if  $\alpha \in \mathbb{N}^{\mathbb{N}}$  is built by the sequence  $(\alpha_n)$ , we obtain  $K \subseteq U_\alpha \subseteq U$ . Thus  $\mathcal{U}$  is a  $\mathfrak{G}$ -base at  $\Delta_K$ .



We shall use the following fact which is proved in the “if” part of the Ascoli theorem [5, 3.4.20].

**Fact 2.8** *Let  $X$  be a space and  $A$  be an evenly continuous (in particular, equicontinuous) pointwise bounded subset of  $C_k(X)$ . Then the closure  $\bar{A}$  of  $A$  in  $\tau_k$  is a compact equicontinuous subset of  $C_k(X)$ .*

If an Ascoli space  $X$  is additionally  $\sigma$ -compact, we can reverse Proposition 2.5.

**Proposition 2.9** *Let  $X$  be a  $\sigma$ -compact space. Then the space  $C_k(X)$  has a compact resolution swallowing compact sets if one of the following conditions holds:*

- (i)  $X$  has an Ascoli uniformity  $\mathcal{U}$  with a  $\mathfrak{G}$ -base;
- (ii)  $X$  is an Ascoli space of  $\mathbb{N}^{\mathbb{N}}$ -uniformly compact type.

*Proof* Let  $X = \bigcup_{n \in \mathbb{N}} C_n$  be the union of an increasing sequence  $\{C_n\}_{n \in \mathbb{N}}$  of compact subsets. For the case (i), let  $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  be a  $\mathfrak{G}$ -base for the Ascoli uniformity  $\mathcal{U}$ . For the case (ii), for every  $n \in \mathbb{N}$ , let  $\{U_{\alpha,n} : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  be a  $\mathfrak{G}$ -base of  $\Delta_{C_n}$ . For every  $\alpha \in \mathbb{N}^{\mathbb{N}}$  with the encoding  $\{\alpha_k\}_{k \in \omega}$ , we define

$$A_\alpha := \bigcap_{k \in \mathbb{N}} \{f \in C(X) : |f(x)| \leq \alpha_0(k) \quad \forall x \in C_k\},$$

$$B_\alpha := \bigcap_{n \in \mathbb{N}} \left\{ f \in C(X) : |f(x) - f(y)| \leq \frac{1}{n} \quad \forall (x, y) \in U_{\alpha_n} \right\}, \quad \text{for case (i),}$$

$$B_\alpha := \bigcap_{n \in \mathbb{N}} \left\{ f \in C(X) : |f(x) - f(y)| \leq \frac{1}{n} \quad \forall (x, y) \in U_{\alpha_n, n} \right\}, \quad \text{for case (ii),}$$

and set  $K_\alpha := A_\alpha \cap B_\alpha$ . Clearly,  $K_\alpha$  is closed in the compact-open topology  $\tau_k$  and  $K_\alpha \subseteq K_\beta$  for every  $\alpha \leq \beta$ . Fix  $\alpha \in \mathbb{N}^{\mathbb{N}}$ . By construction,  $K_\alpha$  is pointwise bounded. We check that the set  $K_\alpha$  is equicontinuous. We distinguish between cases (i) and (ii).

*Case (i).* Given  $\varepsilon > 0$  take  $n \in \mathbb{N}$  such that  $n > 1/\varepsilon$ . Then for every  $f \in K_\alpha$ , by the definition of  $B_\alpha$ , we obtain  $|f(x) - f(y)| \leq \frac{1}{n} < \varepsilon$  whenever  $(x, y) \in U_{\alpha_n}$ . So  $K_\alpha$  is equicontinuous.

*Case (ii).* Fix  $x \in X$ , so  $x \in C_l$  for some  $l \in \mathbb{N}$ . Given  $\varepsilon > 0$  take  $n > l$  such that  $n > 1/\varepsilon$ . Then for every  $f \in K_\alpha$ , by the definition of  $B_\alpha$ , we obtain  $|f(x) - f(y)| \leq \frac{1}{n} < \varepsilon$  whenever  $(x, y) \in U_{\alpha_n, n}$ . Since  $U_{\alpha_n, n}(x)$  is an open neighborhood of  $x$ , the set  $K_\alpha$  is equicontinuous.

Now in both cases (i) and (ii), Fact 2.8 implies that  $K_\alpha$  is a compact subset of  $C_k(X)$ .

Let us show that the family  $\mathcal{K} := \{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  swallows the compact sets of  $C_k(X)$ . Fix a compact subset  $K$  of  $C_k(X)$ . Since  $X$  is an Ascoli space,  $K$  is pointwise bounded and equicontinuous. Define  $\alpha_0 = (\alpha_0(k))_{k \in \mathbb{N}}$  as follows: for every  $k \in \mathbb{N}$ , set

$$\alpha_0(k) := [\sup\{|f(x)| : x \in C_k, f \in K\}] + 1,$$

where  $[t]$  is the integral part of a real number  $t$ . Again we distinguish between cases (i) and (ii).

*Case (i).* Since  $\mathcal{U}$  is Ascoli, for every  $n \in \mathbb{N}$ , take  $\alpha_n \in \mathbb{N}^{\mathbb{N}}$  such that  $|f(x) - f(y)| \leq 1/n$  for every  $f \in K$  and each  $(x, y) \in U_{\alpha_n}$ . If  $\alpha \in \mathbb{N}^{\mathbb{N}}$  is built by the above procedure we obtain  $K \subseteq K_{\alpha}$ .

*Case (ii).* Fix  $n \in \mathbb{N}$ . For every  $x \in C_n$  take an open neighborhood  $U_x$  of  $x$  such that  $|f(x) - f(y)| \leq 1/2n$  for every  $f \in K$  and each  $y \in U_x$ . Set  $W := \bigcup_{x \in C_n} U_x \times U_x$ . Then for every  $(z, y) \in W$  there is  $x \in C_n$  such that  $(z, y) \in U_x \times U_x$  and hence

$$|f(z) - f(y)| \leq |f(x) - f(z)| + |f(x) - f(y)| \leq \frac{1}{n}, \quad \text{for every } f \in K.$$

Since  $X$  is of  $\mathbb{N}^{\mathbb{N}}$ -uniformly compact type, we choose  $\alpha_n \in \mathbb{N}^{\mathbb{N}}$  such that  $\Delta_{C_n} \subseteq U_{\alpha_n, n} \subseteq W$ . If  $\alpha \in \mathbb{N}^{\mathbb{N}}$  is built by the sequence  $(\alpha_n)$ , we obtain  $K \subseteq K_{\alpha}$ .

Also now in both cases (i) and (ii) the family  $\mathcal{K}$  swallows the compact sets of  $C_k(X)$ .

As a corollary we obtain the following strengthening of Christensen's theorem.

**Corollary 2.10** *For a metrizable space  $X$ ,  $C_k(X)$  has a compact resolution swallowing the compact sets of  $C_k(X)$  if and only if  $X$  is  $\sigma$ -compact.*

*Proof* If  $C_k(X)$  has a compact resolution swallowing the compact sets of  $C_k(X)$ , then  $X$  is  $\sigma$ -compact by Corollary 2.3. The converse assertion follows from Propositions 2.7 and 2.9.

In particular, the space  $C_k(\mathbb{Q})$  has a compact resolution swallowing its compact sets.

We conclude this section with the following Christensen's type result.

**Proposition 2.11** *The following assertions are equivalent.*

- (i)  $C_k(X)$  is analytic.
- (ii)  $C_k(X)$  is  $K$ -analytic and  $X$  is  $\sigma$ -compact.
- (iii)  $C_k(X)$  has a compact resolution and  $X$  is  $\sigma$ -compact.

*If additionally  $X$  is first countable, the above conditions are equivalent to*

- (iv)  $X$  is metrizable and  $\sigma$ -compact.

*Proof* First we prove the following claim using some ideas from [9] strongly motivated by Ferrando's Theorem 1 of [6].

*Claim.* *If  $X$  is a  $\sigma$ -compact space, then  $C_p(X)$  admits a stronger metrizable locally convex topology.* Indeed, let  $X = \bigcup_{n=1}^{\infty} K_n$ , where  $K_n$  is a compact subset of  $X$  and  $K_n \subseteq K_{n+1}$  for every  $n \in \mathbb{N}$ . For every  $n \in \mathbb{N}$ , define

$$V_n := \left\{ f \in C(X) : \sup_{x \in K_n} |f(x)| \leq \frac{1}{n} \right\}. \quad (1)$$

Clearly,  $V_{n+1} \subseteq V_n$  and  $\bigcap_{n=1}^{\infty} V_n = \{0\}$ , where 0 stands for the identically null function on  $X$ . Note that the sets  $V_n$  are absorbing since if  $g \in C(X)$ , then there is  $k \in \mathbb{N}$  such that  $\sup_{x \in K_n} |g(x)| \leq k$ , so that  $g \in knV_n$ . Moreover, if

$$U = \left\{ f \in C(X) : \max_{1 \leq i \leq n} |f(x_i)| < \epsilon \right\}$$

and  $p \in \mathbb{N}$  is chosen so that  $x_i \in V_p$  for  $1 \leq i \leq n$  and  $p^{-1} < \epsilon$ , then  $V_p \subseteq U$  and clearly  $V_{2n} \subseteq 2^{-1}V_n$  for each  $n \in \mathbb{N}$ . This shows that  $\{V_n : n \in \mathbb{N}\}$  is a base of neighborhoods of the origin of a locally convex topology on  $C(X)$  stronger than the pointwise topology. The claim is proved.

The implications (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii) are clear; note that if  $C_k(X)$  is analytic, then Calbrix's result, see [19, Theorem 9.7], implies that  $X$  is  $\sigma$ -compact.

(iii)  $\Rightarrow$  (i): If  $X$  is  $\sigma$ -compact, the claim and (1) show that the space  $C(X)$  admits a metrizable locally convex topology weaker (or equal) to the compact-open topology  $\tau_k$ . As  $C_k(X)$  has a compact resolution (by assumption), we apply Talagrand's result, see [19, Proposition 6.3], to deduce that  $C_k(X)$  is analytic.

If  $X$  first countable, (i) is equivalent to (iv) by [21, Theorem 5.7.5].

### 3 Proofs of Theorems 1.2, 1.4 and 1.7

It is well-known that the dual space of  $C_k(X)$  is the space  $M_c(X)$  of all regular Borel measures on  $X$  with compact support. Denote by  $\tau_e$  the topology on  $M_c(X)$  of uniform convergence on the equicontinuous pointwise bounded subsets of  $C(X)$ . For  $A \subseteq C_k(X)$  and  $B \subseteq M_c(X)$ , we set as usual

$$A^\circ = \{\mu \in M_c(X) : |\mu(f)| \leq 1 \forall f \in A\}, \text{ and} \\ B^\circ = \{f \in C_k(X) : |\mu(f)| \leq 1 \forall \mu \in B\}.$$

**Proposition 3.1** *Let  $X$  be an Ascoli space. Then  $(M_c(X), \tau_e)$  has a  $\mathfrak{G}$ -base if and only if  $C_k(X)$  has a compact resolution swallowing compact subsets of  $C_k(X)$ .*

*Proof* Assume that  $C_k(X)$  has a compact resolution swallowing compact subsets of  $C_k(X)$ . Let  $\mathcal{K} = \{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  be a compact resolution swallowing compact sets of  $C_k(X)$ . For every  $\alpha \in \mathbb{N}^{\mathbb{N}}$ , set  $U_\alpha := K_\alpha^\circ$ . We show that the family  $\mathcal{U} := \{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  is a  $\mathfrak{G}$ -base in  $(M_c(X), \tau_e)$ . Indeed, every  $U_\alpha$  is a neighborhood of zero in  $\tau_e$  because  $K_\alpha$  is equicontinuous. Now let  $U$  be a neighborhood of zero in  $(M_c(X), \tau_e)$ . Take an equicontinuous pointwise bounded subset  $A$  of  $C(X)$  such that  $A^\circ \subseteq U$ . By Fact 2.8, the closure  $K$  of  $A$  in  $C_k(X)$  is compact. So there is  $\alpha \in \mathbb{N}^{\mathbb{N}}$  such that  $K \subseteq K_\alpha$ . Clearly,  $U_\alpha = K_\alpha^\circ \subseteq A^\circ \subseteq U$ . Thus  $\mathcal{U}$  is a base of  $\tau_e$ .

Conversely, let  $(M_c(X), \tau_e)$  have a  $\mathfrak{G}$ -base  $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ . For every  $\alpha \in \mathbb{N}^{\mathbb{N}}$ , set  $C_\alpha := U_\alpha^\circ$ . We show that the family  $\mathcal{C} := \{C_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  is a compact resolution in  $C_k(X)$  swallowing the compact sets.

Clearly, if  $\alpha \leq \beta$  then  $C_\alpha \subseteq C_\beta$ . Since  $M_c(X)$  is the dual space of  $C_k(X)$ , every  $C_\alpha$  is closed in  $C_k(X)$ . To show that  $C_\alpha$  is compact in  $C_k(X)$ , take an absolutely convex neighborhood  $V$  of zero in  $M_c(X)$  such that  $\overline{V} \subseteq U_\alpha$  and choose an equicontinuous pointwise bounded subset  $A$  of  $C(X)$  such that  $A^\circ \subseteq V$ . Clearly, the absolutely convex hull  $\text{acx}(A)$  of  $A$  is also an equicontinuous pointwise bounded subset of  $C(X)$ . So, by Fact 2.8, the closure  $K := \overline{\text{acx}(A)}^{\tau_k}$  of  $\text{acx}(A)$  in the compact-open topology  $\tau_k$  is a compact equicontinuous subset of  $C_k(X)$ . Since the bounded convex subsets of  $C(X)$  in  $\tau_k$  and  $\sigma(C(X), M_c(X))$  are the same, the Bipolar theorem implies that  $K = K^{\circ\circ}$ . As

$$C_\alpha \subseteq \overline{V}^\circ \subseteq A^{\circ\circ} = K^{\circ\circ} = K$$

we obtain that  $C_\alpha$  is compact.

Let  $C$  be a compact subset of  $C_k(X)$ . Since  $X$  is Ascoli,  $C$  is equicontinuous and clearly pointwise bounded. Take  $\alpha \in \mathbb{N}^{\mathbb{N}}$  such that  $U_\alpha \subseteq C^\circ$ . Then

$$C \subseteq C^{\circ\circ} \subseteq U_\alpha^\circ = C_\alpha.$$

Thus the family  $\mathcal{C}$  swallows the compact sets of  $C_k(X)$ .

For a space  $X$  we denote by  $\mu X$  the Dieudonné completion of  $X$ . Note that any paracompact space is Dieudonné complete, see [5, 8.5.13(d)]. Now Theorem 1.2 immediately follows from the following more general result.

**Theorem 3.2** *Let  $X$  be a Tychonoff space such that  $\mu X$  is an Ascoli space. Then  $L(X)$  has a  $\mathfrak{G}$ -base if and only if  $C_k(\mu X)$  has a compact resolution swallowing compact subsets. In this case the space  $C_k(\mu X)$  is Lindelöf.*

*Proof* The space  $L(X)$  has a  $\mathfrak{G}$ -base if and only if its (Raikov) completion  $\overline{L(X)}$  has a  $\mathfrak{G}$ -base, see Proposition 2.7 of [18]. It is known that  $\overline{L(X)}$  is  $(M_c(\mu X), \tau_e)$ , see Theorem 5 of [25]. Now Proposition 3.1 applies. To prove the last assertion we note that the space  $C_k(\mu X)$  is  $K$ -analytic by Theorem 9.9 of [19]. So  $C_k(\mu X)$  is Lindelöf by Proposition 3.13 of [19].

We do not know whether the condition on  $X$  to be an Ascoli space is essential in Theorem 1.2.

Question 4.18 in [18] asks whether for a  $k$ -space the existence of a  $\mathfrak{G}$ -base on  $L(X)$  implies that also  $C_k(C_k(X))$  has a  $\mathfrak{G}$ -base. By Ferrando–Kąkol theorem [9] (see also [18, Theorem 4.9]), the space  $C_k(X)$  has a compact resolution swallowing compact subsets if and only if  $C_k(C_k(X))$  has a  $\mathfrak{G}$ -base. Combining this result with Theorem 1.2 we obtain a partial answer to [18, Question 4.18].

**Corollary 3.3** *Let  $X$  be a Dieudonné complete Ascoli space. Then  $L(X)$  has a  $\mathfrak{G}$ -base if and only if the space  $C_k(C_k(X))$  has a  $\mathfrak{G}$ -base.*

Corollary 1.9 follows from the next result.

**Corollary 3.4** *If  $X$  is a countable Ascoli space, then the following assertions are equivalent:*

- (i)  $L(X)$  has a  $\mathfrak{G}$ -base;  
(ii)  $C_k(X)$  has a compact resolution swallowing the compact sets of  $C_k(X)$ ;  
(iii)  $X$  has a  $\mathfrak{G}$ -base.

*Proof* (i) $\Leftrightarrow$ (ii) follows from Theorem 1.2 (recall that any countable space being Lindelöf is Dieudonné complete). (i) $\Rightarrow$ (iii) follows from the fact that  $X$  is a subspace of  $L(X)$ . (iii) $\Rightarrow$ (ii) follows from Propositions 2.7 and 2.9.

Note that Ferrando in [7] gives a direct proof of the implication (iii) $\Rightarrow$ (ii) in Corollary 3.4.

We provide another necessary condition for a space  $X$  to have the space  $L(X)$  with a  $\mathfrak{G}$ -base.

**Proposition 3.5** *If  $L(X)$  has a  $\mathfrak{G}$ -base, then every precompact set in  $L(X)$  (hence also in  $X$ ) is metrizable.*

*Proof* Note that in every locally convex space  $E$  with a  $\mathfrak{G}$ -base every precompact set is metrizable, see [19, Theorem 11.1]. We conclude the proof by noticing that  $X$  embeds into  $L(X)$ .

Recall that a topological space  $X$  is a  $k_\omega$ -space (an  $\mathcal{MK}_\omega$ -space) if  $X$  is the inductive limit of a countable family of compact (compact and metrizable) subsets. We proved in [15] that  $L(X)$  has a  $\mathfrak{G}$ -base for every  $\mathcal{MK}_\omega$ -space  $X$ . Combining this result with Proposition 3.5 we obtain

**Corollary 3.6** *Let  $X$  be a  $k_\omega$ -space. Then  $L(X)$  has  $\mathfrak{G}$ -base if and only if  $X$  is an  $\mathcal{MK}_\omega$ -space.*

**Remark 3.7** In [18, Question 4.19] we ask whether the existence of a  $\mathfrak{G}$ -base in the free abelian group  $A(X)$  over a space  $X$  implies that  $L(X)$  has also a  $\mathfrak{G}$ -base. We answer this question in the negative. Let  $X$  be a discrete space of cardinality  $\omega_1$ . Then the free abelian group  $A(X)$  being discrete has a  $\mathfrak{G}$ -base. On the other hand,  $L(X)$  does not have a  $\mathfrak{G}$ -base by Corollary 1.8.

## References

1. Arhangel'skii A. V., Tkachenko M. G.: Topological groups and related structures, Atlantis Press/World Scientific, Amsterdam-Raris (2008)
2. Banach T., Gabrielyan S.: On the  $C_k$ -stable closure of the class of (separable) metrizable spaces, Monatshefte Math., accepted.
3. Banach T., Leiderman A.:  $\mathfrak{G}$ -bases in free (locally convex) topological spaces and free (abelian) topological groups, preprint.
4. Christensen J. P. R.: Topology and Borel Structure, Math. Studies **10** (1974)
5. Engelking R.: General Topology, Heldermann Verlag, Berlin (1989)
6. Ferrando J. C.: On uniform spaces with a small base and  $K$ -analytic  $C_c(X)$ , Topology Appl. **193** 77-83 (2015)
7. Ferrando J. C.: Private communication, 2016.
8. Ferrando J. C., Kąkol J.: On precompact sets in spaces  $C_c(X)$ , Georgian Math. J. **20** 247-254 (2013)
9. Ferrando J. C., Kąkol J.: Stronger metrizable locally convex topologies on  $C_p(X)$ , preprint.

10. Ferrando J. C., Kąkol J., López Pellicer M., Saxon S. A.: Tightness and distinguished Fréchet spaces, *J. Math. Anal. Appl.* **324** 862–881 (2006)
11. Flood J.: Free locally convex spaces, *Dissertationes Math CCXXI*, PWN, Warszawa, (1984)
12. Floret K.: Weakly compact sets, *Lecture Notes in Mathematics* **801**, Springer-Verlag, Berlin, (1980)
13. Gabrielyan S.: The  $k$ -space property for free locally convex spaces, *Canadian Math. Bull.* **57** 803–809 (2014)
14. Gabrielyan S.: A characterization of free locally convex spaces over metrizable spaces which have countable tightness, *Scientiae Mathematicae Japonicae* **78** 201–205 (2015)
15. Gabrielyan S., Kąkol J.: On topological spaces and topological groups with certain local countable networks, *Topology Appl.* **190** 59–73 (2015)
16. Gabrielyan S., Kąkol J., Kubzdela A., Lopez-Pellicer M.: On topological properties of Fréchet locally convex spaces with the weak topology, *Topology Appl.* **192** 123–137 (2015)
17. Gabrielyan S., Kąkol J., Leiderman A.: The strong Pytkeev property for topological groups and topological vector spaces, *Monatsch. Math.* **175** 519–542 (2014)
18. Gabrielyan S., Kąkol J., Leiderman A.: On topological groups with a small base and metrizability, *Fund. Math.* **229** 129–158 (2015)
19. Kąkol J., Kubiś W., Lopez-Pellicer M.: *Descriptive Topology in Selected Topics of Functional Analysis*, *Developments in Mathematics*, Springer, (2011)
20. Markov A. A.: On free topological groups, *Dokl. Akad. Nauk SSSR* **31** 299–301 (1941)
21. McCoy R. A., Ntantu I.: *Topological Properties of Spaces of Continuous Functions*, *Lecture Notes in Math.* **1315**, Springer, Berlin (1988)
22. Pol R.: Normality in function spaces, *Fund. Math.* **84** 145–155 (1974)
23. Raïkov D. A.: Free locally convex spaces for uniform spaces, *Math. Sb.* **63** 582–590 (1964)
24. Tkachuk V. V.: A space  $C_p(X)$  is dominated by irrationals if and only if it is  $K$ -analytic, *Acta Math. Hung.* **107** 253–265 (2005)
25. Uspenskiĭ V. V.: Free topological groups of metrizable spaces, *Math. USSR-Izv.* **37** 657–680 (1991)