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Abstract

The classic Arens theorem states that the space $C(X)$ of real-valued continuous functions on a Tychonoff space X is metrizable in the compact-open topology τ_k if and only if X is hemicompact. Less demanding but still applicable problem asks whether τ_k has an $\mathbb{N}^{\mathbb{N}}$-decreasing base at zero $(U_\alpha)_{\alpha\in\mathbb{N}^{\mathbb{N}}}$, called in the literature a $\mathfrak{G}$-base. We characterize those spaces X for which $C(X)$ admits a locally convex topology $\mathcal{T}$ between the pointwise topology τ_p and the bounded-open topology τ_b such that $(C(X),\mathcal{T})$ is either metrizable or is an (LM)-space or even has a $\mathfrak{G}$-base.

Keywords: metrizable, (LM)-topology, $\mathfrak{G}$-base, K-analytic, Hewitt realcompactification, functionally bounded set
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1. Introduction

For a Tychonoff space X we denote by $C_p(X)$, $C_k(X)$ and $C_b(X)$ the space $C(X)$ of all real-valued continuous functions on X endowed with the pointwise topology τ_p, the compact-open topology τ_k and the bounded-open topology τ_b, respectively. By τ_w we mean the weak topology of the locally convex space $C_k(X)$.

The interplay among the topological properties of a Tychonoff space X and the locally convex or topological properties of the space $C(X)$ equipped with a locally convex topology $\mathcal{T}$ has been widely studied, mainly for the cases when $\mathcal{T}$ is τ_p or τ_k. For example, classical Nachbin–Shirota theorems provide necessary and sufficient conditions, in terms of X, for the space $C_k(X)$ to be barrelled or bornological, see [13, Theorems 11.7.5 and 13.6.1]. The corresponding characterizations for $C_p(X)$ are due to Buchwalter and Schmets, see [3].

The question about metrizability of $(C(X),\mathcal{T})$ seems also to be attracting and important. The classic Arens theorem states that $C_k(X)$ is a metrizable (metrizable and complete) locally convex space if and

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only if X is hemicompact (and a $k_{\mathbb{R}}$-space), and $C_k(X)$ is a Banach space if and only if X is compact by [1, Theorem 13]. It is also well-known that $C_p(X)$ is metrizable if and only if X is countable. This shows that metrizability for the mentioned topologies on $C(X)$ implies strong conditions on X. The Fréchet–Urysohn property for $C(K)$, a weaker condition than metrizability, have provided another interesting line of research. Pytkeev, Gerlitz and Nagy (see §3 of [2]) characterized those spaces X for which $C_p(X)$ is Fréchet–Urysohn, sequential or a k-space (these properties coincide for the spaces $C_p(X)$).

It is clear that one of the simplest conditions on X which guarantees the metrizability of compact subsets of X is *submetrizability*, i.e., X admits a weaker metrizable topology. It is well known (see [16]) that the space $C_p(X)$ is submetrizable if and only if X is separable, and $C_k(X)$ is submetrizable if and only if X is almost σ-compact. However, in order to show that all compact subsets of $C_k(X)$ or $C_b(X)$ are metrizable it is sufficient to find a metrizable (eventually locally convex) topology $\mathcal{T}$ on $C(X)$ either between τ_p and τ_k or between τ_p and τ_b, respectively. These facts and observations motivate the main result of our paper presented in Theorem 3.1.

The concept of a locally convex space E with a so-called $\mathfrak{G}$-base (of neighborhoods of the origin), which is more general than metrizability (but still yielding angelicity of E), has been successfully adopted to study spaces $C(X)$; we know that $C_p(X)$ has a $\mathfrak{G}$-base if and only if X is countable (cf. [14, Corollary 15.2]) and that $C_k(X)$ has a $\mathfrak{G}$-base if and only if X has a compact resolution swallowing the compact sets (cf. [8, Theorem 2]); the latter one is a nice generalization of mentioned Arens theorem. Recall that a locally convex space E is said to have a $\mathfrak{G}$-*base* if it admits a base of neighborhoods at zero $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ such that $U_\alpha \subseteq U_\beta$ whenever $\beta \le \alpha$ for all $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$. The class of locally convex spaces with a $\mathfrak{G}$-base is rich, contains among others, the class of all (LM)-spaces, particularly all metrizable locally convex spaces. We refer the reader to the monograph [14] for more details. These results motivate our general result, Theorem 2.2, which particularly provides a characterization of spaces X for which $C(X)$ endowed with some set-open topology $\tau_{\mathcal{S}}$ admits a finer locally convex topology $\mathcal{T}$ which is metrizable or has a $\mathfrak{G}$-base, see Corollary 2.3.

2. General results

We start with the following general observation (connected also to the previous facts for concrete cases), which seems to be known but hard to locate.

Proposition 2.1. *A locally convex space E with dual E' admits a metrizable and separable locally convex topology $\mathcal{T}$ weaker than $\sigma(E, E')$ if and only if $\sigma(E', E)$ is separable.*

Proof. For the 'only if' part set $F := (E, \mathcal{T})'$. Then $(F, \sigma(F, E))$ is a σ-compact space with a coarser metrizable topology, the latter because $(E, \sigma(E, F))$ is separable. Now observe that the adjoint of the identity map from $(E, \sigma(E, E'))$ onto $(E, \sigma(E, F))$ has $\sigma(E', E)$-dense rank. □

In what follows we need some notations. Let X be a topological space (all topological spaces in the article are assumed to be Hausdorff). Denote by $\mathfrak{S}(X)$ the family of all collections $\mathcal{S}$ of functionally bounded subsets of X which are directed (that is for every $S_1, S_2 \in \mathcal{S}$ there is $S_3 \in \mathcal{S}$ such that $S_1 \cup S_2 \subseteq S_3$) and $\cup\mathcal{S} = X$. For every $\mathcal{S} \in \mathfrak{S}(X)$, the sets of the form

$$[S, \varepsilon] := \{f \in C(X) : |f(x)| < \varepsilon \ \forall x \in S\}, \qquad \text{where } S \in \mathcal{S} \text{ and } \varepsilon > 0,$$

define a base at the zero function 0 of a Hausdorff locally convex topology $\tau_{\mathcal{S}}$ on $C(X)$, put $C_{\mathcal{S}}(X) := (C(X), \tau_{\mathcal{S}})$. If $\mathcal{S}$ is the family $\mathsf{Fin}(X)$ of all finite subsets of X or the family $\mathsf{Com}(X)$ of all compact subsets of X, we obtain the pointwise topology τ_p and the compact-open topology τ_k on $C(X)$, respectively. For other interesting set-open topologies $\mathcal{T}$ see [19], where some distinguishing examples are also provided. If E is a vector subspace of $C(X)$, a subset A of X is called *E-functionally bounded in X* if every $f \in E$ is bounded on A. $C(X)$-functionally bounded subsets of X are called *functionally bounded in X*. Denote by $\mathsf{FB}(X)$ the family of all functionally bounded subsets in X and let $\tau_b := \tau_{\mathsf{FB}(X)}$. We set $C_p(X) := (C(X), \tau_p)$, $C_k(X) := (C(X), \tau_k)$ and $C_b(X) := (C(X), \tau_b)$. It is clear that $\tau_p \le \tau_{\mathcal{S}} \le \tau_b$ for every $\mathcal{S} \in \mathfrak{S}(X)$, in particular $\tau_p \le \tau_k \le \tau_b$.

Let Ω be a set and let I be a partially ordered set with an order $\le$. A family $\mathcal{A} = \{A_i\}_{i\in I}$ of subsets of Ω is called *I-increasing* (*I-decreasing*) if $A_i \subseteq A_j$ (respectively, $A_i \supseteq A_j$) for every $i \le j$ in I. For two families $\mathcal{B}$ and $\mathcal{C}$ of subsets of Ω we say that $\mathcal{C}$ *swallows* $\mathcal{B}$ if for every $B \in \mathcal{B}$ there is $C \in \mathcal{C}$ such that $B \subseteq C$. If X is a topological space, a family $\mathcal{U} = \{U_i\}_{i\in I}$ is said to be a *local I-base at a point $x \in X$* if $\mathcal{U}$ is an I-decreasing base at x.

Theorem 2.2. *Let X be a Tychonoff space, $\mathcal{S} \in \mathfrak{S}(X)$, E a vector subspace of $C(X)$ and I a partially ordered set.*

- (i) *Assume that E has a locally convex topology $\mathcal{T}$ with a local I-base $\mathcal{U} = \{U_i\}_{i\in I}$ at zero stronger than the relative topology $\tau_{\mathcal{S}}|_E$. Then there exists an I-increasing family of E-functionally bounded subsets in X swallowing $\mathcal{S}$.*
- (ii) *Assume that I is ordered isomorphic to $\mathbb{N} \times \mathcal{A}$ for some partially ordered set $\mathcal{A}$ and there exists an I-increasing family $\{B_i\}_{i\in I}$ of E-functionally bounded subsets in X swallowing $\mathcal{S}$. Then E has a locally convex topology $\mathcal{T}$ with a local I-base at zero stronger than the relative topology $\tau_{\mathcal{S}}|_E$.*

Proof. (i) For each $i \in I$ define

$$B_i := \{x \in X : |f(x)| < 1 \ \forall f \in U_i\}.$$

Let us show that B_i is E-functionally bounded in X. Indeed, if $g \in E$, take $k \in \mathbb{N}$ such that $g \in kU_i$. Then $\sup\{|g(x)| : x \in B_i\} \le k$, and hence B_i is E-functionally bounded. Clearly, the family $\{B_i\}_{i\in I}$ is I-increasing. To show that $\{B_i\}_{i\in I}$ swallows $\mathcal{S}$ take arbitrarily $S \in \mathcal{S}$. By assumption $\tau_{\mathcal{S}}|_E \le \mathcal{T}$. So there is $i \in I$ such that $U_i \subseteq [S, 1] \cap E$, which means that $S \subseteq B_i$.

(ii) We shall identify I with $\mathbb{N} \times \mathcal{A}$. For each $i = (n, \alpha) \in I$, set

$$U_i := \{f \in E : |f(x)| < n^{-1} \ \forall x \in B_i\}.$$

Clearly, $U_i \subseteq U_j$ for every $i \ge j$ in I, and $2U_{i'} \subseteq U_i$ for $i' = (2n, \alpha)$. Moreover, U_i is E-absorbing. Indeed, if $g \in E$ there is $k \in \mathbb{N}$ such that $|g(x)| < k$ for every $x \in B_i$, so $g \in nkU_i$. So the family $\mathcal{U} = \{U_i\}_{i\in I}$ is I-decreasing and defines a locally convex topology $\mathcal{T}$ on E. To show that $\tau_{\mathcal{S}}|_E \le \mathcal{T}$ fix arbitrarily $S \in \mathcal{S}$ and $\varepsilon > 0$. Take $t = (m, \beta) \in I$ such that $S \subseteq B_t$ and $m^{-1} < \varepsilon$. Then clearly $U_t \subseteq [S, \varepsilon] \cap E$. Thus $\tau_{\mathcal{S}}|_E \le \mathcal{T}$. □

Part (ii) of Theorem 2.2 suggests (ii) of the corollary below. If $I = \mathbb{N}$ or $I = \mathbb{N}^{\mathbb{N}}$, then $I = \mathbb{N} \times \{e\}$ (where $\{e\}$ is a singleton with the trivial order) or $I = \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$, respectively. So Theorem 2.2 applies to get the following

Corollary 2.3. *Let X be a Tychonoff space, $\mathcal{S} \in \mathfrak{S}(X)$ and E a vector subspace of $C(X)$. Then:*

- *(i) E admits a metrizable locally convex topology $\mathcal{T}$ stronger than the induced topology $\tau_{\mathcal{S}}|_E$ if and only if there is an increasing sequence $\{B_n\}_{n\in\mathbb{N}}$ of E-functionally bounded subsets of X swallowing $\mathcal{S}$.*
- *(ii) E admits a locally convex topology $\mathcal{T}$ with a $\mathfrak{G}$-base stronger than the induced topology $\tau_{\mathcal{S}}|_E$ if and only if there is an $\mathbb{N}^{\mathbb{N}}$-increasing family $\{B_\alpha\}_{\alpha\in\mathbb{N}^{\mathbb{N}}}$ of E-functionally bounded subsets of X swallowing $\mathcal{S}$.*

Let (E,τ) be a locally convex space covered by an increasing sequence $\mathcal{E}:=\{E_n\}_{n\in\mathbb{N}}$ of vector subspaces of E. We say that E admits an (LM)-*topology* on E associated with the sequence $\mathcal{E}$ if for every $n\in\mathbb{N}$ there is a metrizable topology τ_n on E_n such that $\tau|_{E_n}\leq\tau_n$ and $\tau_{n+1}|_{E_n}\leq\tau_n$. The finest locally convex topology ξ on E such that $\xi|_{E_n}\leq\tau_n$ (which clearly exists and is stronger than τ) is called the (LM)-*topology* on E and the space (E,ξ) is an (LM)-*space* associated with the sequence $\mathcal{E}$. In [15] it is proved that $C_p(X)$ is an (LM)-space if and only if X is countable. Below we consider an analogous question for $C_k(X)$.

If $\Sigma\subseteq\mathbb{N}^{\mathbb{N}}$ and $\mathcal{A}=\{A_\alpha:\alpha\in\Sigma\}$ is a family of subsets of a space X, we set

$$A\left(\alpha|n\right)=\bigcup\{A_\beta:\beta\in\Sigma,\ \beta\left(i\right)=\alpha\left(i\right),\ 1\leq i\leq n\}$$

for $\alpha\in\Sigma$ and $n\in\mathbb{N}$. Since $A\left(\alpha|n\right)=A\left(\beta|n\right)$ whenever $\alpha\left(i\right)=\beta\left(i\right)$ for $1\leq i\leq n$, we have that $\mathcal{M}=\{A\left(\alpha|n\right):\alpha\in\Sigma,\ n\in\mathbb{N}\}$ is a countable family of subsets of X. Following [21, Definition 2.3] the family $\mathcal{M}$ is called the *envelope* of $\mathcal{A}$.

Definition 2.4. Let $\mathcal{E}=\{E_n\}_{n\in\mathbb{N}}$ be an increasing sequence of vector subspaces of $C(X)$ covering $C(X)$. We say that the envelope of a family $\{A_\alpha:\alpha\in\Sigma\}$ of subsets of X with $\Sigma\subseteq\mathbb{N}^{\mathbb{N}}$ is $\mathcal{E}$-bounded if $A\left(\alpha|n\right)$ is E_n-functionally bounded for each $\alpha\in\Sigma$ and $n\in\mathbb{N}$.

The following theorem characterizes spaces $C_{\mathcal{S}}(X)$ which admit stronger (LM)-topologies.

Theorem 2.5. *Let X be a Tychonoff space, $\mathcal{S}\in\mathfrak{S}(X)$ and let $\mathcal{E}=\{E_n\}_{n\in\mathbb{N}}$ be an increasing sequence of vector subspaces of $C(X)$ covering $C(X)$. Then the following assertions are equivalent:*

(i) *$C(X)$ admits an (LM)-topology $\mathcal{T}$ associated with the sequence $\mathcal{E}$ finer than $\tau_{\mathcal{S}}$;*
(ii) *for every $n\in\mathbb{N}$ there exists an increasing sequence $\{B_{i,n}\}_{i\in\mathbb{N}}$ of E_n-functionally bounded subsets of X swallowing $\mathcal{S}$;*
(iii) *X has a resolution swallowing $\mathcal{S}$ with $\mathcal{E}$-bounded envelope.*

Proof. (i)⇒(ii) follows from Corollary 2.3. Let us prove (ii)⇒(i). We proceed by induction. For $n=1$ and every $i\in\mathbb{N}$, set $C_{i,1}:=B_{i,1}$. So, by Corollary 2.3, $\{C_{i,1}\}_{i\in\mathbb{N}}$ defines a metrizable topology τ_1 on E_1. Assume that for every $n=k>1$ we find an increasing sequence $\{C_{i,k}\}_{i\in\mathbb{N}}$ of E_k-functionally bounded subsets of X swallowing $\mathcal{S}$ which defines a metrizable locally convex topology τ_k on E_k such that $\tau_k|_{E_{k-1}}\leq\tau_{k-1}$ and $\tau_{\mathcal{S}}|_{E_k}\leq\tau_k$. Let $\{C'_{i,k+1}\}_{i\in\mathbb{N}}$ be an enumeration of the countable family

$$\{B_{i,k+1}\cap C_{m,k}:i,m\in\mathbb{N}\}.$$

For every $i\in\mathbb{N}$, set $C_{i,k+1}:=\cup_{j\leq i}C'_{j,k+1}$. Clearly, $\{C_{i,k+1}\}_{i\in\mathbb{N}}$ is an increasing sequence of E_{k+1}-functionally bounded subsets of X. Since the sequences $\{B_{i,k+1}\}_{i\in\mathbb{N}}$ and $\{C_{i,k}\}_{i\in\mathbb{N}}$ swallow $\mathcal{S}$, then also $\{C_{i,k+1}\}_{i\in\mathbb{N}}$ swallows $\mathcal{S}$. So, by Corollary 2.3, $\{C_{i,k+1}\}_{i\in\mathbb{N}}$ defines a metrizable locally convex topology τ_{k+1} on E_{k+1} stronger than $\tau_{\mathcal{S}}|_{E_{k+1}}$. Since every $C_{i,k+1}$ is contained in $C_{m,k}$ for some $m\in\mathbb{N}$ we obtain that $\tau_{k+1}|_{E_k}\leq\tau_k$. Finally, the finest locally convex topology ξ on E such that $\xi|_{E_n}\leq\tau_n$, $n\in\mathbb{N}$, is an (LM)-topology on E associated with the sequence $\mathcal{E}$.

(ii)⇒(iii) For $\alpha\in\mathbb{N}^{\mathbb{N}}$ define

$$A_\alpha=\bigcap_{n=1}^{\infty}B_{\alpha(n),n}$$

and note that $A_\alpha\subseteq A_\beta$ whenever $\alpha\leq\beta$ due to the fact that the sequences $\{B_{i,n}:i\in\mathbb{N}\}$ are increasing. If $S\in\mathcal{S}$ for each $k\in\mathbb{N}$ there exists $\gamma\left(k\right)\in\mathbb{N}$ such that $S\subseteq B_{\gamma(k),k}$. Consequently, $S\subseteq A_\gamma$. On the other hand, observe that

$$A\left(\alpha|n\right)=B_{\alpha(1),1}\cap\cdots\cap B_{\alpha(n),n}$$

for $\alpha \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$. Since $A(\alpha|n) \subseteq B_{\alpha(n),n}$, it turns out that $A(\alpha|n)$ is functionally E_n-bounded.

(iii)⇒(ii) Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a resolution in X swallowing $\mathcal{S}$ with $\mathcal{E}$-bounded envelope. So, if we fix $n \in \mathbb{N}$ then $A(\alpha|n)$ is E_n-bounded for every $\alpha \in \mathbb{N}$. Since $A(\alpha|n)$ is completely determined by the n-tuple $(\alpha(1), \dots, \alpha(n))$ and $\mathbb{N}^n$ is isomorphic to $\mathbb{N}$, in case that $(\alpha(1), \dots, \alpha(n)) \mapsto j$ we may define $C_{j,n} := A(\alpha|n)$. If $B_{i,n} := \bigcup_{1 \le j \le i} C_{j,n}$, then the sequence $\{B_{i,n} : i \in \mathbb{N}\}$ is increasing, E_n-functionally bounded and satisfies that $\bigcup_{i \in \mathbb{N}} B_{i,n} = X$. In addition, if $S \in \mathcal{S}$ there exists $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $S \subseteq A_\gamma \subseteq A(\gamma|n) = C_{k,n} \subseteq B_{k,n}$, where $(\gamma(1), \dots, \gamma(n)) \mapsto k$. Thus the sequences $\{B_{i,n} : i \in \mathbb{N}\}$ for $n \in \mathbb{N}$ satisfy the required conditions. □

Corollary 2.6. *Let X be a Tychonoff space and let $\mathcal{E} = \{E_n\}_{n \in \mathbb{N}}$ be an increasing sequence of vector subspaces of $C(X)$ covering $C(X)$. The compact open-topology τ_k is the (LM)-topology associated with the sequence $\mathcal{E}$ only if X is K-analytic and has a compact resolution $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ with $\mathcal{E}$-bounded envelope that swallows the compact sets.*

Proof. By Proposition 2.5, if τ_k is the (LM)-topology on $C(X)$ associated with the sequence $\mathcal{E}$, then X has a resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ with $\mathcal{E}$-bounded envelope that swallows the compact sets. Since $C_k(X)$ is bornological, X is realcompact by virtue of the Nachbin–Shirota theorem and hence $\tau_b = \tau_k$. Thus, setting $K_\alpha = \overline{A_\alpha}$ for each $\alpha \in \mathbb{N}^{\mathbb{N}}$, the family $\mathcal{K} := \{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ satisfies the required conditions. Indeed, if

$$K(\alpha|n) = \bigcup \left\{ K_\beta : \beta \in \mathbb{N}^{\mathbb{N}},\, \beta(i) = \alpha(i),\, 1 \le i \le n \right\}$$

choose $f \in E_n$. Since $\{A_\gamma : \gamma \in \mathbb{N}^{\mathbb{N}}\}$ has $\mathcal{E}$-bounded envelope, let $q \in \mathbb{N}$ be such that

$$\sup_{x \in A(\alpha|n)} |f(x)| < q.$$

In case that there exists $\{x_m\}_{m=1}^\infty \subseteq K(\alpha|n)$ with $|f(x_m)| \ge m$ for each $m \in \mathbb{N}$, there is $\{\beta_m\}_{m=1}^\infty$ with $\beta_m(i) = \alpha(i)$ for $1 \le i \le n$ and $m \in \mathbb{N}$ such that $x_m \in K_{\beta_m}$ for every $m \in \mathbb{N}$. Selecting $y_m \in A_{\beta_m}$ such that $|f(x_m) - f(y_m)| < 1$ for each $m \in \mathbb{N}$ one has that $|f(x_m)| < 1 + q$ for all $m \in \mathbb{N}$ due to the fact that $y_m \in A_{\beta_m} \subseteq A(\alpha|n)$ for all $m \in \mathbb{N}$. Particularly $|f(x_{q+1})| < q + 1$, a contradiction. Therefore $\mathcal{K}$ has an $\mathcal{E}$-bounded envelope, and clearly $\mathcal{K}$ swallows the compact sets of X. By Proposition 3.13 of [14], X is K-analytic. □

Below we apply Corollary 2.6 to prove the following classical result.

Corollary 2.7. *The space $C_k(X)$ is metrizable if and only if X is hemicompact.*

Proof. If X is hemicompact then clearly $C_k(X)$ is metrizable. Conversely, assume that $C_k(X)$ is metrizable. Set $\mathcal{E} = \{E_n\}_{n \in \mathbb{N}}$, where $E_n = C(X)$ for every $n \in \mathbb{N}$. By Corollary 2.6, X is K-analytic and has a compact resolution $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ with $\mathcal{E}$-bounded envelope that swallows the compact sets. By the definition of $\mathcal{E}$, the envelope $\mathcal{M} = \{K(\alpha|n) : \alpha \in \mathbb{N}^{\mathbb{N}}, n \in \mathbb{N}\}$ consists of functionally bounded subsets of X. As X is realcompact, the countable family

$$\overline{\mathcal{M}} = \{\overline{K(\alpha|n)} : \alpha \in \mathbb{N}^{\mathbb{N}}, n \in \mathbb{N}\}$$

consists of compact subsets of X and clearly swallows the compact sets of X. Thus X is hemicompact. □

It is natural to ask whether the topology $\tau_{\mathcal{S}}$ is metrizable. We answer this question in the next proposition which complements Corollary 2.7, its proof is similar to the original proof of the Arens theorem and so it is omitted.

Proposition 2.8. *Let X be a Tychonoff space and $\mathcal{S} \in \mathfrak{S}(X)$. Then the space $C_{\mathcal{S}}(X)$ is metrizable if and only if X is almost hemi-$\mathcal{S}$-compact, i.e., there is a sequence $\{S_n\}_{n\in\mathbb{N}} \subseteq \mathcal{S}$ such that for every $S \in \mathcal{S}$ there exists $n \in \mathbb{N}$ such that $S \subseteq \overline{S_n}$.*

Example 2.9. If X is an uncountable P-space, there is no metrizable locally convex topology $\mathcal{T}$ on $C(X)$ such that $\tau_p \le \mathcal{T} \le \tau_b$. Indeed, otherwise there exists a sequence $\{A_n : n \in \mathbb{N}\}$ of functionally bounded sets in X covering X. But this is impossible, since each functionally bounded set in X is finite (see [11, Problem 4K(3)]).

3. Concrete results

Let X be a Tychonoff space. Denote by υX the Hewitt realcompactification of X. It is clear that a subset B of X is functionally bounded if and only if B is relatively compact in υX. Recall that X is called a *μ-space* if the closure of every functionally bounded subset of X is compact. Every realcompact space is Dieudonné complete, see [6, 8.5.13], and each Dieudonné complete space is a μ-space.

Let us also recall that an $\mathbb{N}^{\mathbb{N}}$-increasing family $\{B_\alpha\}_{\alpha\in\mathbb{N}^{\mathbb{N}}}$ of functionally bounded (compact) subsets of X is called a *functionally bounded* (respectively, *compact*) *resolution* in X if it covers X.

The following theorem deals with the case when a desired topology satisfies inequalities $\tau_p \le \mathcal{T} \le \tau_k$ or $\tau_k \le \mathcal{T} \le \tau_b$, in (vii) we supplement also the list of another results on $C_p(X)$ yielding countability of X, see [21], [22].

Theorem 3.1. *Let X be a Tychonoff space. Then:*

(i) There exists a metrizable locally convex topology $\mathcal{T}$ on $C(X)$ such that $\tau_p \le \mathcal{T} \le \tau_k$ if and only if X is a σ-compact space.

(ii) There exists a metrizable locally convex topology $\mathcal{T}$ on $C(X)$ such that $\tau_k \le \mathcal{T} \le \tau_b$ if and only if there is an increasing sequence $\{B_n\}_{n\in\mathbb{N}}$ of functionally bounded subsets of X swallowing the compact sets of X.

(iii) There exists a metrizable locally convex topology $\mathcal{T}$ on $C(X)$ such that $\tau_p \le \mathcal{T} \le \tau_b$ if and only if there is an increasing sequence $\{B_n\}_{n\in\mathbb{N}}$ of functionally bounded subsets of X covering X.

(iv) There is a metrizable locally convex topology $\mathcal{T}$ on $C(X)$ such that $\tau_p \le \mathcal{T} \le \tau_w$ if and only if X is countable.

(v) *There exists a locally convex topology $\mathcal{T}$ on $C(X)$ with a $\mathfrak{G}$-base such that $\tau_p \le \mathcal{T} \le \tau_k$ if and only if X has a compact resolution.*

(vi) *There exists a locally convex topology $\mathcal{T}$ on $C(X)$ with a $\mathfrak{G}$-base such that $\tau_k \le \mathcal{T} \le \tau_b$ if and only if X has a functionally bounded resolution swallowing the compact sets of X.*

(vii) $C(X)$ admits a locally convex topology $\mathcal{T}$ with a $\mathfrak{G}$-base such that $\tau_p \le \mathcal{T} \le \tau_b$ if and only if X has a functionally bounded resolution.

(viii) There is a locally convex topology $\mathcal{T}$ on $C(X)$ with a $\mathfrak{G}$-base such that $\tau_p \le \mathcal{T} \le \tau_w$ if and only if X is countable.

PROOF. To prove (i) and (v), for every set U in $C(X)$ define $U^\Diamond$ in X by

$$U^\Diamond = \{x \in X : |f(x)| \le 1 \ \forall f \in U\}.$$

Clearly, $U^\Diamond$ is closed in X and $U \subseteq V$ implies that $U^\Diamond \supseteq V^\Diamond$.

Claim. Let $\mathcal{T}$ be a locally convex topology on $C(X)$ such that $\tau_p \le \mathcal{T} \le \tau_k$. If U is a neighborhood of the origin in $(C(X), \mathcal{T})$, then $U^\Diamond$ is compact.

Indeed, as in the proof of [8, Theorem 2], if K is compact and $\epsilon > 0$ then $[K,\epsilon]^{\lozenge} \subseteq K$, since if $x \in X \setminus K$ there is $f \in C(X)$ with $f(x) = 2$ and $f(K) = \{0\}$, so that $f \in [K,\epsilon]$ and $x \notin [K,\epsilon]^{\lozenge}$. If K is compact and $0 < \epsilon \leq 1$, then $K \subseteq [K,\epsilon]^{\lozenge}$ and hence $[K,\epsilon]^{\lozenge} = K$.

Now since $\mathcal{T} \leq \tau_k$ we may choose a compact set K in X such that $[K,\epsilon] \subseteq U$ for some $\epsilon > 0$. Hence $U^{\lozenge} \subseteq [K,\epsilon]^{\lozenge} \subseteq K$, so $U^{\lozenge}$ is compact.

(i) If $X = \bigcup \{K_n : n \in \mathbb{N}\}$ is σ-compact with $K_n \subseteq K_{n+1}$ for each $n \in \mathbb{N}$, we set

$$V_n := \left\{ f \in C(X) : \sup_{x \in K_n} |f(x)| < \frac{1}{n} \right\}$$

Then $\{V_n : n \in \mathbb{N}\}$ is an open decreasing base of absolutely convex neighborhoods of the origin of a metrizable locally convex topology $\mathcal{T}$ on $C(X)$ such that $\tau_p \leq \mathcal{T} \leq \tau_k$.

Conversely, assume that $C(X)$ has a metrizable locally convex topology $\tau_p \leq \mathcal{T} \leq \tau_k$ with a decreasing base $\{U_n : n \in \mathbb{N}\}$ of neighborhoods of the origin. Then, by the claim, the family $\mathcal{K} = \{U_n^{\lozenge} : n \in \mathbb{N}\}$ consists of compact subsets of X. Moreover, if $y \in X$ since $\tau_p \leq \mathcal{T}$, there exists $m \in \mathbb{N}$ such that $U_m \subseteq \{f \in C(X) : |f(y)| \leq 1\}$, which means that $U_m \subseteq [\{y\}, 1]$, so that $y \in U_m^{\lozenge}$. Thus $\mathcal{K}$ is a covering of X and we are done.

(v) If X has a compact resolution $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ set

$$V_\alpha := \left\{ f \in C(X) : \sup_{x \in K_\alpha} |f(x)| < \frac{1}{\alpha(1)} \right\}.$$

Then the family $\{V_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is an open $\mathfrak{G}$-base of a locally convex topology $\mathcal{T}$ on $C(X)$ consisting of absolutely convex sets such that $\tau_p \leq \mathcal{T} \leq \tau_k$.

Conversely, assume that $C(X)$ has a locally convex topology $\tau_p \leq \mathcal{T} \leq \tau_k$ with a $\mathfrak{G}$-base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Then, by the claim, the family $\mathcal{K} = \{U_\alpha^{\lozenge} : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ consists of compact subsets of X. If $y \in X$ since $\tau_p \leq \mathcal{T}$, there exists $\beta \in \mathbb{N}^{\mathbb{N}}$ such that $U_\beta \subseteq \{f \in C(X) : |f(y)| \leq 1\}$. This means that $U_\beta \subseteq [\{y\}, 1]$, so that $\{y\} = [\{y\}, 1]^{\lozenge} \subseteq U_\beta^{\lozenge}$. Thus $y \in U_\beta^{\lozenge}$, which shows that $\mathcal{K}$ is a compact resolution of X.

(ii) and (vi) follow from Corollary 2.3.

(iii) It is known (see [2, Proposition III.2.21]), that X has a sequence of functionally bounded sets covering X if and only if υX is σ-compact. Now the assertion follows from (i) of Corollary 2.3.

(iv) follows from (viii).

(vii) First we note that: (1) by Proposition 3.13 of [14], υX is K-analytic if and only if υX has a compact resolution, and (2) $\{K_\alpha\}_{\alpha \in \mathbb{N}^{\mathbb{N}}}$ is a compact resolution in υX if and only if $\{X \cap K_\alpha\}_{\alpha \in \mathbb{N}^{\mathbb{N}}}$ is a functionally bounded resolution in X. So, if X has a functionally bounded resolution, the space $C(X)$ admits a locally convex topology $\mathcal{T}$ with a $\mathfrak{G}$-base such that $\tau_p \leq \mathcal{T} \leq \tau_b$ by (ii) of Corollary 2.3 applied to $\mathcal{S} = \mathsf{Fin}$.

Assume that $C(X)$ admits a locally convex topology $\mathcal{T}$ with a $\mathfrak{G}$-base $\{V_\alpha\}_{\alpha \in \mathbb{N}^{\mathbb{N}}}$ such that $\tau_p \leq \mathcal{T} \leq \tau_b$. Denote by E the topological dual space of $(C(X), \mathcal{T})$. Then the family $\{V_\alpha^{\circ}\}_{\alpha \in \mathbb{N}^{\mathbb{N}}}$, the polars being taken with respect to E, covers E and is a resolution of E consisting of absolutely convex $\sigma\big(E, C(X)\big)$-compact sets. Denote by $L(X)$ the free vector space over X. Then the space $L_p(X) := \big(L(X), \sigma(L(X), C(X))\big)$ is a vector subspace of $\big(E, \sigma(E, C(X))\big)$. Since every functionally bounded subset of a locally convex space is bounded, we obtain that $\{L(X) \cap V_\alpha^{\circ}\}_{\alpha \in \mathbb{N}^{\mathbb{N}}}$ is a bounded resolution in $L_p(X)$. Thus υX is K-analytic by Lemma 30 of [7].

(viii) The 'if' case is trivial. For the 'only if' case, suppose that such $\mathcal{T}$ exists and proceed by contradiction by assuming that X is uncountable. Let $\mathcal{U} = \{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-base of neighborhoods of the origin of $\mathcal{T}$. Clearly, each $U_\alpha \in \mathcal{U}$ is a neighborhood of the origin for the weak topology τ_w. So the family $\mathcal{M} = \{U_\alpha^{\circ} : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ (polars in $C_k(X)'$) is an $\mathbb{N}^{\mathbb{N}}$-increasing family of subsets of $F := C_k(X)'$ consisting of τ_w-equicontinuous sets. But since $\mathcal{T}$ is stronger than τ_p, $\mathcal{M}$ covers the linear subspace $L(X)$ of F. Note that each U_α° is contained in a finite-dimensional subspace of F. Consequently, we have an $\mathbb{N}^\mathbb{N}$-increasing family $\mathcal{M}$ of subsets of F, covering $L(X)$, consisting of finite-dimensional sets. This implies that each of those sets U_α° meets the canonical copy $\delta(X)$ of X in $\big(F, \sigma(F, C(X)\big)$ in a finite set $U_\alpha^\circ \cap \delta(X)$ (otherwise U_α°, would be infinite-dimensional due to the fact that $\delta(X)$ is a linearly independent set in F). Hence $\mathcal{M}$ meets $\delta(X)$ in a resolution consisting of finite sets. But since X is uncountable, some of these sets must be infinite by Proposition 3.7 of [14]. This contradiction shows that X is countable. □

There are plenty of nonmetrizable locally convex topologies with a $\mathfrak{G}$-base on $C(X)$ as the following example shows.

Example 3.2. Nonmetrizable locally convex topologies on $C(X)$ with a $\mathfrak{G}$-base. If X has a compact resolution but is not σ-compact, then there exists a non-metrizable locally convex topology $\mathcal{T}$ on $C(X)$ with a $\mathfrak{G}$-base such that $\tau_p \leq \mathcal{T} \leq \tau_b$. For instance, if K is an infinite Talagrand compact set, $X := C_p(K)$ is K-analytic but not σ-compact by virtue of Velichko's theorem. So there exists a locally convex topology on $C(X)$ with those characteristics.

Example 3.3. Let κ be the first ordinal of cardinality $2^\mathfrak{c}$. Then κ is a pseudocompact non-compact space whose cofinality is strictly bigger than the continuum $\mathfrak{c}$. As the cofinality of $\mathbb{N}^\mathbb{N}$ is less or equal than $\mathfrak{c}$ we obtain that κ does not have compact resolution, in particular, $C_k(\kappa)$ does not have a $\mathfrak{G}$-base. Clearly, the metrizable topology defined by the sup-norm of $C(\kappa)$ is strictly finer than the compact-open topology.

Example 3.4. Let Z be the subspace of $[0,1]^{\omega_1}$ consisting of transfinite sequences with at most countably many non-zero coordinates and define $X := \bigcup_{n\in\mathbb{N}} nZ \subseteq \mathbb{R}^{\omega_1}$. If τ_{sc} denotes the set-open topology on $C(X)$ defined by the sequentially compact subsets of X, there exists a metrizable locally convex topology $\mathcal{T}$ on $C(X)$ such that $\tau_p < \mathcal{T} < \tau_{sc}$ and neither $\mathcal{T} < \tau_k$ nor $\tau_k < \mathcal{T}$. Indeed, setting $A_n := \bigcup_{k\leq n} kZ$ for $n \in \mathbb{N}$, the sequence $\{A_n : n \in \mathbb{N}\}$ consists of sequentially compact sets and covers X, so it defines a metrizable locally convex topology $\mathcal{T}$ on $C(X)$ such that $\tau_p < \mathcal{T} < \tau_{sc}$. In addition, since each A_n is closed and noncompact, it turns out that $\mathcal{T} \not< \tau_k$. On the other hand, the set $K = \{ne_n : n \in \mathbb{N}\} \cup \{\mathbf{0}\}$, where $e_n(\gamma) = 0$ if $\gamma \neq n$ and $e_n(n) = 1$, $\gamma < \omega_1$, is evidently compact in $\mathbb{R}^{\omega_1}$ and is not contained in any A_n. Therefore $\tau_k \not< \mathcal{T}$.

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References

[1] R.F. Arens, A topology for spaces of transformations, Ann. of Math. **47** (1946), 480–495.

[2] A. V. Arhangelskii, *Topological function spaces*, Math. Appl. **78**, Kluwer Academic Publishers, Dordrecht, 1992.

[3] H. Buchwalter, J. Schmets, Sur quelques propiétés de l′espace $C_s(T)$, J. Math. Pures Appl. **52** (1973), 337–352.

[4] B. Cascales, On K-analytic locally convex spaces, Arch. Math. (Basel) **49** (1987), 232–244.

[5] B. Cascales, J. Orihuela, On compactness in locally convex spaces, Math. Z. **195** (1987), 365–381.

[6] R. Engelking, *General Topology*, PWN, Warszawa, 1977.

[7] J.C. Ferrando, Some characterizations for υX to be Lindelöf Σ or K-analytic in terms of $C_p(X)$, Topology Appl. **156** (2009), 823–830.

[8] J.C. Ferrando, J. Kąkol, On precompact sets in spaces $C_c(X)$, Georgian. Math. J., **20** (2013), 247–254.

[9] K. Floret, *Weakly Compact Sets*, Lecture Notes in Math., 801, Springer, Berlin, 1980.

[10] S. Gabriyelyan, J. Kąkol, A. Leiderman, On topological groups with a small base and metrizability, Fund. Math. **229** (2015), 129–158.

[11] L. Gillman, M. Jerison, *Rings of Continuous Functions*, Van Nostrand, New York, 1960.

[12] W. Govaerts, A productive class of angelic spaces, J. London Math. Soc. **22** (1980), 355–364.

[13] H. Jarchow, *Locally Convex Spaces*, B.G. Teubner Stuttgart, 1981.

[14] J. Kąkol, W. Kubiś, M. Lopez-Pellicer, *Descriptive Topology in Selected Topics of Functional Analysis*, Developments in Mathematics, Springer, 2011.

[15] J. Kąkol, I. Tweddle, Spaces of continuous functions $C_p(X, E)$ as (LM)-spaces, Bull. Belgian Math. Soc. (special volume in honor of N. De Grande-De Kimpe) (2002), 109–117.

[16] R.A. McCoy, I. Ntantu, *Topological Properties of Spaces of Continuous Functions*, Lecture Notes in Math. **1315**, 1988.

[17] L. Narici, E. Beckenstein, *Topological Vector Spaces*, Second Edition, CRC Press, New York, 2011.

[18] J. Orihuela, Pointwise compactness in spaces of continuous functions, J. Lond. Math. Soc. **36** (1987), 143–152.

[19] A.V. Osipov, The C-compact-open topology on function spaces, Topol. Appl. **159** (2012), 3059–3066.

[20] M. Talagrand, Espaces de Banach faiblement K-analytiques, Annals of Math. **110** (1979), 407–438.

[21] V.V. Tkachuk, A space $C_p(X)$ is dominated by irrationals if and only if it is K-analytic, Acta Math. Hung., **107** (2005), 253–265.

[22] B. Cascales, J. Orihuela, V.V. Tkachuk, Domination by second countable spaces and Lindelöf Σ-property, Topology Appl. **158** (2011), 204–214.