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Abstract

We consider the complete Euler system describing the time evolution of a general inviscid compressible fluid. We introduce a new concept of measure–valued solution based on the total energy balance and entropy inequality for the physical entropy without any renormalization. This class of so–called dissipative measure–valued solutions is large enough to include the vanishing dissipation limits of the Navier–Stokes–Fourier system. Our main result states that any sequence of weak solutions to the Navier–Stokes–Fourier system with vanishing viscosity and heat conductivity coefficients generates a dissipative measure-valued solution of the Euler system under some physically grounded constitutive relations. Finally, we discuss the same asymptotic limit for the bi-velocity fluid model introduced by H.Brenner.

Keywords: Euler system, measure–valued solution, weak-strong uniqueness, vanishing dissipation limit

1 Introduction

We consider the *complete Euler system* describing the time evolution of the mass density $\varrho = \varrho(t, x)$, the temperature $\vartheta = \vartheta(t, x)$ and the velocity field $\mathbf{u} = \mathbf{u}(t, x)$ of a compressible inviscid fluid:

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$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = 0, \tag{1.2}$$

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e \right) + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e + p \right) \mathbf{u} \right] = 0. \tag{1.3}$$

The system (1.1–1.3) contains the thermodynamic functions: The pressure $p(\varrho, \vartheta)$ and the (specific) internal energy $e(\varrho, \vartheta)$ depending on the state variables ϱ, ϑ and satisfying Gibbs' relation

$$\vartheta Ds = De + pD \left(\frac{1}{\varrho} \right). \tag{1.4}$$

The new quantity s appearing in (1.4) is the (specific) entropy. It follows from (1.4) that any smooth solution of (1.1–1.3) satisfies also the entropy balance

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) = 0 \text{ or } \partial_t s + \mathbf{u} \cdot \nabla_x s = 0. \tag{1.5}$$

In the context of *weak solutions*, the equation (1.5) is relaxed to the inequality

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) \geq 0, \tag{1.6}$$

see e.g. Benzoni-Gavage, Serre [3], Dafermos [13]. To avoid problems with physical boundaries, we restrict ourselves to the periodic boundary conditions, meaning the underlying physical domain Ω can be identified with the flat torus,

$$\Omega = \left([0, 1]|_{\{0,1\}} \right)^N, \ N = 1, 2, 3.$$

The problem is formally closed by prescribing the initial data

$$\varrho(0, \cdot) = \varrho_0, \ \vartheta(0, \cdot) = \vartheta_0, \ \mathbf{u}(0, \cdot) = \mathbf{u}_0. \tag{1.7}$$

In view of recent results based on the theory of convex integration, see [15], weak solutions of (1.1–1.3), even if supplemented by (1.6), are not uniquely determined by the initial data as long as $N > 1$. As a matter of fact, for any *piecewise constant* initial density ϱ_0 and ϑ_0, there exists $\mathbf{u}_0 \in L^\infty(\Omega; R^N)$, $N = 2, 3$ such that problem (1.1–1.3), (1.6), and (1.7) admits infinitely many weak (distributional) solutions on a given time interval $(0, T)$. This kind of result indicates that the measure–valued solutions that are supposed to capture possible oscillatory behavior of the weak solutions may be a suitable concept for the Euler system.

In [9], we have introduced a concept of dissipative measure–valued (DMV) solution, based on postulating the total energy balance and a renormalized version of the entropy production equation, see Definition 2.4 below. In contrast with the standard approach used e.g. in a series of papers by Fjordholm et al. [20], [21], [22], based on the hypothetical $L^\infty-$bounds on the family of generating solutions, our definition covers a more general class of objects that may be seen as suitable limits of weak solutions (cf. [9]) or even certain numerical schemes. In the present paper, we introduce a slightly different definition of (DMV) solutions without entropy renormalization, see Definition 2.1 below. Our main result asserts that this kind of measure–valued solutions can be recovered as a vanishing viscosity limit of a natural physical approximation - the Navier–Stokes–Fourier system. More specifically, we show that any sequence of weak solutions to the Navier–Stokes–Fourier system, the existence of which is guaranteed by [17], generates a (DMV) solution of the Euler system under some physically grounded constitutive relations. Finally, we discuss the same asymptotic limit for the bi-velocity fluid model introduced by H.Brenner.

The paper is organized as follows. In Section 2 we introduce the basic concepts as well as the known results used in the text. In Section 3, we consider the vanishing dissipation limit of the weak solutions to the Navier–Stokes–Fourier system and show that they generate a (DMV) solution of the Euler system. Finally, in Section 4, we briefly discuss similar issues for a bi-velocity fluid model proposed by H.Brenner.

2 Preliminary results

In this preliminary section, we introduce conservative variables, the concept of (DMV) solution as well as other already known results used in the paper.

2.1 Conservative variables

In certain situations, for instance in numerical simulations, it is more convenient to introduce the conservative variables: The density ϱ, the momentum $\mathbf{m} = \varrho\mathbf{u}$ and the total energy $E = \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e$ converting (1.1–1.3), (1.6) into

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0, \tag{2.1}$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p = 0, \tag{2.2}$$

$$\partial_t E + \mathrm{div}_x \left[(E + p) \frac{\mathbf{m}}{\varrho} \right] = 0, \tag{2.3}$$

$$\partial_t (\varrho s) + \mathrm{div}_x (s\mathbf{m}) \geq 0. \tag{2.4}$$

For the sake of simplicity, we focus on the case of polytropic gas, for which the pressure is related to the internal energy by the caloric equation of state

$$p = (\gamma - 1)\varrho e, \text{ with the adiabatic exponent } \gamma > 1. \tag{2.5}$$

Accordingly, we have

$$p = (\gamma - 1)\left[E - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}\right] \tag{2.6}$$

closing the system of equations (2.1–2.3). Under these circumstances, it is convenient to consider the specific entropy $s = s(\varrho, e)$ as a function of ϱ, e, for which Gibbs' relation (1.4) yields

$$\frac{\partial s}{\partial e}(\varrho, e) = \frac{1}{\vartheta}, \ \frac{\partial s}{\partial \varrho}(\varrho, e) = -\frac{p}{\vartheta \varrho^2},$$

where the first relation can be seen as a definition of the absolute temperature. Moreover, it can be deduced from Gibbs' relation (1.4) and (2.5) that the entropy s can be written in the form

$$s(\varrho, e) = S\left(\frac{p}{\varrho^\gamma}\right) = S\left(\frac{(\gamma - 1)e}{\varrho^{\gamma - 1}}\right) \tag{2.7}$$

for a suitable function S.

2.2 Thermodynamic stability

In the original state variables ϱ, ϑ, the thermodynamic stability hypothesis reads

$$\frac{\partial p}{\partial \varrho}(\varrho, \vartheta) > 0, \ \frac{\partial e}{\partial \vartheta}(\varrho, \vartheta) > 0 \text{ for any } \varrho, \vartheta > 0. \tag{2.8}$$

In the conservative variables $(\varrho, \mathbf{m}, E)$, this is equivalent to the statement that the total entropy

$$\mathcal{S}(\varrho, \mathbf{m}, E) = \varrho S\left(\frac{(\gamma - 1)\left(E - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}\right)}{\varrho^\gamma}\right) \tag{2.9}$$

is a concave function of $(\varrho, \mathbf{m}, E)$, cf. Bechtel, Rooney, and Forest [2]. It is a matter of direct computation to check that, in terms of the function S introduced in (2.7), the condition (2.8) reduces to

$$(1 - \gamma)S'(Z) - \gamma S''(Z)Z > 0 \text{ for all } Z > 0. \tag{2.10}$$

Note that the domain of definition of S may not be $(0, \infty)$. To see this, it is convenient to write p and e interrelated through (2.5) as functions of ϱ and ϑ. Accordingly, Gibbs' equation (1.4) can be written in the form of Maxwell's relation

$$\frac{\partial e}{\partial \varrho} = \frac{1}{\varrho^2}\left(p - \vartheta\frac{\partial p}{\partial \vartheta}\right),$$

which, together with (2.5), gives rise to

$$p(\varrho,\vartheta) = \frac{P(q)}{q^\gamma}\varrho^\gamma \text{ for a certain function } P, \text{ where we have set } q = \frac{\varrho}{\vartheta^{c_v}},\ c_v = \frac{1}{\gamma-1}. \tag{2.11}$$

Furthermore, it follows from the second inequality in (2.8) that

$$q \mapsto \frac{P(q)}{q^\gamma} \text{ is a non–incresing function of } q,$$

in particular,

$$\lim_{\vartheta\to 0+} \frac{p(\varrho,\vartheta)}{\varrho^\gamma} = \overline{p} \geq 0.$$

We infer that the "natural" domain of definition of $S = S(Z)$ is the interval $(\overline{p},\infty)$,

$$S : (\overline{p},\infty) \to R.$$

In addition, we define

$$S(Z) = \left\{ \begin{array}{l} -\infty \text{ for } Z < \overline{p}, \\ \lim_{Z\to\overline{p}+} S(Z) \in [-\infty,\infty) \text{ for } Z = \overline{p}. \end{array} \right.$$

Accordingly, the total entropy $\mathcal{S} = \mathcal{S}(\varrho,\mathbf{m},E)$ is concave upper semi–continuous in $[0,\infty)\times R^N \times [0,\infty)$ ranging in $[-\infty,\infty)$.

Similarly, we observe that the kinetic energy

$$(\varrho,\mathbf{m}) \mapsto \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} \text{ is a convex function for } \varrho > 0;$$

whence we may define

$$\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} = \left\{ \begin{array}{l} 0 \text{ whenever } \mathbf{m} = 0,\ \varrho \geq 0, \\ \infty \text{ for } \varrho = 0,\ \mathbf{m} \neq 0. \end{array} \right.$$

The kinetic energy is therefore a convex lower semi–continuous function defined for $(\varrho,\mathbf{m}) \in [0,\infty)\times R^N$ ranging in $[0,\infty]$.

2.3 Relative energy

The relative energy functional introduced in [18], reads

$$\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\middle|\tilde\varrho,\tilde\vartheta,\tilde{\mathbf{u}}\right) = \frac{1}{2}\varrho|\mathbf{u}-\tilde{\mathbf{u}}|^2 + H_{\tilde\vartheta}(\varrho,\vartheta) - \partial_\varrho H_{\tilde\vartheta}(\tilde\varrho,\tilde\vartheta)(\varrho-\tilde\varrho) - H_{\tilde\vartheta}(\tilde\varrho,\tilde\vartheta)$$
$$H_{\tilde\vartheta}(\varrho,\vartheta) \equiv \varrho\Big(e(\varrho,\vartheta) - \tilde\vartheta s(\varrho,\vartheta)\Big).$$

Passing to the conservative variables,

$$\mathbf{m} = \varrho \mathbf{u},\ E = \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho, \vartheta),\ \tilde{\mathbf{m}} = \tilde{\varrho}\tilde{\mathbf{u}},\ \tilde{E} = \frac{1}{2}\tilde{\varrho}|\tilde{\mathbf{u}}|^2 + \tilde{\varrho} e(\tilde{\varrho}, \tilde{\vartheta}),$$

we can check by a bit tedious but straightforward manipulation that

$$\begin{aligned}
\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \middle| \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}\right) &\equiv \mathcal{E}\left(\varrho, E, \mathbf{m} \middle| \tilde{\varrho}, \tilde{E}, \tilde{\mathbf{m}}\right) \\
&= -\tilde{\vartheta}\Big[\mathcal{S}(\varrho, \mathbf{m}, E) \\
&\qquad - \partial_\varrho \mathcal{S}(\tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{E})(\varrho - \tilde{\varrho}) - \nabla_{\mathbf{m}} \mathcal{S}(\tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{E}) \cdot (\mathbf{m} - \tilde{\mathbf{m}}) - \partial_E \mathcal{S}(\tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{E})(E - \tilde{E}) \\
&\qquad - \mathcal{S}(\tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{E})\Big],
\end{aligned} \tag{2.12}$$

where $\mathcal{S}$ is the total entropy introduced in (2.9). It is worth–noting that the expression in the brackets on the right–hand side of (2.12) coincides with the relative entropy à la Dafermos [12]. In agreement with the discussion in the previous section, the relative energy plays a role of distance between $(\varrho, \vartheta, \mathbf{u})$ and $(\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}})$, or, equivalently, between $(\varrho, E, \mathbf{m})$ and $(\tilde{\varrho}, \tilde{E}, \tilde{\mathbf{m}})$, as long as the thermodynamic stability hypothesis holds.

2.4 Measure-valued solutions

In contrast with our preceding paper [9], we define a *dissipative measure–valued (DMV) solution* with respect to the conservative variables $(\varrho, \mathbf{m}, E)$. Accordingly, the phase space for the associated Young measure is

$$Q = \left\{ (\varrho, \mathbf{m}, E) \ \Big| \ \varrho \in [0, \infty),\ \mathbf{m} \in R^N,\ E \in [0, \infty) \right\}.$$

Definition 2.1. A *dissipative measure–valued solution* to the problem (2.1–2.4) consists of a family of *parameterized probability measures* $\{Y_{t,x}\}_{t \in (0,T), x \in \Omega}$ and a non–negative function $\mathcal{D} \in L^\infty(0,T)$ called *dissipation defect* satisfying:

- $Y \in L^\infty_{\text{weak}-(*)}((0,T) \times \Omega; \mathcal{P}(Q))$, where $\mathcal{P}(Q)$ denotes the set of probability measures on Q;
- $$\int_0^\tau \int_\Omega \left[\langle Y_{t,x}; \varrho \rangle \, \partial_t \varphi + \langle Y_{t,x}; \mathbf{m} \rangle \cdot \nabla_x \varphi \right] \ \mathrm{d}x \, \mathrm{d}t = \left[\int_\Omega \langle Y_{t,x}; \varrho \rangle \, \varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} \tag{2.13}$$
  for a.a. $\tau \in (0,T)$ and any $\varphi \in C^\infty([0,T] \times \Omega)$;
- $$\begin{aligned}
&\int_0^\tau \int_\Omega \left[\langle Y_{t,x}; \mathbf{m} \rangle \cdot \partial_t \boldsymbol{\varphi} + \left\langle Y_{t,x}; \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right\rangle : \nabla_x \boldsymbol{\varphi} + (\gamma - 1) \left\langle Y_{t,x}; \left(E - \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} \right) \right\rangle \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x \, \mathrm{d}t \\
&\quad = \left[\int_\Omega \langle Y_{t,x}; \mathbf{m} \rangle \cdot \boldsymbol{\varphi} \ \mathrm{d}x \right]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mu_C
\end{aligned} \tag{2.14}$$

for a.a. $\tau \in (0,T)$ and any $\boldsymbol{\varphi} \in C^\infty([0,T]\times\Omega; R^N)$, where μ_C is a (vectorial) signed measure on $[0,T]\times\Omega$;

- $$\left[\int_\Omega \langle Y_{t,x}; E \rangle \ \mathrm{d}x \right]_{t=0}^{t=\tau} + \mathcal{D}(\tau) = 0 \tag{2.15}$$

  for a.a. $\tau \in (0,T)$;

- $$\begin{aligned} &\left[\int_\Omega \left\langle Y_{t,x}; \varrho s \left(\varrho, \frac{1}{\varrho}\left(E - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} \right) \right) \right\rangle \varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} \\ &\geq \int_0^\tau \int_\Omega \left[\left\langle Y_{t,x}; \varrho s \left(\varrho, \frac{1}{\varrho}\left(E - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} \right) \right) \right\rangle \right] \partial_t \varphi \ \mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^\tau \int_\Omega \left[\left\langle Y_{t,x}; s \left(\varrho, \frac{1}{\varrho}\left(E - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} \right) \right) \mathbf{m} \right\rangle \cdot \nabla_x \varphi \right] \ \mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{2.16}$$

  for a.a. $\tau \in (0,T)$, any $\varphi \in C^\infty([0,T]\times\Omega)$, $\varphi \geq 0$;

- $$\|\mu_C\|_{\mathcal{M}([0,\tau]\times\Omega)} \leq c \int_0^\tau \mathcal{D}(t) \ \mathrm{d}t \text{ for a.a. } \tau \in (0,T). \tag{2.17}$$

Remark 2.2. The parameterized family of measures $\{Y_{0,x}\}_{x\in\Omega}$ plays the role of initial conditions, cf. (1.7).

Remark 2.3. In Definition 2.1 we tacitly assume that all integrals are finite. In particular, as

$$\varrho s \left(\varrho, \frac{1}{\varrho}\left(E - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} \right) \right) = \mathcal{S}(\varrho, \mathbf{m}, E),$$

where $\mathcal{S}$ is a concave function of all arguments, we have

$$Y_{t,x} \left\{ (\varrho, \mathbf{m}, E) \ \middle|\ \varrho \geq 0,\ E - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} \geq \frac{\overline{p}}{\gamma - 1}\varrho^\gamma \right\} = 1$$

for a.a. $(t,x) \in (0,T)\times\Omega$.

Following [9], we may replace (2.16) by its renormalized version, namely

$$\begin{aligned} &\left[\int_\Omega \left\langle Y_{t,x}; \varrho Z \left(s \left(\varrho, \frac{1}{\varrho}\left(E - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} \right) \right) \right) \right\rangle \varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} \\ &\geq \int_0^\tau \int_\Omega \left[\left\langle Y_{t,x}; \varrho Z \left(s \left(\varrho, \frac{1}{\varrho}\left(E - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} \right) \right) \right) \right\rangle \right] \partial_t \varphi \ \mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^\tau \int_\Omega \left[\left\langle Y_{t,x}; Z \left(s \left(\varrho, \frac{1}{\varrho}\left(E - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} \right) \right) \right) \mathbf{m} \right\rangle \cdot \nabla_x \varphi \right] \ \mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{2.18}$$

for a.a. $\tau \in (0,T)$, any $\varphi \in C^1([0,T] \times \Omega)$, $\varphi \geq 0$, where $Z \in C(R)$ is a nondecreasing concave function and $Z(s) \leq Z_\infty$ for any $s \in R$.

Definition 2.4. A *renormalized (DMV) solution* to (2.1–2.4) consists of a family of parameterized probability measures $\{Y_{t,x}\}_{t\in(0,T),x\in\Omega}$ and a non–negative function $\mathcal{D} \in L^\infty(0,T)$ that satisfy all the requirements of Definition 2.1 except (2.16), which is replaced by (2.18).

The concept of renormalized (DMV) solution is motivated by a similar definition of the weak solutions introduced in Chen and Frid [11]. As shown in [9, Section 2.1.1], the renormalized solutions enjoy certain minimum principle, in particular

$$Y_{0,x}(\{s(\varrho, \mathbf{m}, E) \geq s_0\}) = 1 \text{ implies } Y_{t,x}(\{s(\varrho, \mathbf{m}, E) \geq s_0\}) = 1 \text{ for a.a. } (t,x).$$

2.5 Weak–strong uniqueness

As shown in [9, Theorem 3.3], the renormalized (DMV) solutions coincide with strong solutions emanating from the same initial data. The same result can be shown for the (DMV) solutions in the sense of Definition 2.1. The proof requires only obvious modification. In this context, the weak–strong uniqueness principle reads:

Theorem 2.5. **[Weak (measure-valued) - strong uniqueness principle]**

Let the thermodynamic functions p, e, and s satisfy Gibbs' equation (1.4), and the thermodynamic stability condition (2.8). In addition, let the pressure be related to the internal energy through the caloric equation of state (2.6). Suppose that the Euler system (2.1–2.4) admits a smooth (C^1) solution $(\varrho, \mathbf{m}, E)$ in $[0,T)$ originating from the initial data $(\varrho_0, \mathbf{m}_0, E_0)$,

$$\varrho_0 \in C^1(\Omega),\ \varrho_0 > 0,\ \mathbf{m}_0 \in C^1(R^N; R^N),\ E_0 \in C^1(R),\ E_0 - \frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} > \frac{\overline{p}}{\gamma - 1}\varrho_0^\gamma.$$

Let $[\{Y_{t,x}\}, \mathcal{D}]$ be a (DMV) solution (renormalized (DMV) solution) in the sense of Definition 2.1 (Definition 2.4) starting from the same initial data, meaning

$$Y_{0,x} = \delta_{(\varrho_0(x), \mathbf{m}_0(x), E_0(x))} \text{ for a.a. } x \in \Omega.$$

Then $\mathcal{D} = 0$, and

$$Y_{t,x} = \delta_{(\varrho(t,x), \mathbf{m}(t,x), E(t,x))} \text{ for a.a. } (t,x) \in (0,T) \times \Omega.$$

3 Measure–valued solutions generated in the vanishing dissipation limit

The measure–valued solutions are natural candidates for describing the zero dissipation limits of more complex systems of Navier–Stokes type. Here, we show two results in this direction. We restrict ourselves to the physically relevant case $N = 3$ although the same proof can be adapted to the case $N = 1, 2$.

3.1 Vanishing dissipation limit of the Navier–Stokes–Fourier system

The full compressible Navier–Stokes–Fourier system describing the motion of a general viscous and heat conducting fluid reads:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{3.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S}, \tag{3.2}$$

$$\partial_t (\varrho e) + \mathrm{div}_x (\varrho e \mathbf{u}) + \nabla_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - p \mathrm{div}_x \mathbf{u}. \tag{3.3}$$

The viscous stress $\mathbb{S}$ and the heat flux $\mathbf{q}$ are determined by Stokes' law

$$\mathbb{S} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \tag{3.4}$$

and Fourier's law

$$\mathbf{q} = -\kappa \nabla_x \vartheta. \tag{3.5}$$

Finally, the relevant entropy balance is

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \nabla_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right). \tag{3.6}$$

3.2 Weak solutions

We adopt the concept of weak solution to the Navier–Stokes–Fourier system introduced in [17, Chapters 2, 3].

Definition 3.1. We say that $(\varrho, \vartheta, \mathbf{u})$ is a *weak solution* of the Navier–Stokes–Fourier system if:

- $\varrho \geq 0$, $\vartheta > 0$ a.a. in $(0, T) \times \Omega$;
- the equations (3.1), (3.2) are satisfied in the sense of (space periodic) distributions;

- the entropy balance (3.6), is relaxed to the inequality

$$\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) \geq \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right)$$

  satisfied in the sense of distributions;

- the total energy balance

$$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right](\tau,\cdot)\ \mathrm{d}x = \int_\Omega \left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0,\vartheta_0)\right]\ \mathrm{d}x \tag{3.7}$$

  holds for a.a. $\tau \in [0,T]$.

Remark 3.2. Furthermore, a weak solution $(\varrho,\vartheta,\mathbf{u})$ must belong to a certain regularity class for the weak formulation to make sense. The reader may consult [17, Chapters 2, 3] for details.

3.3 Constitutive equations

In analogy with Section 2.1, we consider the pressure of the monoatomic gas related to the internal energy through

$$p = \frac{2}{3}\varrho e,\ \text{meaning } \gamma = \frac{5}{3}. \tag{3.8}$$

As shown in [17, Chapter 2], relation (3.8) implies that

$$p(\varrho,\vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) \tag{3.9}$$

for some function P, cf. (2.11). In agreement with [17, Chapter 3], we further assume that $P \in C^1[0,\infty) \cap C^5(0,\infty)$ satisfies

$$P(0) = 0,\ P'(q) > 0 \text{ for all } q \geq 0, \tag{3.10}$$

$$0 < \frac{\frac{5}{3}P(q) - P'(q)q}{q} < c \text{ for all } q > 0,\ \lim_{q\to\infty}\frac{P(q)}{q^{5/3}} = \overline{p} > 0. \tag{3.11}$$

All the above requirements are just consequences of the thermodynamic stability hypothesis (2.8), except the stipulation $\overline{p} > 0$. Note that the standard pressure law $p = \varrho\vartheta$ corresponds to $P(Z) = Z$, $\overline{p} = 0$.

In agreement with (3.8), we set

$$e(\varrho,\vartheta) = \frac{3}{2}\vartheta\left(\frac{\vartheta^{3/2}}{\varrho}\right) P\left(\frac{\varrho}{\vartheta^{3/2}}\right), \tag{3.12}$$

and, by virtue of Gibbs' relation (1.4),

$$s(\varrho,\vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right), \tag{3.13}$$

where

$$S'(q) = -\frac{3}{2} \frac{\frac{5}{3} P(q) - P'(q) q}{q^2} < 0. \tag{3.14}$$

We also impose the third law of thermodynamics in the form

$$\lim_{q \to \infty} S(q) = 0. \tag{3.15}$$

Remark 3.3. In the context of viscous fluids, it is convenient to work with the variables $(\varrho, \vartheta, \mathbf{u})$. The function S in (3.13) is therefore not the same as its counterpart expressed in the conservative variables in (2.7). Indeed the function $S = S(Z)$ in (2.7) is expressed in terms of $Z = p/\varrho^\gamma$, while S in (3.13) is a function of $q = \varrho/\vartheta^{c_v}$. If, for instance, $p = \varrho\vartheta$, we obtain $q = Z^{1-\gamma}$.

Finally, we suppose that the transport coefficients μ, η and κ are continuously differentiable functions of the temperature ϑ,

$$\mu, \eta \in C^1([0,\infty)),\ |\mu'(\vartheta)| \leq c,\ \underline{\mu}(1+\vartheta) \leq \mu(\vartheta),\ 0 \leq \eta(\vartheta) \leq \overline{\eta}(1+\vartheta) \text{ for all } \vartheta \geq 0 \tag{3.16}$$

for certain constants $\underline{\mu} > 0$, $\overline{\eta} > 0$ and

$$\kappa \in C^1([0,\infty)),\ \underline{\kappa}(1+\vartheta^3) \leq \kappa(\vartheta) \leq \overline{\kappa}(1+\vartheta^3) \text{ for all } \vartheta \geq 0 \tag{3.17}$$

for certain constants $\underline{\kappa} > 0$, $\overline{\kappa} > 0$.

3.4 Existence of weak solutions for the Navier–Stokes–Fourier system

Motivated by the existence theory developed in [17, Chapter 3], we consider the following system:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{3.18}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(p + a p_R) = \nu \mathrm{div}_x \mathbb{S}, \tag{3.19}$$

$$\partial_t(\varrho(e + a e_R)) + \mathrm{div}_x(\varrho(e + a e_R)\mathbf{u}) + \omega \nabla_x \mathbf{q} = \nu \mathbb{S} : \nabla_x \mathbf{u} - p \mathrm{div}_x \mathbf{u} - \lambda(\vartheta - \overline{\vartheta})^3. \tag{3.20}$$

Here p_R, e_R are the radiation pressure and internal energy introduced in [17, Chapter 3],

$$p_R = \frac{1}{3}\vartheta^4,\ e_R = \frac{\vartheta^4}{\varrho}.$$

The radiation components are multiplied by a (small) constant $a > 0$. They provide a regularizing effect necessary for the existence theory developed in [17, Chapter 3]. The internal energy (3.20) contains a source term $-\lambda(\vartheta - \overline{\vartheta})^3$, $\lambda > 0$ that may be interpreted as radiative "cooling" above a threshold temperature $\overline{\vartheta}$. The presence of this term provides a certain stabilizing effect necessary to perform the vanishing dissipation limit, cf. also [16]. Note that the associated total energy and entropy balance read

$$\left[\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e + a\vartheta^4\right)\ \mathrm{d}x\right]_{t=0}^{t=\tau} + \lambda \int_0^\tau \int_\Omega (\vartheta - \overline{\vartheta})^3\ \mathrm{d}x\,\mathrm{d}t = 0, \tag{3.21}$$

$$\partial_t(\varrho(s+as_R))+\mathrm{div}_x(\varrho(s+as_R)\mathbf{u})+\omega\mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right)\geq\frac{1}{\vartheta}\left(\nu\mathbb{S}:\nabla_x\mathbf{u}-\omega\frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right)+\lambda\frac{(\overline{\vartheta}-\vartheta)^3}{\vartheta} \quad (3.22)$$

with

$$s_R=\frac{4}{3}\frac{\vartheta^3}{\varrho}.$$

As stated in [17, Chapter 3, Theorem 3.1], the problem (3.18–3.20) admits a global-in-time weak solution in the sense of Definition 3.1, whenever a, ν, ω, and λ are positive and the constitutive restrictions specified in Section 3.3 hold.

3.5 The asymptotic limit

Our goal is to send $a\to 0$, $\nu\to 0$, $\omega\to 0$, and $\lambda\to 0$ to recover a dissipative measure–valued solution of the Euler system. To this end, the following issues will be addressed:

- Uniform bounds based on the energy estimates that will guarantee boundedness of the state variables $(\varrho,\mathbf{m},E)$.
- Showing that the dissipation terms vanish in the asymptotic limit.
- Identifying the dissipation defect $\mathcal{D}$ as well as the Young measure $\{Y_{t,x}\}$ associated to the family of weak solutions.

3.5.1 Uniform bounds

The total energy balance (3.21) yields immediately

$$\operatorname*{ess\,sup}_{t\in(0,T)}\int_\Omega\left[\frac{|\mathbf{m}|^2}{\varrho}+\varrho e+a\vartheta^4\right]\ \mathrm{d}x+\lambda\int_0^T\int_\Omega\vartheta^3\ \mathrm{d}x\,\mathrm{d}t\leq c(a,\mathrm{data}). \quad (3.23)$$

In addition, it follows from hypotheses (3.10), (3.11), and (3.12) that

$$\varrho e(\varrho,\vartheta)\gtrsim\varrho\vartheta+\varrho^{5/3};$$

whence, in view of (3.23),

$$\operatorname*{ess\,sup}_{t\in(0,T)}\|\varrho(t,\cdot)\|_{L^{5/3}(\Omega)}\leq c(a,\mathrm{data}),\ \operatorname*{ess\,sup}_{t\in(0,T)}\|\varrho\vartheta(t,\cdot)\|_{L^1(\Omega)}\leq c(a,\mathrm{data}). \quad (3.24)$$

Finally, writing $\mathbf{m}=\sqrt{\varrho}\sqrt{\varrho}\mathbf{u}$, we deduce from (3.23), (3.24) that

$$\operatorname*{ess\,sup}_{t\in(0,T)}\|\mathbf{m}(t,\cdot)\|_{L^{5/4}(\Omega)}\leq c(a,\mathrm{data}). \quad (3.25)$$

Here and hereafter, the symbol $a\lesssim b$ means $a\leq cb$ for a certain constant $c>0$.

Next, we have to handle the terms in the entropy balance. Writing

$$\varrho|s| \leq \frac{1}{2}\varrho + \frac{1}{2}\varrho s^2, \ \ \varrho|s|\mathbf{u} \leq \frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{2}\varrho s^2$$

we can see that it is enough to control ϱs^2 in L^q for some $q > 1$. To this end, we use the third law of thermodynamics encoded in hypothesis (3.15):

$$|s| \stackrel{<}{\sim} 1 \text{ whenever } \vartheta^{3/2} \leq \varrho.$$

If $\varrho < \vartheta^{3/2}$, we deduce from (3.14) that

$$\varrho s^2 \stackrel{<}{\sim} \varrho|\log(\varrho)|^2 + \varrho|\log(\vartheta)|^2,$$

where $\varrho \log(\varrho)$ is controlled in the full range of ϱ's by (3.24). As for $\varrho|\log(\vartheta)|^2$ it is dominated by $\varrho\vartheta$ as long as $\vartheta \geq 1$. Thus it remains to control $\varrho|\log(\vartheta)|^2$ in the range $\varrho < \vartheta^{3/2}$, $\vartheta < 1$:

$$\varrho|\log(\vartheta)|^2 \leq \varrho|\log(\varrho^{2/3})| = \frac{2}{3}\varrho|\log(\varrho)|.$$

We may infer that

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\varrho s\|_{L^q(\Omega)} + \operatorname*{ess\ sup}_{t\in(0,T)} \|\varrho s\mathbf{u}\|_{L^q(\Omega;R^3)} \leq c(a, \text{data}) \text{ for some } q > 1. \tag{3.26}$$

Note that all estimates obtained so far are uniform with respect to the parameters ν, ω, λ, and a as long as $a \stackrel{<}{\sim} 1$. They are strong enough to pass to the limit in the system (3.18–3.20) to generate a (DMV) solution of the limit Euler system as soon as we show that the dissipative terms vanish in the asymptotic regime. First observe that (3.22), together with hypotheses (3.16), (3.17), gives rise to the bound

$$\nu \int_0^T \int_\Omega \left| \nabla_x \mathbf{u} + \nabla_x \mathbf{u}^t - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I} \right|^2 \ \mathrm{d}x\,\mathrm{d}t \leq c(a, \text{data}),$$

which, after a simple by parts integration, yields

$$\nu \int_0^T \int_\Omega |\nabla_x \mathbf{u}|^2 \ \mathrm{d}x \leq c(a, \text{data}). \tag{3.27}$$

Seeing that the total mass of the fluid is conserved,

$$\int_\Omega \varrho(t,\cdot) \ \mathrm{d}x = \int_\Omega \varrho_0 \ \mathrm{d}x \text{ for any } t \geq 0, \tag{3.28}$$

we may use a version of Poincare's inequality to deduce from (3.24), (3.27), (3.28) that

$$\nu \int_0^T \int_\Omega |\mathbf{u}|^2 \ \mathrm{d}x\,\mathrm{d}t \stackrel{<}{\sim} \nu \left[\int_0^T \int_\Omega \varrho|\mathbf{u}|^2 \ \mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega |\nabla_x \mathbf{u}|^2 \ \mathrm{d}x\,\mathrm{d}t \right] \leq c(a, \text{data});$$

whence

$$\nu \int_0^T \|\mathbf{u}\|^2_{W^{1,2}(\Omega;R^3)}\,\mathrm{d}t \leq c(a,\mathrm{data}). \tag{3.29}$$

Applying the same treatment to ϑ, we get

$$\omega \int_0^T \int_\Omega \left(\frac{1}{\vartheta^2} + \vartheta \right) |\nabla_x \vartheta|^2 \ \mathrm{d}x\,\mathrm{d}t \lesssim c(a,\mathrm{data}), \tag{3.30}$$

and, using (3.24),

$$\omega \int_0^T \int_\Omega |\vartheta|^2 \ \mathrm{d}x\,\mathrm{d}t \lesssim \omega \left[\int_0^T \left(\int_\Omega \varrho\vartheta \ \mathrm{d}x \right)^2 \mathrm{d}t + \int_0^T \int_\Omega |\nabla_x \vartheta|^2 \ \mathrm{d}x\,\mathrm{d}t \right] \leq c(a,\mathrm{data});$$

whence

$$\omega \int_0^T \|\vartheta\|^2_{W^{1,2}(\Omega)}\,\mathrm{d}t \leq c(a,\mathrm{data}). \tag{3.31}$$

3.5.2 Vanishing dissipation limit

For the sake of simplicity, we suppose that

$$\nu = \omega = \varepsilon,\ a = \varepsilon^\alpha,\ \lambda = \varepsilon^\beta$$

for suitable $\alpha > 0$, $\beta > 0$ fixed below. For $(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon)_{\varepsilon>0}$ - a sequence of weak solutions of problem (3.18–3.20) in the sense of Definition 3.1 - we set

$$(\varrho_\varepsilon, \mathbf{m}_\varepsilon, E_\varepsilon)_{\varepsilon>0},\ \mathbf{m}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon,\ E_\varepsilon = \frac{1}{2}\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon).$$

In addition, we suppose that the initial data

$$\left(\varrho_{0,\varepsilon}, \varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon}, \frac{1}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2 + \varrho_{0,\varepsilon} e(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) \right)_{\varepsilon>0}$$

generate a Young measure $Y_{0,x}$, specifically,

$$\begin{aligned}
\int_\Omega \varrho_{0,\varepsilon}\phi \ \mathrm{d}x &\to \int_\Omega \langle Y_{0,x}; \varrho \rangle\, \phi \ \mathrm{d}x \text{ for any } \phi \in C_c^\infty(\Omega);\\
\int_\Omega \varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon} \cdot \boldsymbol{\phi} \ \mathrm{d}x &\to \int_\Omega \langle Y_{0,x}; \mathbf{m} \rangle \cdot \boldsymbol{\phi} \ \mathrm{d}x \text{ for any } \boldsymbol{\phi} \in C_c^\infty(\Omega;R^3),\\
\int_\Omega \left[\frac{1}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2 + \varrho_{0,\varepsilon} e(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) + \varepsilon^\alpha \vartheta_{0,\varepsilon}^4 \right] \phi \ \mathrm{d}x &\to \int_\Omega \langle Y_{0,x}; E \rangle\, \phi \ \mathrm{d}x \text{ for any } \phi \in C_c^\infty(\Omega);\\
\int_\Omega \left[\varrho_{0,\varepsilon} s(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) + \varepsilon^\alpha \frac{4}{3} \vartheta_{0,\varepsilon}^3 \right] \phi \ \mathrm{d}x &\to \int_\Omega \langle Y_{0,x}; \varrho s(\varrho, \mathbf{m}, E) \rangle\, \phi \ \mathrm{d}x \text{ for any } \phi \in C_c^\infty(\Omega).
\end{aligned} \tag{3.32}$$

In view of the uniform bounds (3.23–3.25) and the fundamental theorem of the theory of Young measures (see e.g. Ball [1]), there is a subsequence of $(\varrho_\varepsilon, \mathbf{m}_\varepsilon, E_\varepsilon)_{\varepsilon>0}$ (not relabeled here) that generates a Young measure $\{Y_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$. Moreover, passing to the limit in the total energy balance (3.21), we obtain

$$\left[\int_\Omega \langle Y_{\tau,x}; E\rangle \ \mathrm{d}x\right]_{t=0}^{\tau} + \mathcal{D}(\tau) = 0; \tag{3.33}$$

for a.a. $\tau \in (0,T)$, where

$$\mathcal{D}(\tau) \geq \liminf_{\varepsilon\to 0} \int_\Omega \left[\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) + \varepsilon^\alpha \vartheta_\varepsilon^4\right] \ \mathrm{d}x - \int_\Omega \langle Y_{\tau,x}; E\rangle \ \mathrm{d}x \text{ for a.a } \tau \in (0,T). \tag{3.34}$$

Indeed the extra term in (3.21) can be handled as

$$\lambda \int_0^\tau \int_\Omega (\vartheta_\varepsilon - \overline{\vartheta})^3 \ \mathrm{d}x \gtrsim -\varepsilon^\beta \int_0^\tau \int_\Omega \left[\vartheta_\varepsilon^2 \overline{\vartheta} + \overline{\vartheta}^2 \vartheta_\varepsilon + \overline{\vartheta^3}\right] \ \mathrm{d}x\,\mathrm{d}t \approx -\varepsilon^{\beta/3} \to 0, \tag{3.35}$$

where we have used the bound (3.23). Moreover, it is easy to pass to the limit in the weak formulation of (3.18), to obtain (2.13).

Next, seeing that

$$\nu\mathbb{S}(\vartheta_\varepsilon, \nabla_x \mathbf{u}_\varepsilon) \approx \varepsilon\vartheta_\varepsilon|\nabla_x \mathbf{u}_\varepsilon| = \varepsilon^{1/2}\vartheta_\varepsilon \varepsilon^{1/2}|\nabla_x \mathbf{u}_\varepsilon|,$$

we may use the bounds (3.23), (3.27) to conclude that

$$\|\nu\mathbb{S}(\vartheta_\varepsilon, \nabla_x \mathbf{u}_\varepsilon)\|_{L^1((0,T)\times\Omega)} \lesssim \varepsilon^{\frac{1}{2}-\frac{\beta}{3}}.$$

Thus choosing $0 < \beta < \frac{3}{2}$ we can pass to the limit in the momentum balance (3.19) to recover (2.14). Note that the measure μ_C contains the concentration defect of the terms

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon,\ p(\varrho_\varepsilon, \mathbf{u}_\varepsilon),\ \text{and } \varepsilon^\alpha \vartheta_\varepsilon^4$$

and, by virtue of (3.34), it is controlled by $\mathcal{D}$ exactly as required in (2.17).

Finally, it remains to perform the limit in the entropy balance (3.22) to obtain (2.16). First, by virtue of the same argument as in (3.35), we get

$$\lambda \frac{(\overline{\vartheta} - \vartheta_\varepsilon)^3}{\vartheta_\varepsilon} \gtrsim -\varepsilon^{\beta/3} \to 0.$$

Next, the entropy heat flux can be treated as

$$\omega \frac{\mathbf{q}(\vartheta_\varepsilon)}{\vartheta_\varepsilon} = \varepsilon \frac{\kappa_1(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \vartheta_\varepsilon + \varepsilon \frac{\kappa_2(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \vartheta_\varepsilon,$$

where κ_1 is bounded and $\kappa_2 \approx \vartheta^3$. In view of (3.30), we get

$$\varepsilon \left| \frac{\kappa_1(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \vartheta_\varepsilon \right|_{L^2((0,T)\times\Omega)} \lesssim \varepsilon \| \, |\nabla_x \log(\vartheta_\varepsilon)| + |\nabla_x \vartheta_\varepsilon| \, \|_{L^2((0,T)\times\Omega)} \lesssim \varepsilon^{1/2}.$$

Moreover,

$$\varepsilon \vartheta_\varepsilon^2 \nabla_x \vartheta_\varepsilon = \varepsilon \frac{1}{3} \nabla_x \vartheta_\varepsilon^3,$$

where, in accordance with (3.23)

$$\varepsilon \vartheta_\varepsilon^3 \to 0 \text{ in } L^1((0,T)\times\Omega) \text{ as soon as } 0 < \beta < 1.$$

Seeing that the terms $\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)$ and $\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)\mathbf{u}_\varepsilon$ are controlled via (3.26), it remains to show

$$a \left(\operatorname*{ess\,sup}_{t\in(0,T)} \|\vartheta_\varepsilon^3\|_{L^1(\Omega)} + \left\|\vartheta_\varepsilon^3 \mathbf{u}_\varepsilon\right\|_{L^1((0,T)\times\Omega)} \right) \to 0.$$

Seeing that, in view of the energy bound (3.23),

$$\operatorname*{ess\,sup}_{t\in(0,T)} a\|\vartheta_\varepsilon^3\|_{L^1(\Omega)} \to 0,$$

we have to handle only the second term. To this end, write

$$a\vartheta_\varepsilon^3 \mathbf{u}_\varepsilon = \varepsilon^\alpha \nu^{-1/2} \vartheta_\varepsilon^3 \nu^{1/2} \mathbf{u}_\varepsilon.$$

In view of (3.29), it is enough to show that

$$\varepsilon^{\alpha - \frac{1}{2}} \operatorname*{ess\,sup}_{t\in(0,T)} \|\vartheta_\varepsilon^3\|_{L^{4/3}(\Omega)} \to 0.$$

However, this follows from the energy bound (3.23) as soon as $\alpha > 2$.

We have shown the following result:

Theorem 3.4. *Suppose that the thermodynamic functions p, e, and s satisfy (3.8–3.15), and the transport coefficients μ, η and κ satisfy (3.16), (3.17). Let $(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon)_{\varepsilon>0}$ be a family of weak solutions to the Navier–Stokes–Fourier system (3.18–3.20), where*

$$\nu = \omega = \varepsilon,\ a = \varepsilon^\alpha,\ \alpha > 2,\ \lambda = \varepsilon^\beta,\ 0 < \beta < 1.$$

Finally, let the initial data $(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon})_{\varepsilon>0}$ generate a Young measure $Y_{0,x}$ specified in (3.32).

Then (at least for a suitable subsequence)

$$\left(\varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon, \frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) \right)_{\varepsilon>0}$$

generates a Young measure $\{Y_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$ and a dissipation defect $\mathcal{D}$ specified in (3.34) that represent a dissipative measure–valued solution of the Euler system (2.1–2.4) in the sense of Definition 2.1.

In view of Theorem 2.5, we immediately obtain the following corollary that can be seen as a version of the result in [14].

Corollary 3.5. *In addition to the hypotheses of Theorem 3.4 suppose that the limit Euler system (2.1–2.3) admits a smooth (C^1) solution $(\varrho, \mathbf{m}, E)_{\varepsilon>0}$ in $[0,T] \times \Omega$.*

Then

$$\varrho_\varepsilon \to \varrho,\ \varrho_\varepsilon \mathbf{u}_\varepsilon \to \mathbf{m}, \frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) \to E \text{ in } L^1((0,T) \times \Omega).$$

Indeed the fact that the limit DMV solution is represented by the Dirac masses implies (up to a subsequence) strong a.a. pointwise convergence. In addition, the limit defect $\mathcal{D}$ vanishes which implies strong convergence in the L^1-norm.

4 Vanishing dissipation limit of Brenner's model

The measur–valued solutions constructed via the Navier–Stokes–Fourier system do not, or at least are not known, to satisfy the renormalized entropy balance (2.18). In addition, the hypotheses imposed in Theorem 3.4 on the constitutive relations and the transport coefficients are rather awkward and, unfortunately, do not cover the mostly used Boyle–Mariotte law $p = \varrho\vartheta$, with $e = c_v\vartheta$. It is interesting to see that a model proposed by H.Brenner [4], [5], [6] behaves actually better in the vanishing dissipation limit and does not suffer the above mentioned drawbacks.

Brenner's model consists of the following equations:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{v}_m) = 0, \tag{4.1}$$

$$\partial_t(\varrho \mathbf{v}) + \mathrm{div}_x\left(\varrho \mathbf{v} \otimes \mathbf{v}_m\right) + \nabla_x p(\varrho, \vartheta) = \varepsilon \mathrm{div}_x \mathbb{S}, \tag{4.2}$$

$$\begin{aligned} \partial_t \left(\varrho(\frac{1}{2}|\mathbf{v}|^2 + e(\varrho,\vartheta)) \right) + \mathrm{div}_x \left(\varrho(\frac{1}{2}|\mathbf{v}|^2 + e(\varrho,\vartheta))\mathbf{v}_m \right) & \\ + \mathrm{div}_x(p(\varrho,\vartheta)\mathbf{v}) + \varepsilon \mathrm{div}_x \mathbf{q} = & \ \varepsilon \mathrm{div}_x(\mathbb{S}\mathbf{v}), \end{aligned} \tag{4.3}$$

where, similarly to the preceding section, $\varepsilon > 0$ is a small parameter supposed to vanish in the zero dissipation limit. Brenner's main idea was to introduce two velocity fields - $\mathbf{v}$ and $\mathbf{v}_m$ - interrelated through

$$\mathbf{v} - \mathbf{v}_m = \varepsilon K \nabla_x \log(\varrho), \tag{4.4}$$

where $K \geq 0$ is a purely *phenomenological* coefficient. Similarly to the previous part, we suppose

$$\mathbf{q} = -\kappa \nabla_x \vartheta, \tag{4.5}$$

and

$$\mathbb{S} = \mu \left(\nabla_x \mathbf{v} + \nabla_x^T \mathbf{v} - \frac{2}{3} \mathrm{div}_x \mathbf{v} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{v} \mathbb{I}. \tag{4.6}$$

As a matter of fact, Brenner's model has been thoroughly criticized and its relevance to fluid mechanics questioned in Oettinger et al. [23]. On the other hand, it is mathematically tractable and yields essentially better theory than the standard Navier–Stokes–Fourier system, see e.g. [19], Cai, Cao, Sun [10]. Recently, the interest in "two velocity models" has been revived in Bresch et al. [7], [8].

Leaving apart the conceptual difficulties of the model, we claim that it generates in the vanishing dissipation limit a *renormalized* (DMV) solution of the Euler system (2.1–2.4). The crucial aspect of the analysis is a specific form of the coefficient K in (4.4). Note that K is taken constant in [19] as well as in Cai et al. [10], while Brenner proposed $K = \frac{\kappa}{c_p \varrho}$, see [5], where c_p denotes the specific heat at constant *pressure*. Further, we assume Boyle–Mariotte law

$$p = \varrho \vartheta, \ e = c_v \vartheta, \ s = \log(\vartheta^{c_v}) - \log(\varrho), \tag{4.7}$$

where $c_v = \frac{1}{\gamma - 1}$ is the specific heat at constant *volume*. As we show below, a convenient form of K reads

$$K = \frac{\kappa}{c_v \varrho}. \tag{4.8}$$

4.1 Estimates of entropy

To begin, let us say frankly that there is no rigorous existence result for the Brenner model if K is given by (4.8). For this reason, we restrict ourselves to showing the argument how the renormalized entropy inequality is obtained. Furthermore, we shall suppose that we deal with a smooth solution of (4.1–4.3).

Taking the scalar product of the momentum equation (4.2) on $\mathbf{v}$ and subtracting the resulting expression from (4.3), we obtain the internal energy balance

$$\partial_t(\varrho e(\varrho,\vartheta)) + \mathrm{div}_x(\varrho e(\varrho,\vartheta)\mathbf{v}_m) - \varepsilon \mathrm{div}_x\left(\kappa \nabla_x \vartheta\right) = \varepsilon \mathbb{S} : \nabla_x \mathbf{u} - p(\varrho,\vartheta)\mathrm{div}_x \mathbf{v}. \tag{4.9}$$

Dividing (4.9) on ϑ and keeping in mind the constitutive relations (4.7), (4.8), we obtain

$$\partial_t(\varrho \log(\vartheta^{c_v})) + \mathrm{div}_x\left(\varrho \log(\vartheta^{c_v})\mathbf{v}_m\right) - \varepsilon \mathrm{div}_x\left(\frac{\kappa}{\vartheta}\nabla_x \vartheta\right) = \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{v} + \frac{\kappa}{\vartheta}|\nabla_x \vartheta|^2\right) - \varrho \mathrm{div}_x \mathbf{v}.$$

Furthermore, in accordance with (4.4),

$$\varrho \mathrm{div}_x \mathbf{v} = \varrho \mathrm{div}_x(\mathbf{v} - \mathbf{v}_m) + \varrho \mathrm{div}_x \mathbf{v}_m = \varepsilon \varrho \mathrm{div}_x\left(\frac{\kappa}{\varrho c_v}\nabla_x \log(\varrho)\right) - \partial_t(\varrho \log(\varrho)) - \mathrm{div}_x\left(\varrho \log(\varrho)\mathbf{v}_m\right).$$

Thus we have deduced the entropy balance

$$\begin{aligned}
\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x\left(\varrho s(\varrho,\vartheta)\mathbf{v}_m\right) - \varepsilon \mathrm{div}_x&\left[\frac{\kappa}{c_v}\nabla_x\left(\log(\vartheta^{c_v}) - \log(\varrho)\right)\right] \\
&= \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{v} + \frac{\kappa}{\vartheta}|\nabla_x \vartheta|^2\right) + \varepsilon \frac{\kappa}{c_v}|\nabla_x \log(\varrho)|^2,
\end{aligned}$$

or, in a slightly different form,

$$\begin{aligned}
\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x\left(\varrho s(\varrho,\vartheta)\mathbf{v}_m\right) - \varepsilon \mathrm{div}_x&\left[\frac{\kappa}{c_v}\nabla_x s(\varrho,\vartheta)\right] \\
&= \frac{1}{\vartheta}\mathbb{S} : \nabla_x \mathbf{v} + \kappa|\nabla_x \log \vartheta|^2 + \varepsilon \frac{\kappa}{c_v}|\nabla_x \log(\varrho)|^2.
\end{aligned} \tag{4.10}$$

Multiplying (4.10) on $Z'(s)$, where Z is a non-decreasing concave function, we obtain the desired relation

$$\begin{aligned}
&\partial_t(\varrho Z(s(\varrho,\vartheta))) + \mathrm{div}_x\left(\varrho Z(s(\varrho,\vartheta))\mathbf{v}_m\right) - \varepsilon \mathrm{div}_x\left[\frac{\kappa}{c_v}\nabla_x Z(s(\varrho,\vartheta))\right] \\
&= \frac{Z'(s)}{\vartheta}\mathbb{S} : \nabla_x \mathbf{v} + \kappa Z'(s)|\nabla_x \log \vartheta|^2 + \varepsilon Z'(s)\frac{\kappa}{c_v}|\nabla_x \log(\varrho)|^2 - Z''(s)\frac{\kappa}{c_v}|\nabla_x s(\varrho,\vartheta)|^2.
\end{aligned} \tag{4.11}$$

In particular, by virtue of the parabolic maximum principle, we may deduce from (4.11) that

$$\varrho^\gamma \leq c(s_0) c_v \varrho \vartheta,$$

where $s_0 > -\infty$ is the initial entropy, see [9, Section 2.1.1].

Similarly to [9, Section 2.1] we could show that a family of (strong) solutions $(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon)_{\varepsilon>0}$ of Brenner's model generates a measure–valued solution of the Euler system (2.1–2.4) satisfying (2.18) in the asymptotic limit $\varepsilon \to 0$. As the result is only formal (we do not know whether the strong or even weak solutions exist) we do not state it as a theorem and leave the interested reader to work out the details.

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