On the approximation of a virtual coarse space for domain decomposition methods in two dimensions

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Abstract. A new extension operator for a virtual coarse space is presented which can be used in domain decomposition methods for nodal elliptic problems in two dimensions. In particular, a two-level overlapping Schwarz algorithm is considered and a bound for the condition number of the preconditioned system is obtained. This bound is independent of discontinuities across the interface. The extension operator saves computational time compared to previous studies where discrete harmonic extensions are required and it is suitable for general polygonal meshes and irregular subdomains. Numerical experiments that verify the result are shown, including some with regular and irregular polygonal elements and with subdomains obtained by a mesh partitioner.

Key words. domain decomposition, overlapping Schwarz algorithms, nodal elliptic problems, irregular subdomain boundaries, virtual element methods

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1. Introduction. We will consider the scalar elliptic problem in two dimensions

\[(1)\quad -\nabla \cdot (\rho(x)\nabla u(x)) = f(x), \quad x \in \Omega, \quad \rho(x) > 0, \quad f \in L^2(\Omega),\]

posed in \(H^1(\Omega) := \{v \in L^2(\Omega) : \|\nabla v\|_{L^2(\Omega)} < \infty\}\) and with homogeneous Dirichlet boundary conditions. We will work as usual with a weak formulation for problem (1), namely: Find \(u \in H^1_0(\Omega)\) such that

\[(2)\quad a(u, v) = F(v) \quad \forall \; v \in H^1_0(\Omega),\]

where \(a(u, v) := \int_\Omega \rho \nabla u \cdot \nabla v \, dx\), \(F(v) := (f, v)_{0, \Omega}\) is the inner product in \(L^2(\Omega)\) and \(H^1_0(\Omega)\) is the subspace of \(H^1(\Omega)\) with vanishing trace.

We consider a general polygonal triangulation \(T_h\) of \(\Omega\) and discretize problem (2) with the novel Virtual Element Method (VEM) [1, 2]. The associated stiffness matrix is then obtained by defining a finite-dimensional space \(V_h\) and an computable, consistent and stable bilinear form \(a_h(\cdot, \cdot)\) defined in \(V_h \times V_h\).

We then construct a two-level overlapping additive Schwarz preconditioner for the associated linear system; see, e.g., [18, Chapter 3]. These methods were introduced in \([10, 11, 12]\). For this purpose, the domain \(\Omega\) is decomposed in \(N\) non-overlapping subdomains \(\{\Omega_i\}_{i=1}^N\) of diameter \(H_i\) which are the union of elements of the triangulation \(T_h\). The preconditioner has local contributions that correspond to homogeneous Dirichlet solvers on extensions \(\Omega'_i \supseteq \Omega_i\), and a coarse component that involves a global solver of modest size. The dimension of our coarse space is equal to the number of subdomain vertices (the nodes of the triangulation that belong to three or more subdomains).

In general, there is no straightforward approach to define coarse basis functions in the presence of irregular subdomains. In this setting, the virtual space seems to be a natural choice since it can accommodate polygonal elements with no complications. Thus, general polygonal subdomains can be considered in the decomposition. Furthermore, the lowest-order space is characterized by functions that are harmonic in the interior of the subdomains, which have minimum \(H^1\)-seminorm amog the functions with the same values on the boundary. We note that previous studies related to
irregular subdomains are based on discrete harmonic extensions starting with [8]; see also [7, 16, 19, 9] where John [13] and Jones [14] subdomains are considered.

Our virtual coarse space was introduced in [6], where two variants were considered: the full coarse virtual space $V_0^R$ defined on the partition $\{\Omega_i\}$ of $\Omega$, and a reduced version $V_0^R \subset V_0$. In the VEM terminology, functions are virtual in the sense that they are never required to be constructed explicitly; the only available information is their degrees of freedom. Hence, in order to compute the coarse component of our preconditioner, we need to approximate the coarse functions in the virtual element space.

The first approach considered in [6] consists in defining the operator $R_0^T : V_0^R \to V_h$ by evaluating a piecewise linear interpolant. In the present study, we modify this operator and consider polynomial approximations inside the subdomains (with degree greater than or equal to two). These projectors are used in the VEM methods when assembling the stiffness matrices and can be computed by only knowing the degrees of freedom of the virtual functions. They provide a good approximation and allow to deduce a similar upper bound for the condition number of the preconditioned system. The bound obtained in Theorem 4.1 is also independent of discontinuities of $\rho$ across the interface, and we will assume that $\rho$ is constant in each subdomain $\Omega_i$.

We note that the main advantage of our approach with respect to previous studies is that no discrete harmonic extensions are required in the algorithm, saving computational time. We also aim to contribute and enrich the literature related to iterative solvers for VEM discretizations, since there is a lack of theoretical analysis for such problems.

The rest of this paper is organized as follows. In Section 2 we introduce the notation that will be used throughout our analysis. In Section 3 we present the basic theory of VEM and the discretization of our problem. The coarse space is described in Section 4 along with the new extension operators that approximate the coarse basis functions. Then, Section 5 includes some technical tools and the proof of the bound for the condition number of the preconditioned system. We include some implementation details and report on some numerical experiments in Section 6. Finally, we present some closing remarks in Section 7.

2. Notation. Given $h > 0$, we will divide the domain $\Omega$ into simply connected polygons $K$ (not necessarily similar) of diameter $h_K \leq h$; see some examples in Section 6. These polygons will be called elements and later we will introduce two assumptions on them. This polygonal triangulation will be called fine mesh and will be denoted by $T_h$. The set of nodes of a partition $T_h$ contains the vertices of all the elements $K \in T_h$ and will be represented by $S_N$, while the set $S_N^K$ will include the nodes of $T_h$ that are on the boundary of $K$. We note that the set $S_N^K$ can contain, besides vertices of $K$, also hanging nodes, i.e., vertices of other polygons that lie on an edge of $K$. The bilinear form (2) restricted to an element $K$ will be denoted by

$$a^K(u, v) := \int_K \rho \nabla u \cdot \nabla v \, dx.$$ 

Furthermore, the domain $\Omega$ will be partitioned into $N$ non-overlapping and simply connected subdomains $\{\Omega_i\}_{i=1}^N$ of diameter $H_i$ which are the union of elements of the triangulation $T_h$; see, e.g., Figure 1. The partition formed by $\{\Omega_i\}$ will be called coarse mesh and will be denoted by $T_H$. In a similarly way, the set of nodes of $T_H$ contains the vertices of all the subdomains $\Omega_i$ and will be represented by $S_N^H$, while the set $S_N^{H,K}$ will include only the nodes of $T_H$ that are on the boundary of $\Omega_i$.
hanging nodes are allowed on the coarse mesh. We will assume also that the coefficient \( \rho(x) \) is constant and equal to \( \rho_i \) inside each subdomain \( \Omega_i \). We will then construct overlapping subdomains \( \Omega'_i \supset \Omega_i \) by adding layers of elements that are external to \( \Omega_i \), and we will denote by \( \delta_i \) the maximum width of the region \( \Omega'_i \setminus \Omega_i \).

We will also partition the set of nodes \( S_N \) into disjoint sets. The nodes that belong to exactly one subdomain \( \Omega_i \) are its \textit{interior nodes}. A \textit{subdomain edge} \( E_{ij} \) will be the interior of the intersection of the closure of two neighboring subdomains \( \Omega_i \) and \( \Omega_j \). If such intersection has more than one component, each open component will be considered as a subdomain edge. Then, the endpoint nodes of \( E_{ij} \) will belong to the set of \textit{subdomain vertices}, which will be denoted by \( S_V \). We will write \( \mathcal{E} \) instead of \( E_{ij} \) if there is no need to identify the two subdomains \( \Omega_i \) and \( \Omega_j \). The \textit{interface} of the decomposition will include the closure of all the subdomain edges.

We remark the distinction between \( S_{\Omega_i}^N \) (nodes that are vertices of the polygonal subdomains) and \( S_{\Omega_i}^V \) (nodes that belong to at least three subdomains); see Figure 1.

It is clear that \( S_{\Omega_i}^V \subset S_{\Omega_i}^N \).

![Fig. 1: A METIS decomposition for the unit square with square elements. The colored subdomain is a dodecagon with six subdomain edges. (a) Subdomain nodes, \( S_{\Omega_i}^N \) (left) (b) Subdomain vertices, \( S_{\Omega_i}^V \) (right)](image)

We also define some polynomial spaces that will be used in the description of the algorithm. Given two non-negative integers \( \alpha_1, \alpha_2 \), we will use the standard notation for a multi-index \( \alpha = (\alpha_1, \alpha_2) \), with \( |\alpha| := \alpha_1 + \alpha_2 \) and \( x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \) for any point in the plane \( x = (x_1, x_2) \). The space of polynomials of degree less than or equal to \( k \) defined on \( D \) will be denoted by \( P_k(D) \). We recall that \( \dim P_k(D) = (k+1)(k+2)/2 \) for two-dimensional domains.

Let \( m_\alpha \) be the scaled monomial of degree \( |\alpha| \) in the domain \( D \), defined by

\[
m_\alpha(x) := \left( \frac{x - x_D}{h_D} \right)^\alpha,
\]

where \( x_D \) is the barycenter of \( D \) and \( h_D \) its diameter. The set of such polynomials of degree less than or equal to \( k \) will be denoted by

\[
(3) \quad \mathcal{M}_k(D) := \{ m_\alpha : |\alpha| \leq k \}.
\]

Let \( D \) represent an element \( K \in \mathcal{T}_h \) or a subdomain \( \Omega_i \). We distinguish between \( \varpi_D \), the standard \textit{nodal average} of \( v \), and \( \hat{v}_D \), the \textit{mean value} of \( v \), defined respectively
by

\[ \nabla_D := \frac{1}{|S_D^N|} \sum_{x \in S_D^N} w(x), \quad \text{and} \quad \hat{v}_D := \frac{1}{|D|} \int_D w(x) \, dx. \]

For a positive integer \( k \), we will also consider the set

\[ \mathcal{B}_k(\partial D) := \{ v \in C^0(\partial D) : v|_e \in \mathcal{P}_k(e) \quad \forall \ e \subset \partial D \} \]

where \( e \) represents any edge on the boundary of \( D \), and the local virtual element space

(4) \[ V^D_k := \{ v \in H^1(D) : v \in \mathcal{B}_k(\partial D), \Delta v \in \mathcal{P}_{k-2}(D) \} \]

We note that we consider \( \mathcal{P}_{-1}(D) = 0 \). Hence, a function in \( V^D_1 \) is continuous and piecewise linear on the boundary of \( D \) and harmonic in the interior, and it is completely determined by its values at the nodes in \( S_D^N \).

Finally, we define the operator \( \Pi_{D,k}^V : V^D_k \to \mathcal{P}_k(D) \subset V^D_k \) where \( \tilde{v} := \Pi_{D,k}^V v \) is the polynomial that satisfies

(5) \[ \begin{cases} a^D(\tilde{v}, q) = a^D(v, q) & \forall \ q \in \mathcal{P}_k(D), \\ P_0(\tilde{v}) = P_0(v). \end{cases} \]

Here, \( P_0 : V^D_k \to \mathbb{R} \) is a projection operator onto constants, which is chosen as the nodal average if \( k = 1 \) or the mean value if \( k \geq 2 \); see Subsection 6.1 for implementation details related to the operator \( \Pi_{D,k}^V \).

3. Virtual Element Methods and discretization. We describe the basic theory of the VEM introduced in [1]; see also [2] for implementation details. We discretize (2) with the lowest-order VEM and we restrict in this section our presentation to this case. Nevertheless, for the approximation of the coarse space we will use the projectors \( \Pi_{D,k}^V \) for \( k \geq 2 \); see Subsection 4.2. It is assumed that the elements satisfy the following two assumptions, where \( h \) denotes the maximum of the diameters of the elements in \( T_h \):

Assumption 1. There exists \( \gamma > 0 \) such that for all \( h \), each element \( K \) in \( T_h \) is star-shaped with respect to a ball of radius greater than or equal to \( \gamma h_K \).

Assumption 2. There exists \( \gamma > 0 \) such that for all \( h \) and for each element \( K \) in \( T_h \), the distance between any two vertices of \( K \) is greater than or equal to \( \gamma h_K \).

The global space of lowest-order virtual element functions is defined as

\[ V_h := \{ v \in H^1_0(\Omega) : v|_K \in V^K_1 \quad \forall \ K \in T_h \}, \]

where \( V^K_1 \) is defined in (4). The dimension of \( V_h \) is equal to the number of nodes of \( T_h \) in the interior of \( \Omega \). It is easy to check that they are unisolvent ([1, Proposition 4.1]) and \( V_h \) reduces to the first-order Lagrange finite element space in the case of a triangular mesh.

We consider the canonical basis \( \{ \phi^h_{x_i} \} \) of \( V_h \) such that, for any node \( x_j \in S_N \), we have that \( \phi^h_{x_j}(x) = \delta_{ij} \). For a continuous function \( u \), we can define the natural interpolant onto \( V_h \) given by

\[ I_h u := \sum_{x \in S_N} u(x) \phi^h_{x}. \]
We note that in general $a(\phi_h^j, \phi_h^k)$ cannot be evaluated as discussed in [1, Section 4.5]. Therefore, we consider the local bilinear form $a_h^K : V^1_h \times V^1_h \rightarrow \mathbb{R}$ defined by

$$a_h^K(u, v) := a^K(\Pi^K_{K,1} u, \Pi^K_{K,1} v) + S^K(u - \Pi^K_{K,1} u, v - \Pi^K_{K,1} v),$$

where $S^K : V^1_h \times V^1_h \rightarrow \mathbb{R}$ is a stabilizing term, which can be chosen as

$$S^K(u, v) := \sum_{x \in S_N^h} \rho(x) u(x)v(x).$$

From (6) it is clear that $a_h^K$ satisfies the consistency property

$$a_h^K(p, v) = a^K(p, v) \quad \forall p \in \mathcal{P}_1(K), \quad \forall v \in V^1_h,$$

and the stability property

$$\alpha_1 a^K(v, v) \leq a_h^K(v, v) \leq \alpha_2 a^K(v, v) \quad \forall v \in V^1_h,$$

where $\alpha_1, \alpha_2$ are independent of $\rho$, $h_K$ and $K$; see [1, Theorem 4.1]. The discrete bilinear form and right hand side are then defined by

$$a_h(u, v) := \sum_{K \in \mathcal{T}_h} a_h^K(u, v), \quad F_h(v) := \sum_{K \in \mathcal{T}_h} |K| \hat{f} \hat{v}.$$

Finally, the discrete formulation for problem (2) is: Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h.$$

We refer to [1, 5] for an a priori estimate and approximation properties, [2] for implementation details and [17] for an implementation in Matlab.

4. Overlapping Schwarz methods. We briefly describe the two-level additive overlapping Schwarz methods; for a complete study see [18, Chapters 2, 3]. We also describe the reduced virtual coarse space $V^R_0$ introduced in [6, Section 6] and the new operator $R_0^\alpha$ in detail.

4.1. The coarse space. Consider the partition $\mathcal{T}^H$ formed by the subdomains $\{\Omega^i\}_{i=1}^N$. For each subdomain, let $V^i_1 := V^{1\Omega^i}_1$ be the lowest-order local virtual space defined in (4). The degrees of freedom are chosen again as the values at the nodes in $S^N_{\Omega^i}$. Then, the global virtual space on $\mathcal{T}^H$ is defined as

$$V_0 := \left\{ v \in H^1(\Omega) : v|_{\Omega^i} \in V^i_1, \ 1 \leq i \leq N \right\}.$$

For each subdomain vertex $x_0 \in S_V$ we define a coarse function $\psi_{x_0}^H \in V_0$ by choosing appropriately its degrees of freedom. First, we set $\psi_{x_0}^H(x) = 0$ for all the subdomain vertices $x$, except at $x_0$ where $\psi_{x_0}^H(x_0) = 1$. Second, we set the degrees of freedom related to the nodal values on each subdomain edge. If $x_0$ is not an endpoint of $\mathcal{E}$, then $\psi_{x_0}^H$ vanishes on that edge. If $\mathcal{E}$ has endpoints $x_0$ and $x_1$, let $d_\mathcal{E}$ be the unit vector with direction from $x_1$ to $x_0$. Consider any node $\bar{x} \in \mathcal{E}$. If $0 \leq (\bar{x} - x_1) \cdot d_\mathcal{E} \leq |x_0 - x_1|$, we then set

$$\psi_{x_0}^H(\bar{x}) = \frac{(\bar{x} - x_1) \cdot d_\mathcal{E}}{|x_0 - x_1|}.$$
It is clear that $\psi^H_{x_0}(x_0) = 1$, $\psi^H_{x_0}(x_1) = 0$ and that the function varies linearly in the direction of $d_x$ for such nodes. If $(x - x_1) \cdot d_x < 0$ or $(x - x_1) \cdot d_x > |x_0 - x_1|$, we then set $\psi^H_{x_0}(x) = 0$ or $\psi^H_{x_0}(x) = 1$, respectively. In this way, we define all the degrees of freedom of $\psi^H_{x_0} \in V_0$. By construction it is clear that $0 \leq \psi^H_{x_0} \leq 1$ and the sum $\sum_{x_0} \psi^H_{x_0}$ of all these coarse basis functions is equal to one.

We then define the reduced coarse space as the span of $\{\psi^H_{x_0}\}$, i.e.,

$$V_0^R := \left\{ v \in H^1_0(\Omega) : v = \sum_{x_0 \in S_V} \alpha_{x_0} \psi^H_{x_0} \right\} \subset V_0,$$

where $\alpha_{x_0} \in \mathbb{R} \forall x_0 \in S_V$. We note that there is only one degree of freedom per subdomain vertex, and that functions are piecewise linear on the subdomain edges and harmonic in the interior of each subdomain. We can naturally define a linear interpolant $I^H : V_h \to V_0^R$ by

$$I^H u := \sum_{x_0 \in S_V} u(x_0) \psi^H_{x_0},$$

and it is easy to deduce that $I^H$ reproduces linear polynomials.

### 4.2. Extension operators.

The crucial aspect in the construction of the pre-conditioner is to define an operator $R^G_0 : V_0^R \to V_h$ that approximates functions in the coarse space by elements in $V_h$. The straightforward approach would be to evaluate each coarse basis function $\psi^H_{x_0}$ at the internal nodes of the subdomains, but these coarse functions are virtual and we only know the value of their degrees of freedom. We could consider also the discrete harmonic extension for each subdomain $\Omega_i$ by using the bilinear form $a(\cdot, \cdot)$ restricted to $\Omega_i$. This is the usual approach in most of the existing literature; see, e.g., [8, 7, 16, 19, 9]. A different approach is presented in [6], where $R^G_0$ is constructed by considering piecewise-linear contributions.

Instead, we take advantage of the operators $\Pi_{\Omega_i}^V$ defined in (5). We start by describing the virtual space when $k \geq 2$. For each subdomain $\Omega_i$, the local virtual space of degree $k$ is defined by $V^i_k := V^i_{\omega_i} \cap \mathbb{P}_k^i$; see (4). It is easy to deduce that $\dim V^i_k = n_i k + (k - 1)/2$, where $n_i$ is the number of edges of the polygon $\Omega_i$. The degrees of freedom of a function $v \in V^i_k$ can be chosen as:

- (a) The values of $v$ at the $n_i$ vertices of the polygon $\Omega_i$.
- (b) The values of $v$ at $k - 1$ internal points of each edge of $\Omega_i$.
- (c) The moments $\frac{1}{|\Omega_i|} \int_{\Omega_i} m(x)v(x)\ dx$ for $m \in \mathcal{M}_{k-2}(\Omega_i)$, where $\mathcal{M}_{k-2}$ is the set of scaled monomials defined in (3).

Then, the global virtual space on $T^H$ of degree $k$ is defined as

$$V^H_k := \left\{ v \in H^1_0(\Omega) : v|_{\Omega_i} \in V^i_k, \ 1 \leq i \leq N \right\}.$$

Clearly $V_0^R \subset V_0 \subset V^H_k$. Thus, given a coarse basis function $\psi^H_{x_0}$, we can identify it as an element of $V^H_k$ by computing its degrees of freedom (a), (b) and (c). We note that the nodal degrees of freedom given in (a) and (b) are straightforward to compute by using (9). We then need to compute the moments (c), namely

$$\frac{1}{|\Omega_i|} \int_{\Omega_i} m_\alpha(x)\psi^H_{x_0}(x)\ dx, \ \forall m_\alpha \in \mathcal{M}_{k-2}(\Omega_i),$$

which might seem difficult. However, we recall that $\psi^H_{x_0}$ is harmonic in $\Omega_i$. Thus, it is enough to choose these moments in such way that $a^H(\psi^H_{x_0}, \psi^H_{x_0})$ is minimized among
all the functions in $V^k$ with the prescribed boundary data. As we will show, this is equivalent to solving a linear system with $k(k-1)/2$ unknowns for each subdomain $\Omega_i$; see Subsection 6.1 for implementation details.

We then define the action of $R^T_0$ on each basis function of $V^R_0$. We note that $R^T_0$ depends on $k \geq 2$, but for simplicity we omit this dependence. We set the degrees of freedom of $R^T_0 \psi_{x_0}^H \in V_h$ as follows:

(a') If $x$ is a node of $T_h$ in the interior of $\Omega_i$, we set
$$
(R^T_0 \psi_{x_0}^H)(x) := (\Pi^H_{\Omega_i,k} \psi_{x_0}^H)(x).
$$

(b') For all the other nodes in $T_h$ (that belong to the interface), we set
$$
(R^T_0 \psi_{x_0}^H)(x) := \psi_{x_0}^H(x).
$$

Therefore, given $u_0 = \sum_{x \in S} \alpha_x \psi_x^H \in V^R_0$, we can write the restriction of $R^T_0$ to a subdomain $\Omega_i$ as
$$
(R^T_0 u_0)|_{\Omega_i} = I^h (\Pi^H_{\Omega_i,k} u_0) + \sum_{x \in S' \cap \partial \Omega_i} (u_0 - \Pi^H_{\Omega_i,k} u_0)(x) a^h_x.
$$

Finally, we consider the bilinear form defined in $V^R_0 \times V^R_0$ by
$$
\tilde{a}_{h,0}(u_0, v_0) := a_h(R^T_0 u_0, R^T_0 v_0) = \sum_{K \in T_h} a^K_0(u_0, v_0).
$$

We note that as we increase $k$, the possibility of choosing more moments (c) allows us to obtain a better approximation $R^T_i \psi_{x_0}^H$ for $\psi_{x_0}^H$ by just solving a small linear system. In practice, results for $k = 2$ and $k = 3$ are quite competitive as it is shown in Section 6.

4.3. Local spaces. We consider the usual local virtual spaces $V_i$, $1 \leq i \leq N$, defined by
$$
V_i := \{ v \in H_0^1(\Omega_i') : v|_K \in B_1(\partial K), \Delta v|_K = 0 \text{ in } K, \forall K \subset \Omega_i' \}.
$$

Thus, the degrees of freedom are the values at the nodes of $T_h$ in the interior of $\Omega_i'$. We also consider the natural operators $R^T_i : V_i \rightarrow V_i$ given by the zero extension from the subdomain $\Omega_i'$ to $\Omega$, $1 \leq i \leq N$. We use exact solvers for the local spaces, i.e., we define the bilinear forms $\tilde{a}_{h,i} : V_i \times V_i \rightarrow \mathbb{R}$ given by
$$
\tilde{a}_{h,i}(u_i, v_i) := a_h(R^T_i u_i, R^T_i v_i) = \sum_{K \in T_i} a^K_i(u_i, v_i), \quad 1 \leq i \leq N.
$$

4.4. Algorithm. In this subsection we define an additive preconditioner for the ill-conditioned sparse linear system $A \lambda = g$ obtained from problem (8), where $A_{i,j} = a_h(\phi_{x_i}^h, \phi_{x_j}^h)$, $g_i = F_h(\phi_{x_i}^h)$ and $\lambda$ is the vector of coordinates of the solution with respect to the basis $\{ \phi_{x_i}^h \}$ of $V_h$, i.e., $u_h = \sum \lambda_i \phi_{x_i}^h$.

Consider the matrix representation of the operators $R^T_i$ denoted again by $R^T_i$. We define the stiffness matrices $A_i = R_i A R_i^T$, $0 \leq i \leq N$, and then consider the Schwarz projections $P_i = R_i^T A_i^{-1} R_i A$, $0 \leq i \leq N$. The additive preconditioned operator is defined by
$$
P_{ad} := \sum_{i=0}^N P_i = A_{ad}^{-1} A, \quad A_{ad}^{-1} = \sum_{i=0}^N R_i^T A_i^{-1} R_i.
$$
Multiplicative and hybrid preconditioners can be considered as well; see [18, Section 2.2]. For the preconditioned system $A^{-1}_{ad}A\lambda = A^{-1}_{ad}g$, we have the main theorem of this paper. Its proof is presented in Section 5.

**Theorem 4.1.** There exists a constant $C$, independent of $H$, $h$ and $\rho$, such that the condition number of the preconditioned system $\kappa(A^{-1}_{ad}A)$ satisfies

$$
\kappa(A^{-1}_{ad}A) \leq C \left(1 + \log \frac{H}{h}\right) \left(1 + \frac{H}{\delta}\right),
$$

where the ratios $H/h$ and $H/\delta$ denote their maximum value over all the subdomains.

5. Technical Tools. We collect some tools that are needed in the proof of Theorem 4.1. For simplicity we write $\omega_i := 1 + \log(\frac{H_i}{h_i})$ and $\omega := \max_i \omega_i$. We recall the following Poincaré and discrete Sobolev inequalities:

**Lemma 5.1.** Let $\Omega$ be Lipschitz continuous with diameter $H$. Then, there exists a constant $C$ that depends only on the shape of $\Omega$ but not on $H$, such that

$$
||u||^2_{L^2(\Omega)} \leq CH^2||u||^2_{H^1(\Omega)},
$$

for $u \in H^1(\Omega)$ with vanishing mean value.

*Proof.* See [18, Corollary A.15].

**Lemma 5.2.** Given $v \in V_h$ with zero mean value, there exists a constant $C$ such that

$$
||v||^2_{L^\infty(\Omega_i)} \leq C\omega_i||v||^2_{H^1(\Omega_i)},
$$

where $C$ is independent of $H_i$ and $h_i$.

*Proof.* See [3], [4, Section 4.9] for a proof with domains satisfying an interior cone condition. This inequality holds for more general subdomains; see, e.g., [7, Lemma 3.2] for a proof with John domains.

We then have the following estimates for the interpolation operators $I^h$ and $\Pi^\nabla_{\Omega_i,k}$:

**Lemma 5.3.** Let $K \in T_h$ and $v \in H^2(K)$. Then, there exists $C$, independent of $h_K$, such that

$$
|v - I^hv|_{H^s(K)} \leq C h_K^{2-s}|v|_{H^2(K)},
$$

with $s \in \{0,1\}$.

*Proof.* See [5, Lemma 3.3.4].

**Lemma 5.4.** Let $K \in T_h$ and suppose that there exists a triangular partition $T_K$ of $K$ such that $u \in H^1(\Omega)$ is a quadratic function in each $T \in T_K$. Then, there exists a constant $C$ such that

$$
|I^hu|_{H^1(K)} \leq C |u|_{H^1(K)},
$$

where $C$ is independent of $h_K$.

*Proof.* This is a modification of [18, Lemma 3.9]; see the details in [6, Lemma 4.6].
Lemma 5.5. Given $k \geq 2$ and $u_0 \in V^i_k$, there exists a constant $C$ such that

$$\|\Pi_{\Omega_i,k}^\nabla u_0\|_{H^1(\Omega_i)}^2 \leq |u_0|_{H^1(\Omega_i)}^2,$$

$$\|u_0 - \Pi_{\Omega_i,k}^\nabla u_0\|_{L^2(\Omega_i)}^2 \leq CH_i^2|u_0|_{H^1(\Omega_i)}^2,$$

where $C$ is independent of $H_i$.

Proof. Let $\tilde{u}_0 := \Pi_{\Omega_i,k}^\nabla u_0 \in P_k(\Omega_i)$. The first inequality follows straightforward from the definition of $\Pi_{\Omega_i,k}^\nabla$, since $a^{\Omega_i}(u_0 - \tilde{u}_0, u_0 - \tilde{u}_0) \geq 0$ and $a^{\Omega_i}(u_0 - \tilde{u}_0, \tilde{u}_0) = 0$. The second inequality is a consequence of Lemma 5.1, since $u_0$ and $\tilde{u}_0$ have the same mean value over $\Omega_i$, by definition of the operator for $k \geq 2$.

We now obtain some bounds for the operators introduced in Section 4:

Lemma 5.6. Given $u \in V_h$, let $u_0 := I^H u \in V_0^k$. Then, there exists a constant $C$ such that

$$|u_0|_{H^1(\Omega_i)}^2 \leq C\omega_i \|u\|_{H^1(\Omega_i)}^2,$$

where $C$ is independent of $H_i$ and $h_i$.

Proof. See [6, Lemma 4.4 and Theorem 6.1].

Lemma 5.7. Given $u \in V_h$, let $u_0 := I^H u \in V_0^k$. If $k \geq 2$, then there exists a constant $C$ such that

$$\|u - R_T^g u_0\|_{L^2(\Omega_i)}^2 \leq CH_i^2\omega_i |u|_{H^1(\Omega_i)}^2,$$

$$|R_T^g u_0|_{H^1(\Omega_i)}^2 \leq Cw_i \|u\|_{H^1(\Omega_i)}^2,$$

where $C$ is independent of $H_i$ and $h_i$.

Proof. Define $\bar{u}_0 := \Pi_{\Omega_i,k}^\nabla u_0$. By triangle inequality and (10), we have that

$$\|u - R_T^g u_0\|_{L^2(\Omega_i)}^2 \leq 5 \left( \|u - \hat{u}\|_{L^2(\Omega_i)}^2 + \|u_0 - \hat{u}\|_{L^2(\Omega_i)}^2 + \|u_0 - u_0\|_{L^2(\Omega_i)}^2 \right),$$

where $\hat{u}$ is the mean value of $u$ over $\Omega_i$ and

$$g := \sum_{x \in S_{k',0}(\partial \Omega_i)} (u_0 - \bar{u}_0)(x) \phi_{x,k}^H \in V_h.$$

We bound each term of the last sum separately. The first term is easily bounded by Lemma 5.1. In turn, for the second term we have that

$$\|u - \hat{u}\|_{L^2(\Omega_i)}^2 = \|I^H (u - \hat{u})\|_{L^2(\Omega_i)}^2 \leq CH_i^2\|u - \hat{u}\|_{L^2(\Omega_i)}^2,$$

and we use then Lemma 5.2 to obtain the required bound. Here, we have used that $I^H (u - \hat{u})$ is harmonic in $\overline{\Omega}_i$ and that the values (9) are uniformly bounded. For the third term, we recall that $u_0 - \bar{u}_0$ has zero mean value. Thus,

$$\|u_0 - u_0\|_{L^2(\Omega_i)}^2 \leq CH_i^2|\bar{u}_0 - u_0|_{H^1(\Omega_i)}^2,$$

and we then use Lemma 5.5 and Lemma 5.6.
Next, the fourth term is bounded by using Lemma 5.3, an inverse inequality for polynomials, Lemma 5.5 and Lemma 5.6. Finally, we note that \( g \) vanishes on all the elements \( K \) that do not intersect \( \partial \Omega_i \). For the remaining elements,

\[
\|g\|_{L^2(K)}^2 \leq C h_K^2 \|u_0 - \tilde{u}_0\|_{L^\infty(\Omega)}^2 \leq C h_K^2 \|\tilde{u}_0 - u_0\|_{H^1(\Omega_i)}^2.
\]

By adding all the contributions, we get

\[
\|g\|_{L^2(\Omega_i)}^2 \leq C H_i^2 \|\tilde{u}_0 - u_0\|_{H^1(\Omega_i)}^2
\]

and we conclude by using Lemma 5.5 and Lemma 5.6.

In order to deduce the second inequality, by Lemma 5.5 and Lemma 5.6 we note that it is enough to bound the \( H^1 \)-seminorm of \( R_0^T u_0 - \tilde{u}_0 \). We have that

\[
|R_0^T u_0 - \tilde{u}_0|_{H^1(\Omega_i)}^2 \leq |I_h \tilde{u}_0 - \tilde{u}_0|_{H^1(\Omega_i)}^2 + |g|_{H^1(\Omega_i)}^2.
\]

The first term is easily bounded by \( |u_0|_{H^1(\Omega_i)}^2 \), where we use Lemma 5.3 and an inverse inequality for polynomials. Finally, since the energy of each basis function \( \phi^k_x \) is uniformly bounded, for an element with an edge on \( \partial \Omega_i \) we have that

\[
|g|_{H^1(K)}^2 \leq C H^{-1}_K \|u_0 - \tilde{u}_0\|_{L^2(\partial K)}^2
\]

\[
\leq C h_K^{-1} \|u_0 - \tilde{u}_0\|_{L^2(K)} \|u_0 - \tilde{u}_0\|_{H^1(K)}
\]

\[
\leq C \|u_0 - \tilde{u}_0\|_{H^1(\Omega_i)}^2,
\]

where we have used a standard trace estimate; see, e.g., [4, Theorem 1.6.6]. We then add all the contributions and use Lemma 5.1, Lemma 5.5 and Lemma 5.6 to conclude the proof of our lemma.

We now present the proof of our main theorem:

**Proof.** (Theorem 4.1) Given \( u \in V_h \), we define

\[
u_i := I_h^T u \in V_0^R \quad \text{and} \quad u_i := R_i(I_h^T(\tilde{\theta}_i(u - R_0^T u_0))) \in V_i,\]

with \( \tilde{\theta}_i \) a typical partition of unity for the overlapping subdomains \( \Omega_i' \); see [18, Lemma 3.4]. It is straightforward to verify that \( \sum_{i=0}^N R_i^T u_i = u \). By [18, Theorem 3.13], if there exists a constant \( C_0 \) such that

\[
\sum_{i=0}^N \tilde{a}_{h,i}(u_i, u_i) \leq C_0^2 a_h(u, u),
\]

then it holds that \( \kappa(P_{ad}) \leq (N_C + 1)C_0^2 \), where \( N_C \) is the minimum number of colors needed to paint the overlapping subdomains \( \Omega_i' \) such that no pair of subdomains of the same color intersect.

For the sake of completeness we present the main ideas required to estimate \( C_0 \); we refer to [6, Theorem 3.1] for more details. From (11), (7) and Lemma 5.7 it is easy to deduce that

\[
\tilde{a}_{h,0}(u_0, u_0) \leq C \frac{\alpha_2}{\alpha_1} \omega_h(u, u).
\]

For any element \( K \in \mathcal{T}_h \), we consider a fixed triangular mesh \( \mathcal{T}_K \) of \( K \). Denote by \( I_K \) the piecewise-linear interpolant onto \( \mathcal{T}_K \). Define then \( \theta^i \), \( R_0^T u_0 \) \( \in H^1(\Omega_i) \) element by element by

\[
\theta^i|_K := I_K \tilde{\theta}_i \quad \text{and} \quad w^i|_K := I_K(u - R_0^T u_0).
\]
It is clear that $\theta^\ell_i$ satisfies $\|\nabla \theta^\ell_i\|_{L^\infty(K)} \leq \|\nabla \tilde{\theta}_i\|_{L^\infty(K)} \leq C/\delta_i$. From (12), (7) and Lemma 5.4 we deduce that

$$\tilde{a}_{h,i}(u_i, u_i) \leq C\alpha_2 \left( \int_{\Omega'_i} \rho|\theta^\ell_i \nabla \theta^\ell_i|^2 \, dx + \int_{\Omega'_i} \rho|\theta^\ell_i \nabla \theta^\ell_i|^2 \, dx \right).$$

We have that $|w^\ell_i|^2_{H^1(K)} \leq C\|u - R^T_0 u_0\|^2_{H^1(K)}$ by using an inverse inequality. Since $|\theta^\ell_i| \leq 1$, the first term in the sum of (16) can be bounded then by

$$\int_{\Omega'_i} \rho|\theta^\ell_i \nabla \theta^\ell_i|^2 \, dx \leq C \sum_{j \in \Xi_i} \rho_j \|u - R^T_0 u_0\|_{H^1(\Omega_j)} \leq C\omega \sum_{j \in \Xi_i} a^{\Omega_j}(u, u),$$

where we have used Lemma 5.7. Here, $\Xi_i := \{j : \Omega_j \cap \Omega_i \neq \emptyset\}$.

In order to estimate the last term in (16), we note that the gradient of $\theta^\ell_i$ is not zero only in a neighborhood of $\partial \Omega_i$ of width $\max_{j \in \Xi, \delta_j}$. The number of sets $\Omega_i'$ that intersect $\Omega_i$ is uniformly bounded, and therefore we need to consider the contribution from only one of them. We write $w^\ell_i = w_1^\ell + w_2^\ell$, with $w_1^\ell|_{\Omega_j} := I_K(u - u_0)$ and $w_2^\ell|_{\Omega_j} := I_K(u_0 - R^T_0 u_0)$ for each element $K$.

Since $\|w_1^\ell\|^2_{L^\infty(\Omega_1)} \leq C\|u - \tilde{u}\|_{L^\infty(\Omega_1)}$, we have that

$$\int_{\Omega'_1 \cap \Omega_i} \rho_i |w_1^\ell \nabla \theta^\ell_i|^2 \, dx \leq C \frac{\rho_i}{\delta_i} \|u - \tilde{u}\|^2_{L^\infty(\Omega_1)} \leq C \frac{H_i}{\delta_i} \omega a^{\Omega_1}(u, u).$$

For the remaining term $w_2^\ell$, we cover $\Omega'_i \cap \Omega_i$ with square patches with sides on the order of $\delta_i$ and note that on the order of $H_i/\delta_i$ of them will suffice. For a square $\pi_k$ we can bound

$$\int_{\pi_k} \rho_i |w_2^\ell \nabla \theta^\ell_i|^2 \, dx \leq C \frac{\rho_i}{\delta_i} \|w_2^\ell\|^2_{L^2(\pi_k)} \leq C \frac{\rho_i}{\delta_i} |w_2^\ell|_{H^1(\pi_k)}$$

where we have used a Friedrich’s inequality, since $w_2^\ell$ vanishes on $\partial \Omega_i$. By adding all the contributions from the squares $\pi_k$ we can conclude that

$$\int_{\Omega'_i} \rho|w^\ell_i \nabla \theta^\ell_i|^2 \, dx \leq C \left( 1 + \frac{H}{\delta} \right) \omega \sum_{j \in \Xi_i} a^{\Omega_j}(u, u).$$

Thus, by substituting (17) and (19) in (16), we obtain

$$\tilde{a}_{h,i}(u_i, u_i) \leq C \frac{\alpha_2}{\alpha_1} \omega \left( 1 + \frac{H}{\delta} \right) \sum_{j \in \Xi, \delta_j} \sum_{K \subset \Omega_j} a^K_h(u, u).$$

From (15) and (20) we conclude that (14) holds with

$$C^2_h := C \frac{\alpha_2}{\alpha_1} \left( 1 + \frac{H}{\delta} \right) \left( 1 + \log \frac{H}{h} \right),$$

and our proof is complete.

6. Experimental results. In this section we discuss some implementation details and present numerical results for our two-level overlapping additive algorithm with $\Omega = [0, 1]^2$. 
6.1. Implementation. We first describe how to compute the operators $\Pi_{\Omega_i,k}^\Sigma$; see [2] for more details. Consider a subdomain $\Omega_i$ and the basis of scaled monomials $\mathcal{M}_{k-2}(\Omega_i)$. We use the natural ordering for the multi-indices

$$\alpha_1 = (0, 0), \alpha_2 = (1, 0), \alpha_3 = (0, 1), \ldots, \alpha_{N_M} = (0, k - 2),$$

with $N_M := k(k - 1)/2$ the dimension of $\mathcal{M}_{k-2}$. We also order the degrees of freedom of $V_k^i$ and denote by $\text{dof}_p(u)$ the functional that computes the $p$-th degree of freedom of a given function $u$, $1 \leq p \leq N_V$, with $N_V := \dim V_k^i$. In this way, we define the canonical basis $\{\varphi_p\}$ of $V_k^i$ such that $\text{dof}_p(\varphi_q) = \delta_{pq}$. We then define the $N_V \times N_M$ matrix $D$, where its entries are given by

$$D_{pq} = \text{dof}_p(m_{\alpha_q}),$$

and the $N_M \times N_V$ matrix $B$ given by

$$B := \begin{bmatrix} P_0 \varphi_1 & \cdots & P_0 \varphi_{N_V} \\ (\nabla m_{\alpha_2}, \nabla \varphi_1)_{0,\Omega_i} & \cdots & (\nabla m_{\alpha_2}, \nabla \varphi_{N_V})_{0,\Omega_i} \\ \vdots & \ddots & \vdots \\ (\nabla m_{\alpha_{N_M}}, \nabla \varphi_1)_{0,\Omega_i} & \cdots & (\nabla m_{\alpha_{N_M}}, \nabla \varphi_{N_V})_{0,\Omega_i} \end{bmatrix}.$$ 

By simple linear algebra, it is easy to show that the matrix representation of the operator $\Pi_{\Omega_i,k}^\Sigma$ acting from $V_k^i$ to $P_k(\Omega_i)$ in the basis $\mathcal{M}_k(\Omega_i)$ is given by

$$\Pi_{\Omega_i,k}^\Sigma := (BD)^{-1}D.$$ 

We can compute $B$ by using the formula

$$\int_{\Omega_i} \nabla m_{\alpha_i} \cdot \nabla \varphi_p \, dx = - \int_{\Omega_i} \varphi_p \Delta m_{\alpha_i} \, dx + \int_{\partial \Omega_i} \varphi_p \frac{\partial m_{\alpha_i}}{\partial n} \, ds.$$ 

The first integral in the sum can be computed with the moments of $\varphi_p$ since $\Delta m_{\alpha_i} \in \mathcal{P}_{k-2}$. For the last integral, $\varphi_p$ is a polynomial on the boundary of the subdomain. Thus, the entries of $B$ can be computed exactly up to machine precision.

Next, we describe how to compute the matrix $R_0^T$. We note that there is one column for each coarse basis function. Given a subdomain vertex $x_0$, its column contains the values of the degrees of freedom of $R_0^T \psi_{x_0}$ in $V_k$. The values of $\psi_{x_0}$ at the nodes on the interface are computed from the definition of $\psi_{x_0}^H$. For the interior nodes of a subdomain $\Omega_i$, we compute the degrees of freedom as follows. Consider the local matrix $A_{(i)}$ written in block form

$$A_{(i)} = \begin{bmatrix} A_{II}^{(i)} & A_{IB}^{(i)} \\ A_{BI}^{(i)} & A_{BB}^{(i)} \end{bmatrix}$$

where $I$ and $B$ stand for interior and boundary degrees of freedom, respectively. Similarly, we write the projector matrix $\Pi_{\Omega_i}^\Sigma$ in block form

$$\Pi_{\Omega_i}^\Sigma = [\Pi_{\Omega_i,1}, \Pi_{\Omega_i,2}],$$

where the first block $\Pi_{\Omega_i,1}$ includes the columns related to the nodal degrees of freedom, and the second block $\Pi_{\Omega_i,2}$ has the columns related to the degrees of freedom for the moments. Similarly, we write the degrees of freedom of $\psi_{x_0}^H$ as a column vector.
with \( D_2 \) the unknown moments. Therefore, the coefficients of the polynomial \( \Pi^{\nabla} \psi_{x_0}^{H} \) on the scaled monomial basis \( M_{\Omega_i}(\psi_{x_0}) \) are given by \( \Pi^{\nabla} [D_1, D_2]^T \). If we denote by \( M \) the matrix whose columns are the scaled monomials evaluated at the interior nodes of \( \Omega_i \) and by \( v_B \) the values of \( \psi_{x_0}^{H} \) at the nodes of \( T_h \) on \( \partial \Omega_i \), when finding the minimum of \( a^{\Omega_i}(\psi_{x_0}^{H}, \psi_{x_0}^{H}) \) we obtain the linear system with \( k(k-1)/2 \) unknowns given by

\[
(\Pi_{i,2}^T M^T A^{(i)}_{11} M \Pi_{*,2}) D_2 = -(v_B^T A^{(i)}_{B1} M \Pi_{*,2} + \Pi_{*,2}^T M^T A^{(i)}_{11} M \Pi_{*,1} D_1).
\]

After solving for \( D_2 \), the degrees of freedom inside the subdomain are given by

\[
M (\Pi_{*,1} D_1 + \Pi_{*,2} D_2).
\]

Fig. 2: Different approximations for a coarse basis function with square elements and METIS subdomains (\( \rho = 1 \)).

We show the time required to assemble the matrix \( R_0^T \) as a function of \( H/h \) in Figure 3 for different approaches. We consider the case of discrete harmonic extensions as considered in [9], the case of piecewise-linear approximations studied in [6] and our operator \( R_0^T \) for \( k = 2 \) and \( k = 3 \). The assembling times are obtained by a serial code implemented in Matlab. For simplicity, we consider square elements and just four METIS subdomains. The computed time includes the factorization of the local matrices in the case of discrete harmonic extensions, and the computation of the local projectors \( \Pi^{\nabla} \) for our method. In all the cases, the values on the interface are computed similarly.

Fig. 3: Assembling time (in seconds) for \( R_0^T \).
6.2. Examples. We present some experiments to verify the bound obtained in Theorem 4.1. We consider different polygonal meshes with square and irregular subdomains (created with the mesh-partitioner software METIS [15]); see Figure 4 for a triangulation with hexagons and irregular polygons. We estimate the condition number $\kappa$ and compute the number of iterations $I_2$ (for $k = 2$), $I_3$ (for $k = 3$) and $I_H$ (for the coarse space [9] based on discrete harmonic extensions) for each experiment. We note that our operator $R_0$ recovers the bilinear $Q_1$ coarse space if $\Omega_i$ is a square and for this reason we only compute $I_2$ when considering square subdomains. For METIS subdomains we compare the different approximations.

Fig. 4: Solution with virtual elements for hexagonal (left) and irregular (right) meshes and constant coefficient $\rho = 1$.

We solve the resulting linear systems using the preconditioned conjugate gradient method to a relative residual tolerance of $10^{-6}$. We compute the right-hand side such that the exact solution is $u(x_1, x_2) = \sin(\pi x_1) \sin(\pi x_2)$ when $\rho = 1$. When referring to discontinuous coefficients, we generate random numbers $r_i \in [-3, 3]$ with a uniform distribution and assign $\rho_i = 10^{r_i}$ to each individual element inside $\Omega_i$.

6.2.1. Triangular mesh. For this case, the VEM corresponds to the first-order Lagrange space $P_1$ and our analysis provides a new approximation for the coarse space when using irregular subdomains. We verify that our algorithm is scalable and observe the logarithmic factor when $\rho$ is discontinuous across the interface; see results in Table 1. We also confirm the linear growth in the condition number as we increase $H/\delta$; see Figure 5.

6.2.2. Square mesh. For this case we present two sets of results. First, we discretize problem (2) with the $Q_1$ standard space of bilinear elements in order to compare our results to the ones in [9]; see Table 2. In this setting, our coarse space provides a new approach for irregular subdomains. Second, we solve the discrete problem (8) obtained by VEM; see Table 3. In the latter case, results are essentially the same when $\rho = 1$ and we omit them. We also verify the linear dependence on $H/\delta$ in both cases; see Figure 6.

6.2.3. Hexagonal and irregular polygons. For this set of examples we consider polygonal triangulations with METIS subdomains as in Figure 4. The irregular mesh is obtained from a Voronoi diagram for a given set of random numbers in the unit square and contains polygons with different number of edges. We solve the linear system that arises from Equation (8) and we note that results are quite similar if we use discrete harmonic extensions or our spaces even for $k = 2$. We confirm the scalability in Table 4; see also Table 5 and Figure 7 for the dependence on $H/h$ and
Table 1: Number of iterations $I$ and condition number $\kappa$ (in parenthesis) for our problem with triangular elements and $N$ subdomains. $I_2$, $I_3$ and $I_H$ correspond to $k = 2$, $k = 3$ and discrete harmonic extensions, respectively. $N_V$ is the dimension of the coarse space.

<table>
<thead>
<tr>
<th>$N$</th>
<th>Test 1: $H/h = 16$, $H/\delta = 4$, $\rho = 1$</th>
<th>Test 2: $N = 64$, $H/\delta = 4$, $\rho = 1$</th>
<th>Test 3: $N = 64$, $H/\delta = 4$, $\rho$ disc</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$N_V$</td>
<td>$I_2$ ($\kappa$)</td>
<td>$N_V$</td>
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<tr>
<td>$12^2$</td>
<td>121</td>
<td>14 (6.0)</td>
<td>243</td>
</tr>
<tr>
<td>$16^2$</td>
<td>225</td>
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<td>711</td>
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<tr>
<td>$24^2$</td>
<td>529</td>
<td>13 (4.8)</td>
<td>1054</td>
</tr>
</tbody>
</table>

$H/h$ respectively.

6.2.4. Discontinuous coefficients across the interface. We conclude with two experiments where we have discontinuities inside the subdomains, even though our theory does not cover these cases. We consider first a distribution $\rho_C$ as in Figure 8a, where four channels go through the subdomains. We then consider the extreme case of different values $\rho_K$ for each element; see Figure 8b. Results are presented in Table 6, where we have used a square mesh and METIS subdomains.
Table 2: Number of iterations $I$ and condition number $\kappa$ (in parenthesis) for our problem with bilinear elements (square mesh) and $N$ subdomains. $I_2$, $I_3$ and $I_H$ correspond to $k = 2$, $k = 3$ and discrete harmonic extensions, respectively. $N_V$ is the dimension of the coarse space.

<table>
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<tr>
<th>$N$</th>
<th>$H/h$</th>
<th>Test 1: $H/h = 16$, $H/\delta = 4$, $\rho = 1$</th>
<th>$N_V$</th>
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<th>$I_3(\kappa)$</th>
<th>$I_H(\kappa)$</th>
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<tbody>
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<tr>
<th>$H/h$</th>
<th>Test 2: $N = 64$, $H/\delta = 8$, $\rho = 1$</th>
<th>$N_V$</th>
<th>$I_2(\kappa)$</th>
<th>$I_3(\kappa)$</th>
<th>$I_H(\kappa)$</th>
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<tr>
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<td>17 (8.3)</td>
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<th>$I_3(\kappa)$</th>
<th>$I_H(\kappa)$</th>
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<tr>
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<td>30 (20.4)</td>
<td>96</td>
<td>35 (19.5)</td>
<td>32 (12.2)</td>
</tr>
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Table 3: Number of iterations $I$ and condition number $\kappa$ (in parenthesis) for our problem with VEM (square mesh) and $N$ subdomains. $I_2$, $I_3$ and $I_H$ correspond to $k = 2$, $k = 3$ and discrete harmonic extensions, respectively. $N_V$ is the dimension of the coarse space.

<table>
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<th>$N$</th>
<th>$H/h$</th>
<th>Test 1: $H/h = 16$, $H/\delta = 4$, $\rho$ disc</th>
<th>$N_V$</th>
<th>$I_2(\kappa)$</th>
<th>$I_3(\kappa)$</th>
<th>$I_H(\kappa)$</th>
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<tr>
<td>$16^2$</td>
<td>225</td>
<td>30 (12.8)</td>
<td>450</td>
<td>27 (9.5)</td>
<td>26 (8.7)</td>
<td>25 (7.9)</td>
</tr>
<tr>
<td>$20^2$</td>
<td>361</td>
<td>32 (13.5)</td>
<td>721</td>
<td>29 (10.5)</td>
<td>26 (8.5)</td>
<td>25 (7.9)</td>
</tr>
<tr>
<td>$24^2$</td>
<td>529</td>
<td>33 (14.4)</td>
<td>1058</td>
<td>31 (11.1)</td>
<td>29 (10.2)</td>
<td>28 (9.9)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$H/h$</th>
<th>Test 2: $N = 64$, $H/\delta = 8$, $\rho$ disc</th>
<th>$N_V$</th>
<th>$I_2(\kappa)$</th>
<th>$I_3(\kappa)$</th>
<th>$I_H(\kappa)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>49</td>
<td>23 (11.4)</td>
<td>96</td>
<td>27 (9.7)</td>
<td>27 (9.5)</td>
</tr>
<tr>
<td>16</td>
<td>49</td>
<td>24 (14.7)</td>
<td>96</td>
<td>28 (10.2)</td>
<td>27 (9.7)</td>
</tr>
<tr>
<td>32</td>
<td>49</td>
<td>26 (17.6)</td>
<td>96</td>
<td>31 (12.2)</td>
<td>30 (10.4)</td>
</tr>
<tr>
<td>64</td>
<td>49</td>
<td>26 (20.4)</td>
<td>96</td>
<td>35 (19.5)</td>
<td>32 (12.2)</td>
</tr>
</tbody>
</table>
Fig. 6: Condition number as a function of $H/\delta$ for a square mesh discretized with VEM. $N = 64$, $H/h = 32$, $\rho = 1$ (left) and $\rho$ discontinuous (right). The stiffness matrix has 66049 degrees of freedom, $N_V$ is 49 and 96 for square and METIS subdomains, respectively.

Table 4: Number of iterations and condition number (in parenthesis) with hexagonal and irregular elements and METIS subdomains. $N_V$ is the dimension of the coarse space, $N$ the number of subdomains, $k = 2$, $H/h \approx 8$ and $H/\delta \approx 4$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>Hexagonal elements</th>
<th>Irregular elements</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$N_V$ $I_2(\kappa)$ $I_2(\kappa)$</td>
<td>$N_V$ $I_2(\kappa)$ $I_2(\kappa)$</td>
</tr>
<tr>
<td>$12^2$</td>
<td>239 20(7.1) 24(7.8)</td>
<td>203 24(9.1) 26(10.3)</td>
</tr>
<tr>
<td>$16^2$</td>
<td>450 20(6.1) 25(8.4)</td>
<td>398 23(8.9) 28(10.0)</td>
</tr>
<tr>
<td>$20^2$</td>
<td>723 20(6.2) 24(7.3)</td>
<td>580 25(9.9) 32(13.3)</td>
</tr>
<tr>
<td>$24^2$</td>
<td>1064 21(7.2) 26(8.0)</td>
<td>983 36(10.0) 33(13.0)</td>
</tr>
</tbody>
</table>

Table 5: Number of iterations and condition number (in parenthesis) with hexagonal and irregular elements and METIS subdomains. $N_V$ is the dimension of the coarse space, $k = 3$, $N = 16$ and $H/\delta \approx 4$.

<table>
<thead>
<tr>
<th>$H/h$</th>
<th>Hexagonal elements</th>
<th>Irregular elements</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$N_V$ $I_3(\kappa)$ $I_3(\kappa)$</td>
<td>$N_V$ $I_3(\kappa)$ $I(\kappa)$</td>
</tr>
<tr>
<td>8</td>
<td>19 16(4.5) 17(4.6)</td>
<td>18 20(7.1) 22(6.9)</td>
</tr>
<tr>
<td>16</td>
<td>18 17(5.3) 19(5.2)</td>
<td>19 23(8.7) 26(8.9)</td>
</tr>
<tr>
<td>32</td>
<td>18 21(6.7) 23(6.6)</td>
<td>18 27(10.3) 27(10.2)</td>
</tr>
<tr>
<td>64</td>
<td>18 21(6.2) 22(7.2)</td>
<td>18 29(11.0) 30(11.3)</td>
</tr>
</tbody>
</table>
Fig. 7: Condition number as a function of $H/\delta$ for different polygonal meshes discretized with VEM. $N = 16$, $H/h = 32$, $\rho = 1$ (left) and $\rho$ discontinuous (right).

Fig. 8: Two discontinuous coefficients $\rho$ considered in Subsection 6.2.4; see results in Table 6a.

Table 6: Number of iterations and condition number (in parenthesis) for our problem with VEM (square elements), METIS subdomains and $\rho$ discontinuous as in Figure 8.

<table>
<thead>
<tr>
<th>$H/h$</th>
<th>$\rho_C$</th>
<th>$\rho_K$</th>
<th>$N$</th>
<th>$\rho_C$</th>
<th>$\rho_K$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$I_3(\kappa)$</td>
<td>$I_3(\kappa)$</td>
<td>$I_3(\kappa)$</td>
<td>$I_3(\kappa)$</td>
<td>$I_3(\kappa)$</td>
</tr>
<tr>
<td>8</td>
<td>27(51.9)</td>
<td>43(87.9)</td>
<td>12</td>
<td>32(33.7)</td>
<td>74(106)</td>
</tr>
<tr>
<td>16</td>
<td>30(85.3)</td>
<td>47(101)</td>
<td>16</td>
<td>35(20.8)</td>
<td>92(155)</td>
</tr>
<tr>
<td>32</td>
<td>32(125)</td>
<td>48(109)</td>
<td>20</td>
<td>39(18.8)</td>
<td>99(180)</td>
</tr>
<tr>
<td>64</td>
<td>34(150)</td>
<td>50(112)</td>
<td>24</td>
<td>37(14.9)</td>
<td>102(211)</td>
</tr>
</tbody>
</table>
7. Conclusions. In this paper we have introduced a new operator $R^T_0 : V^R_0 \rightarrow V_h$ for approximating virtual coarse functions that belong to the reduced space $V^R_0$. This approach is particularly useful in the presence of irregular subdomains and relies on the construction of a projector operator into the space of polynomials of a prescribed degree $k \geq 2$. Our approach is faster than previous studies based on discrete harmonic extensions as shown in Figure 3, and provides similar number of iterations and estimates for the condition number of the preconditioned system even for $k = 2$, as confirmed in Section 6.

We have obtained a theoretical upper bound for the condition number of the preconditioned system by using a two-level overlapping Schwarz method, where we have used VEM for the discretization of problem (2). Results are competitive and independent of jumps of the coefficient across the subdomains, and the method allows to handle irregular subdomains as the ones obtained by mesh partitioners. We have also tested cases not covered by our theory in Subsection 6.2.4, where we have discontinuities inside the subdomains. In such cases, a reasonable number of iterations is obtained even for extreme cases of discontinuities and jumps across the elements.

REFERENCES

[15] G. Karypis and V. Kumar, A fast and high quality multilevel scheme for partitioning irregular...


