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Abstract. A rational frictionless contact model similar to the normal compliance one with limited interpenetration in viscoelastodynamics is formulated and the existence of its solutions is proved. The convergence of its solutions to a solution of the Signorini contact (without interpenetration) is proved as well provided the depth of the prescribed interpenetration tends to zero.

Keywords. Rational contact model, limited interpenetration, viscoelastodynamics, existence of a solution, approximate problems, limit process, maximal monotonicity, convergence to the Signorini contact.

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1 Introduction

Contact problems represent an important tool in applied mathematics with a lot of applications in different applied physical sciences and engineering. Already since 1930s the basic model of Signorini is used (the mostly cited reference to it is [11]). It describes the contact of a deformable body with a rigid foundation and respects the impenetrability of mass. Due to the mostly dynamic character of contact problems, the main task for mathematical analysts is to study them. Their investigation started in late seventies with one-dimensional objects as strings. However, the one-dimensional case making possible compact imbedding of Sobolev space H^1 into continuous functions and making the boundary very simple is too special to be immediately extended to a higher space dimensions. The first result in higher dimensions was [7], it treated a dynamic contact of an elastic half–space with a flat foundation but it did not indicated a possibility to extend it to general bodies. The solvability of this problem for general viscoelastic bodies was proved in 1996 ([4]). It has been further investigated in [10] while in [9] the question of the existence of an energy-conserving solution has been solved for the case of a string. However, there is still a substantial open question of solvability of such problems for purely elastic material, although an amount of existence results for dynamic contact of two dimensional thin structures as plates and shells was proved, where the purely elastic case was solved for their fourth-order models.

The real material has never a perfect geometrical form, it has small surface asperities to be deformed and small holes to be filled. These microscopical facts can be macroscopically described as some kind of interpenetration between the body and the foundation. Since 1980s

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models of contact with unlimited interpenetration have been studied, cf e.g. [8]. The contact term has there usually the form of a compact perturbation of corresponding contactless problems which is advantageous for mathematical analysis. However, they are obviously not realistic from the point of view of physics, although they can be useful for numerical approximation of problems from technical practice. In 2007 the first model with a limited interpenetration was suggested in [6], the limit of the interpenetration is prescribed and reachable.

In [3] a new physically reasonable model has been introduced which describes a coercive static contact of a body with a foundation which is rigid, but allows some prescribed limited interpenetration into its surface. Both the frictionless and the frictional problem has been treated. The limit of the interpenetration is prescribed and cannot be reached outside a set of zero measure which seems to be physically more suitable than the model in [6]. The semicoercive static version of this problem was treated in [5]. However, the decisive step to some wider applicability of such a model is the proof of the solvability of its dynamic version at least for some sufficiently wide class of materials. A physically well posed viscosity is of a great help. To prove the existence of solutions to such problems is the aim of this paper.

Our contact model, although easily formulated, is mathematically quite complex and thus we introduce and solve first approximate problems similar to penalized ones formulated for the Signorini contact. After an easy proof of their solvability we have to carry out an appropriate limit process based on further estimates of solutions to the approximate problems. Our task here is to prove that the limit satisfies the non-compact normal–compliance type relation. The main argument we use here is the maximal monotonicity of the superposition operator involved.

2 Problem formulation and approximation

We assume the constitutive law for the stress tensor $\boldsymbol{\sigma}$ given by Hooke's law of linear viscoelasticity

$$\boldsymbol{\sigma} = \mathscr{A}^{(1)}\dot{\boldsymbol{\varepsilon}} + \mathscr{A}^{(0)}\boldsymbol{\varepsilon} \tag{1}$$

with a possibly space–dependent tensors $\mathscr{A}^{(\iota)} = (a^{(\iota)}_{ijk\ell})^d_{i,j,k,\ell=1}$ of the fourth order, $\iota = 0, 1$. Both tensors are assumed to be symmetric and to have measurable entries,

$$a^{(\iota)}_{ijk\ell} = a^{(\iota)}_{k\ell ij} = a^{(\iota)}_{jik\ell}, \quad a^{(\iota)}_{ijk\ell} \in L_\infty(\Omega) \text{ for every } i, j, k, \ell \in \{1, \dots, N\},\ \iota = 0, 1$$

on a domain Ω which is a bounded connected set in $\mathbb{R}^d$ with a boundary Γ of the class $C^{3/2}$. The tensor $\boldsymbol{\varepsilon}$ is the linearized strain tensor $\varepsilon_{ij} : \boldsymbol{u} \mapsto 1/2(\partial u_i/\partial u_j + \partial u_j/\partial u_i)$ and the dot $\dot{}$ denotes here and in the sequel the time derivative.

Moreover, the entries $a^{(\iota)}_{ijk\ell}$ are supposed to be positive definite and bounded in the sense

$$\sum_{i,j,k,\ell=1}^{d} a^{(\iota)}_{ijk\ell}(\boldsymbol{x})\, \xi_{ij}\, \xi_{k\ell} \geq a^{(\iota)}_0 |\boldsymbol{\xi}|^2 \text{ and } \sum_{i,j,k,\ell=1}^{d} a^{(\iota)}_{ijk\ell}(\boldsymbol{x})\, \xi_{ij}\, \eta_{k\ell} \leq A^{(\iota)}_0 |\boldsymbol{\xi}||\boldsymbol{\eta}|, \iota = 0, 1 \tag{2}$$

for every symmetric tensors $\boldsymbol{\xi} = (\xi_{ij})^d_{i,j=1}$, $\boldsymbol{\eta} = (\eta_{ij})^d_{i,j=1}$ with norm $|\boldsymbol{\xi}| = \sqrt{\sum_{i,j=1}^d |\xi_{ij}|^2}$ and with constants $a^{(\iota)}_0, A^{(\iota)}_0 > 0$ independent of $\boldsymbol{x} \in \Omega$ for $\iota = 0, 1$.

The normal component of the boundary traction is denoted by $\sigma_\nu = \boldsymbol{\sigma\nu} \cdot \boldsymbol{\nu}$, where $\boldsymbol{\nu}$ is the unit outward normal.

In the sequel, we use the following notation for the spaces employed: by $H^k(M)$ with $k \geq 0$ the Sobolev (for a noninteger k the Sobolev-Slobodetskii) spaces of the Hilbert type are denoted provided they are defined on a domain or an appropriate manifold M. If M is a time-space domain we will use the anisotropic spaces with the notation $H^k(M)$ ($k = (k_1, k_2) \in \mathbb{R}_+^2$) and it signifies that k_1 is related to the time while k_2 to the spacial variables.

The extension of this notation for the Bochner–type spaces is the following: $H^{k_1}(I; H^{k_2}(\Omega))$ stands for the space of mappings $u : I \to H^{k_2}(\Omega)$ having square integrable time derivatives up to the order k_1 in Sobolev space $H^{k_2}(\Omega)$.

If M is an interval, by $H_{00}^{1/2}(M)$ is denoted the space of functions extendable by zero to $H^{1/2}(\mathbb{R})$. By $\mathring{H}^k(M)$ we denote the spaces with zero traces on ∂M if $k > 1/2$. By $H^{-k}(M)$ their duals are denoted.

$\mathscr{D}(M)$ is used for the space of infinitely differentiable functions with compact support in M and $\mathscr{D}'(M)$ for the space of distributions on M.

For the finite-dimensional vectors and spaces of vector-valued functions the bold symbols are consequently used throughout the paper. For the sake of notation simplicity this does not apply for zero elements in infinitely-dimensional spaces.

The contact model shall be a normal compliance law of the type

$$\sigma_\nu = -p(u_\nu - g) \tag{3}$$

with a function $p : \mathbb{R} \to \overline{\mathbb{R}_+}$, where $\overline{\mathbb{R}_+} = [0, +\infty]$. The natural assumptions for p are $p(z) = 0$ for $z \le \alpha$ with a constant α, p is monotone, $\lim_{z\to\beta-} p(z) = +\infty$ with a $\beta > \alpha$, and $p(z) = +\infty$ for $z \ge \beta$. The third requirement here means that the interpenetration in the normal compliance model is limited by β. The value of α may describe the contact of the first asperities, the value of β the total flattening of the boundary such that no further interpenetration is possible.

In the analysis that is presented here we use slightly weaker conditions, we require that

$$\begin{aligned} &p : \mathbb{R} \to \overline{\mathbb{R}_+}, \quad p_{|(-\infty,\beta)} \in C(-\infty, \beta), \quad \lim_{z\to-\infty} p(z) = 0, \\ &\lim_{z\to\beta} p(z) = +\infty, \quad p \text{ is monotone on } (-\infty, \beta) \text{ and } \int_{-\infty}^{y} p(z)\, dz < +\infty \text{ for } y < \beta. \end{aligned} \tag{4}$$

Moreover, we assume that there is a decreasing sequence

$$\begin{aligned} &(\lambda_n)_{n=1}^{+\infty} \text{ with } \lim_{n\to+\infty} \lambda_n = 0 \text{ such that the derivatives } p'(\beta - \lambda_n) \text{ exist,} \\ &\text{the sequence } (p'(\beta - \lambda_n))_{n=1}^{+\infty} \text{ is increasing and } \lim_{n\to+\infty} p'(\beta - \lambda_n) = +\infty. \end{aligned} \tag{5}$$

The dynamic contact problem to be studied has the following classical formulation:
Look for a displacement $\boldsymbol{u}$ such that the following conditions are satisfied

$$\ddot{\boldsymbol{u}} - \operatorname{Div}(\boldsymbol{\sigma}(\boldsymbol{u})) = \boldsymbol{f} \qquad \text{in } Q = (0, T) \times \Omega, \tag{6}$$

$$\boldsymbol{u} = \boldsymbol{U} \qquad \text{on } S_1 = (0, T) \times \Gamma_1, \tag{7}$$

$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu} = \boldsymbol{b} \qquad \text{on } S_2 = (0, T) \times \Gamma_2, \tag{8}$$

$$\sigma_\nu(\boldsymbol{u}) + p(u_\nu - g) = 0 \tag{9}$$

$$\boldsymbol{\sigma}_\tau(\boldsymbol{u}) = 0 \qquad \text{on } S_3 = (0, T) \times \Gamma_3 \tag{10}$$

$$\boldsymbol{u}(0, \boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \dot{\boldsymbol{u}}(0, \boldsymbol{x}) = \boldsymbol{u}_1(\boldsymbol{x}) \qquad \text{for a. e. } \boldsymbol{x} \in \Omega. \tag{11}$$

We assume that Γ_i for $i = 1, 2, 3$ are pairwise disjoint subsets of $\Gamma = \partial\Omega$ which are open in the relative topology induced on the boundary and $\bigcup_{i=1}^3 \overline{\Gamma_i} = \Gamma$. We assume that Γ_3 has a positive surface measure and that Γ_i for $i = 1, ..., 3$ have Lipschitz relative boundaries with respect to the relative topology on Γ. With an abuse of notation we will extend function g defined originally on Γ_3 to S_3 as $g(t, x) = g(x)$ for all $(t, x) \in S_3$.

The nonlinear superposition operator $\tilde{p}$ generated by p is a mapping which is defined on a set

$$\operatorname{dom}(\tilde{p}) = \left\{ z \in H^{1/4,\,1/2}(S_3)\,;\, p(z) \in (H^{1/4,\,1/2}(S_3))^* \right\}$$

as

$$\tilde{p}(z)(w) = \int_{S_3} p(z(t,x))w(t,x)ds_x dt = \langle p(z(t,x)), w(t,x)\rangle_{S_3} \text{ for } w \in H^{1/4,1/2}(S_3).$$

By $\langle f(t,x), g(t,x)\rangle_G$ we denote the ($L_2(G)$ based) duality pairing on an indicated set G. As we shall want to have the boundary tractions to solutions of the problem solved on the contact zone to be functions at least in $\boldsymbol{L}_1(S_3)$, we shall assume that if $\Gamma_1 \neq \emptyset$, then $\operatorname{dist}(\Gamma_1, \Gamma_3) > 0$.

Observe that unlike the static situation the acceleration prevents everytimes the semicoercive phenomenon here, hence the nonemptiness of Γ_1 is negligible.

We introduce the space $\boldsymbol{H}^1_{\boldsymbol{U}}(\Omega) \equiv \{\boldsymbol{w} \in \boldsymbol{H}^1(\Omega); \boldsymbol{w}_{|\Gamma_1} = \boldsymbol{U}\}$. (Note that in case $\boldsymbol{U} \equiv \boldsymbol{0}$ just defined space $\boldsymbol{H}^1_0(\Omega)$ consists of functions having zero traces on Γ_1 while space of functions with zero traces on $\partial\Omega$ is denoted by $\mathring{\boldsymbol{H}}^1(\Omega)$.) The corresponding weak formulation of original problem is given by

Find $\boldsymbol{u} \in L_\infty\big(I; \boldsymbol{H}^1_{\boldsymbol{U}}(\Omega)\big)$ *with* $u_\nu - g \in \operatorname{dom}(\tilde{p})$, $\dot{\boldsymbol{u}} \in L_\infty(I; \boldsymbol{L}_2(\Omega)) \cap L_2(I; \boldsymbol{H}^1(\Omega))$, $\boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0$, *and* $\dot{\boldsymbol{u}}(0,\cdot) = \boldsymbol{u}_1$ *in* Ω *such that for every* $\boldsymbol{v} \in \boldsymbol{H}^1_0(Q)$

$$\begin{aligned} \langle \dot{\boldsymbol{u}}(T), \boldsymbol{v}(T)\rangle_\Omega - \langle \boldsymbol{u}_1, \boldsymbol{v}(0)\rangle_\Omega - \langle \dot{\boldsymbol{u}}, \dot{\boldsymbol{v}}\rangle_Q + \langle \boldsymbol{\sigma}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_Q \\ + \langle p(u_\nu - g), v_\nu\rangle_{S_3} = \langle \boldsymbol{\ell}, \boldsymbol{v}\rangle_Q. \end{aligned} \tag{12}$$

Here $\langle \boldsymbol{\ell}, \boldsymbol{v}\rangle = \langle \boldsymbol{f}, \boldsymbol{v}\rangle_Q + \langle \boldsymbol{b}, \boldsymbol{v}\rangle_{S_2}$.

For $\boldsymbol{f}$, $\boldsymbol{b}$, $\boldsymbol{U}$ and g we will impose the requirements

$$\begin{aligned} &\boldsymbol{f} \in L_2\big(I; \boldsymbol{H}^1(\Omega)^*\big) \text{ such that } \boldsymbol{f}(t)_{|\Gamma} = 0 \text{ in } \boldsymbol{H}^{1/2}(\Gamma)^* \text{ for a.e. } t \in (0,T),\\ &\boldsymbol{b} \in L_2\big(I; \boldsymbol{H}^{1/2}(S_2)^*\big), \quad g \in H^{1/2}(S_3),\\ &\boldsymbol{U} \in H^1\big(I; \boldsymbol{H}^1(\Omega)\big) \cap H^2(I; \boldsymbol{L}_2(\Omega)), \quad U_\nu = 0 \text{ a.e. on } S_3,\\ &\boldsymbol{u}_0 \in \boldsymbol{H}^1(\Omega) \text{ and } \boldsymbol{u}_1 \in \boldsymbol{L}_2(\Omega),\\ &\lim_{t\to 0+} ||\boldsymbol{U}(t,.) - \boldsymbol{u}_0(.)||_{\boldsymbol{H}^1(\Omega)} = 0,\\ &\lim_{t\to 0+} ||\dot{\boldsymbol{U}}(t,.) - \boldsymbol{u}_1(.)||_{\boldsymbol{L}_2(\Omega)} = 0. \end{aligned} \tag{13}$$

For the precise meaning of the requirement for $\boldsymbol{f}$ see Appendix. We remark that it is satisfied e.g. by any $\boldsymbol{f} \in \boldsymbol{L}_2(Q)$.

As in the static case [3] the first step of our analysis is the monotone approximation of p. Let λ_n be a fixed sequence from (5). For $n \in \mathbb{N}$ let p_n be family of functions approximating p such that

$$p_n : z \mapsto \begin{cases} p(z), \ z \le \beta - \lambda_n \\ \min\big\{p(\beta - \lambda_n) + p'(\beta - \lambda_n)(z - \beta + \lambda_n),\ p(z)\big\}, z > \beta - \lambda_n. \end{cases} \tag{14}$$

Then the approximate contact problem is given by the weak formulation (we denote its solution $\boldsymbol{u}_n$).

Find $\boldsymbol{u}_n \in L_\infty\big(I;\boldsymbol{H}^1_{\boldsymbol{U}}(\Omega)\big)$ *with* $\dot{\boldsymbol{u}}_n \in L_\infty(I;\boldsymbol{L}_2(\Omega)) \cap L_2(I;\boldsymbol{H}^1(\Omega))$, $\ddot{\boldsymbol{u}}_n \in L_2\big(I;\boldsymbol{H}^1_0(\Omega)^*\big)$, $\boldsymbol{u}_n(0,\cdot)=\boldsymbol{u}_0$, *and* $\dot{\boldsymbol{u}}_n(0,\cdot)=\boldsymbol{u}_1$ *in* Ω *such that for every* $\boldsymbol{v} \in L_2(I;\boldsymbol{H}^1_0(\Omega))$

$$\langle \ddot{\boldsymbol{u}}_n, \boldsymbol{v}\rangle_Q + \langle \boldsymbol{\sigma}(\boldsymbol{u}_n), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_Q + \langle p_n((u_n)_\nu - g), v_\nu\rangle_{S_3} = \langle \boldsymbol{\ell}, \boldsymbol{v}\rangle_Q. \tag{15}$$

Observe that unlike to original problem no integration by parts in time has been performed here as we prove that solutions of approximate problem have second time derivative $\ddot{\boldsymbol{u}}_n \in L_2(I;\boldsymbol{H}^1_0(\Omega)^*)$. The classical formulation of this approximate problem differs from original one only in (9), where p is replaced by p_n.

We first prove the existence of a solution to the approximate problem.

Theorem 1 *Let* Ω *be a bounded Lipschitz domain satisfying the requirements below formula* (11), *let* $\boldsymbol{f}$, $\boldsymbol{b}$, $\boldsymbol{U}$, $\boldsymbol{u}_\iota$, $\iota=0,1$, *and* g *satisfy the requirements* (13) *and* p *satisfy the requirements* (4–5). *Then problem* (15) *has a solution. This solution satisfies the a priori estimate*

$$\|\dot{\boldsymbol{u}}_n\|_{L_2(I;\boldsymbol{H}^1(\Omega))} + \|\boldsymbol{u}_n\|_{L_\infty(I;\boldsymbol{H}^1(\Omega))} + \|\dot{\boldsymbol{u}}_n\|_{L_\infty(I;\boldsymbol{L}_2(\Omega))} + \|P_n((u_n)_\nu - g)\|_{L_\infty(I;L_1(\Gamma_3))} \le C, \tag{16}$$

where $P_n(z)=\int_{-\infty}^z p_n(y)\,dy$ *and the constant* C *depends on* Ω, Γ_j, $j=1,2,3$, $\boldsymbol{f}$, $\boldsymbol{b}$, $\boldsymbol{U}$, $\boldsymbol{u}_0$, $\boldsymbol{u}_1$ *and* g.

Proof The proof is done by a Galerkin approximation. To avoid the double indexation we omit in this proof the index n connected with approximation of p by p_n.

Let $\{\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots\}$ be a basis of $\boldsymbol{H}^1_0(\Omega)$ and $\mathscr{V}_m=\operatorname{span}\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_m\}$. For simplicity of the presentation we assume that the basis is orthogonal with respect to the scalar products in $\boldsymbol{H}^1_0(\Omega)$ and $\boldsymbol{L}_2(\Omega)$. (Such a basis can be constructed via the eigenfunctions of the corresponding boundary value problem to the Laplace operator.) Then the orthogonal projection π_m of $\boldsymbol{H}^1_0(\Omega)$ to $\mathscr{V}_m$ is bounded in $\boldsymbol{L}_2(\Omega)$ and in $\boldsymbol{H}^1(\Omega)$ norms. The Galerkin approximation searches for a function $\boldsymbol{u}_m$ with $\boldsymbol{u}_m-\boldsymbol{U}: I\to\mathscr{V}_m$, $\boldsymbol{u}_m(0,\cdot)=\pi_m\boldsymbol{u}_0$, and $\dot{\boldsymbol{u}}_m(0,\cdot)=\pi_m\boldsymbol{u}_1$ such that for every $\boldsymbol{v}\in\mathscr{V}_m$ and almost every time $t\in I$

$$\langle \ddot{\boldsymbol{u}}_m, \boldsymbol{v}\rangle_\Omega + \langle \boldsymbol{\sigma}(\boldsymbol{u}_m), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_\Omega + \langle p_n(u_{m\nu} - g), v_\nu\rangle_{\Gamma_3} = \langle \boldsymbol{\ell}, \boldsymbol{v}\rangle_\Omega. \tag{17}$$

This problem can be rewritten as a system of ordinary differential equations of second order with globally Lipschitz lower order terms for the coefficients in the representation of $\boldsymbol{u}_m$ with respect to the basis of $\mathscr{V}_m$. (Note that the Lipschitz constants depend on the approximation p_n of p.) The existence and uniqueness of solutions to this system follow from the theory of ordinary differential equations.

We now derive an energy estimate by inserting the test function $\boldsymbol{v}=\dot{\boldsymbol{u}}_m-\dot{\boldsymbol{U}}$ in the discretized equations (17) and integrating with respect to $t\in(0,t_0)$. After standard calculation we obtain

$$\begin{aligned}
&\int_0^{t_0} \|\dot{\boldsymbol{u}}_m(t)\|^2_{E_1}\,dt + \tfrac12\|\dot{\boldsymbol{u}}_m(t_0)\|^2_{L_2(\Omega)} + \tfrac12\|\boldsymbol{u}_m(t_0)\|^2_{E_0} + \|P_n(u_{m\nu}-g)(t_0)\|_{L_1(\Gamma_3)}\\
&= \tfrac12\|\dot{\boldsymbol{u}}_m(0)\|^2_{L_2(\Omega)} + \tfrac12\|\boldsymbol{u}_m(0)\|^2_{E_0} + \|P_n(u_{m\nu}-g)(0)\|_{L_1(\Gamma_3)}\\
&\quad + \int_0^{t_0}\Big[\langle \ell, \dot{\boldsymbol{u}}_m-\dot{\boldsymbol{U}}\rangle_\Omega + \langle \ddot{\boldsymbol{u}}_m, \dot{\boldsymbol{U}}\rangle_\Omega + \langle \boldsymbol{\sigma}(\boldsymbol{u}_m), \boldsymbol{\varepsilon}(\dot{\boldsymbol{U}})\rangle_\Omega\Big]dt
\end{aligned}$$

with energy seminorms $\|\boldsymbol{v}\|_{E_\iota}=\sqrt{\langle \mathscr{A}^{(\iota)}\boldsymbol{\varepsilon}(\boldsymbol{v}),\boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_\Omega}$, $\iota=0,1$. Note that the last term $\langle p_n(u_{m\nu}-g), \dot g - \dot U_\nu\rangle_{\Gamma_3}$ is zero because of assumption on g and $\boldsymbol{U}$. We use the integration by parts for the acceleration term on the right hand side

$$\int_0^{t_0}\langle \ddot{\boldsymbol{u}}_m, \dot{\boldsymbol{U}}\rangle_\Omega\,dt = \langle \dot{\boldsymbol{u}}_m(t_0), \dot{\boldsymbol{U}}(t_0)\rangle_\Omega - \langle \dot{\boldsymbol{u}}_m(0), \dot{\boldsymbol{U}}(0)\rangle_\Omega - \int_0^{t_0}\langle \dot{\boldsymbol{u}}_m, \ddot{\boldsymbol{U}}\rangle_\Omega\,dt.$$

The application of suitable Hölder inequalities and the Gronwall Lemma together with the well-known coerciveness of strains (cf. [2], Thm.1.2.1) leads to the estimate

$$\|\dot{\boldsymbol{u}}_m\|_{L_2(I;\boldsymbol{H}^1(\Omega))} + \|\dot{\boldsymbol{u}}_m\|_{L_\infty(I;\boldsymbol{L}_2(\Omega))} + \|\boldsymbol{u}_m\|_{L_\infty(I;\boldsymbol{H}^1(\Omega))} + \|P_n(u_{m\nu} - g)\|_{L_\infty(I;L_1(\Gamma_3))} \le C \tag{18}$$

with C independent of m and of n. The properties of p_n yields the estimate for $\|\ddot{\boldsymbol{u}}_m\|_{L_2(I;H_0^1(\Omega)^*)}$ independent of m but dependent on n. As a consequence we have

$$\begin{aligned}
\boldsymbol{u}_m &\rightharpoonup \boldsymbol{u} && \text{in } L_2\big(I;\boldsymbol{H}^1(\Omega)\big) \text{ and strongly in } \boldsymbol{L}_2(S_3),\\
\dot{\boldsymbol{u}}_m &\rightharpoonup \dot{\boldsymbol{u}} && \text{in } L_2\big(I;\boldsymbol{H}^1(\Omega)\big),\\
\ddot{\boldsymbol{u}}_m &\rightharpoonup \ddot{\boldsymbol{u}} && \text{in } L_2(I;\boldsymbol{H}_0^1(\Omega)^*),\\
\boldsymbol{u}_m(T) &\rightharpoonup \boldsymbol{u}(T) && \text{in } \boldsymbol{H}^1(\Omega),\\
\dot{\boldsymbol{u}}_m(T) &\rightharpoonup \dot{\boldsymbol{u}}(T) && \text{in } \boldsymbol{L}_2(\Omega)
\end{aligned}$$

with the limit $\boldsymbol{u} \in L_\infty\big(I;\boldsymbol{H}_U^1(\Omega)\big) \cap W_\infty^1(I;\boldsymbol{L}_2(\Omega))$. Since p_n for fixed n is continuous and has linear growth, theorem on Nemytskii operators (see [12] Thm. A2) implies $p_n(u_{m\nu} - g) \to p_n(u_\nu - g)$ in $L_2(S_3)$. Passing to the limit for $m \to +\infty$ we find that $\boldsymbol{u}$ is a solution to (15) for all test functions $\boldsymbol{v} \in L_2(I;\mathscr{V}_m)$ with arbitrary $m \in \mathbb{N}$. Since $\bigcup_{m\in\mathbb{N}} \mathscr{V}_m$ is dense in $\boldsymbol{H}_0^1(\Omega)$ and $\boldsymbol{u} \in L_\infty\big(I;\boldsymbol{H}^1(\Omega)\big) \cap W_\infty^1(I;\boldsymbol{L}_2(\Omega))$, equation (15) is satisfied for every $\boldsymbol{v} \in L_2(I;\boldsymbol{H}_0^1(\Omega))$. □

3 Further estimates of $\boldsymbol{u}_n$

We proceed by proving further estimates. Let us return to the notation of the solution of the problem with the approximate function p_n by $\boldsymbol{u}_n$. The original *a priori* estimates (16) yield that there is a *dual* estimate of the acceleration equivalent to the estimate

$$\|\dot{\boldsymbol{u}}_n\|_{H^1(I;\boldsymbol{H}^{-1}(\Omega))} \le C \tag{19}$$

which is independent of n. This is easily verified by taking an arbitrary $\boldsymbol{v} \in L_2(I;\mathring{\boldsymbol{H}}^1(\Omega))$ as a test function in (15) since then the contact term is zero. Interpolating this with the a priori estimate (16), which can be written as

$$\|\dot{\boldsymbol{u}}_n\|_{L_2(I;\boldsymbol{H}^1(\Omega))} \le C,$$

we get that

$$\|\dot{\boldsymbol{u}}_n\|_{H^{1/2}(I;\boldsymbol{L}_2(\Omega))} \le C \tag{20}$$

with C independent of n. These interpolation results can be obtained by the extension technique for Sobolev-Slobodetskii spaces from bounded domains to whole $\mathbb{R}^N$ for any dimension N and the partial Fourier transform in the time variable.

This technique allows to reformulate further the estimate (20) as

$$\|\ddot{\boldsymbol{u}}_n\|_{(H^{1/2}(I;\boldsymbol{L}_2(\Omega)))^*} \le C. \tag{21}$$

The contact term $\langle p_n(u_{n\nu} - g), v_\nu\rangle_{S_3}$ can be represented via equation (15) by

$$\langle p_n(u_{n\nu} - g), v_\nu\rangle_{S_3} = \langle \boldsymbol{\ell}, \boldsymbol{v}\rangle_Q + \langle \dot{\boldsymbol{u}}_n, \dot{\boldsymbol{v}}\rangle_Q - \langle \boldsymbol{\sigma}(\boldsymbol{u}_n), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_Q.$$

providing $\boldsymbol{v} \in \boldsymbol{H}_0^{1/2,1}(Q) \cap H_{00}^{1/2}(I;\boldsymbol{L}_2(\Omega)),\ n \in \mathbb{N}$. Due to (16) this implies

$$\|p_n(u_{n\nu} - g)\|_{(H^{1/4,1/2}(S_3))^*} \le C \tag{22}$$

with a constant C independent of n. (We employ here the well-known fact that $\mathscr{D}(I)$ is dense in $H^{1/4}(I)$.) Moreover, using in (15) a test function $\boldsymbol{v} \in L_2\big(I; \boldsymbol{H}_0^1(\Omega)\big)$ with $v_\nu = 1$ on S_3 and the fact that $p_n(u_{n\nu} - g)$ is nonnegative proves

$$\|p_n(u_{n\nu} - g)\|_{L_1(S_3)} \leq C$$

with C independent of n.

4 Solving the original problem

In this section we denote $\boldsymbol{u}_n$ solutions to the approximate problem with p_n while $\boldsymbol{u}$ stands for a solution to the original contact problem (12) with p and prove

Theorem 2 *Let Ω be a bounded Lipschitz domain satisfying the requirements below formula* (11)*, let $\boldsymbol{f}$, $\boldsymbol{b}$, $\boldsymbol{U}$, $\boldsymbol{u}_\iota$, $\iota = 0, 1$, and g satisfy the requirements* (13) *and p satisfy the requirements* (4–5)*. Then problem* (12) *has a solution.*

To prove it, we proceed to the convergence process for the index $n \to +\infty$. As a consequence of all above estimates there is a subsequence which will be with an abuse of notation denoted as the original sequence and which satisfies following convergences

$$
\begin{aligned}
\boldsymbol{u}_n &\rightharpoonup \boldsymbol{u} && \text{weakly in } \boldsymbol{H}^{3/2,1}(Q), \text{ strongly in } C(I; \boldsymbol{L}_2(\Omega)) \cap H^{1/2}(I; \boldsymbol{L}_2(\Omega)),\\
\boldsymbol{u}_n &\to \boldsymbol{u} && \text{strongly in } \boldsymbol{L}_2(S_3) \text{ and a.e. on } S_3,\\
\dot{\boldsymbol{u}}_n &\rightharpoonup \dot{\boldsymbol{u}} && \text{weakly in } \boldsymbol{H}^{1/2,1}(Q), \text{ and strongly in } \boldsymbol{L}_2(Q),\\
\dot{\boldsymbol{u}}_n(T) &\rightharpoonup \dot{\boldsymbol{u}}(T) && \text{weakly in } \boldsymbol{L}_2(\Omega)\\
p_n(u_{n\nu} - g) &\rightharpoonup \theta && \text{weakly in } (H^{1/4,1/2}(S_3))^* \text{ and weakly}^* \text{ in } L_\infty^*(S_3)
\end{aligned}
\tag{23}
$$

with $\theta \in H^{1/4,1/2}(S_3)^* \cap L_\infty^*(S_3)$.

¿From the convergence $u_{n\nu} \to u_\nu$ a.e on S_3 and the fact that $p_n(u_{n\nu} - g)$ is bounded in $L_1(S_3)$ we prove (with the help of the Yegorov theorem as in [3]) that $u_\nu - g < \beta$ a.e. in S_3.

In fact, denote for $\delta > 0$ by M_δ set of points $z = [t, x] \in S_3$ such that $u_\nu(z) - g(z) \geq \beta - \delta$. As the sequence of measures $\{\operatorname{mes} M_\delta\}$ is non decreasing it is enough to prove that $\operatorname{mes} M_\delta \to 0$ for $\delta \to 0+$. Assume by contradiction that there is a positive η such that $\operatorname{mes} M_\delta \geq \eta$ for all $\delta > 0$. By Egorov theorem there is a set $S_{3\eta} \subset S_3$ such that $\operatorname{mes} S_3 \setminus S_{3,\eta} < \eta/2$ and at the same time $u_{n\nu} \to u_\nu$ uniformly on $S_{3,\eta}$. Thus we choose n_1 so that $|u_{n,\nu} - u_\nu| < \delta$ on $S_{3,\eta}$ for all $n > n_1$ and n_2 so that $p_n(y) = p(y)$ for all $y \leq \beta - 2\delta$ and $n \geq n_2$. Thus for $n \geq \max\{n_1, n_2\}$ we have

$$
\begin{aligned}
C &\geq \|p_n(u_\nu - g)\|_{L_1(S_3)} \geq \|p_n(u_\nu - g)\|_{L_1(M_\delta \cap S_{3\eta})}\\
&\geq \operatorname{mes}(M_\delta \cap S_{3\eta)})p_n(\beta - 2\delta) = \frac{\eta}{2} p(\beta - 2\delta).
\end{aligned}
\tag{24}
$$

As $\lim_{\delta \to 0} p(\beta - 2\delta) = \infty$ it gives the contradiction and thus $p_n(u_{n\nu} - g) \to p(u_\nu - g)$ a.e. in S_3 for $n \to \infty$.

For any test function $v \in \boldsymbol{H}^{1/2}(S_3)$ with non negative v_ν we get

$$0 \leq p_n(u_{n\nu} - g)v_\nu \leq p(u_\nu - g)v_\nu$$

a.e. on S_3 and from Fatou lemma

$$\begin{aligned}\langle\theta,v_\nu\rangle_{S_3}&=\liminf_{n\to\infty}\langle p_n(u_{n\nu}-g),v_\nu\rangle_{S_3}\\&\ge\langle\liminf_{n\to\infty}p_n(u_{n\nu}-g),v_\nu\rangle_{S_3}=\langle p(u_\nu-g)v_\nu\rangle_{S_3}.\end{aligned}\tag{25}$$

i.e $\theta\ge p(u_\nu-g)$ in the dual sense.

Using the integration by parts in the first term in (15) and passing to the limit $n\to+\infty$ there, we obtain

$$\begin{aligned}&\langle\dot{\boldsymbol u}(T),\boldsymbol v(T)\rangle_\Omega-\langle\boldsymbol u_1,\boldsymbol v(0)\rangle_\Omega-\langle\dot{\boldsymbol u},\dot{\boldsymbol v}\rangle_Q\\&\quad+\langle\boldsymbol\sigma(\boldsymbol u),\boldsymbol\varepsilon(\boldsymbol v)\rangle_Q+\langle\theta,v_\nu\rangle_{S_3}=\langle\boldsymbol\ell,\boldsymbol v\rangle_Q\end{aligned}\tag{26}$$

Observe that it is valid also for such $\boldsymbol v\in\boldsymbol H^{1/2,1}(Q)$ which are continuous in the endpoints as e.g. $\dot{\boldsymbol u}$.

Furthermore, by putting $\boldsymbol v=\boldsymbol u_n$ in (15) and using convergences in (23), the lower semicontinuity of the term $\langle\boldsymbol\sigma(\boldsymbol u_n),\boldsymbol\varepsilon(\boldsymbol u_n)\rangle_Q$ and the fact that the time level term, the linear part and the $\langle\dot{\boldsymbol u}_n,\dot{\boldsymbol u}_n\rangle_Q$ tend to the corresponding limits due to (23) we get from (26)

$$\langle\theta,u_\nu\rangle_{S_3}\ge\limsup_{n\to+\infty}\langle p_n(u_{n\nu}-g),u_{n\nu}\rangle_{S_3}.\tag{27}$$

Then the monotonicity of functions p_n yields that for any $\boldsymbol w\in H^{1/2,1}(\Omega)$ such that its trace is in $\operatorname{dom}\tilde p$ it holds

$$\langle\theta-p(w_\nu-g),u_\nu-w_\nu\rangle_{S_3}\ge 0.\tag{28}$$

Denote

$$\operatorname{dom}(\tilde P)=\big\{w\in H^{1/4,1/2}(S_3);P(w)\in L^1(S_3)\big\}$$

and for $w\in\operatorname{dom}\tilde P$ by

$$\tilde P(w)=\int_{S_3}P(w(z))ds_xdt$$

and extend $\tilde P$ outside $\operatorname{dom}(\tilde P)$ by $+\infty$. Then $\tilde P$ is convex with non empty domain $\operatorname{dom}(\tilde P)$, lower semicontinuous, and for all $v\in\operatorname{dom}(\tilde p)$ we have that operator $v_\nu\mapsto\tilde p(v_\nu-g)$ is the derivative, hence the subdifferential of the functional $\tilde P$. Moreover the norm of the space $H^{1/4,1/2}(\Gamma_3)$ is of the Hilbert type, hence it is Fréchet differentiable everywhere outside 0. The space is also uniformly convex, hence its dual is uniformly smooth. By Theorem 5.1.7 of the monograph [1] $\tilde p$ is maximal monotone, $u_\nu\in\operatorname{dom}\tilde p$ and the identity $\theta=p(u_\nu-g)$ holds.[1] This identity finishes the proof.

5 Relation to the Signorini contact

Next we turn to solutions of Signorini problem. Let us have a sequence $\beta_k\searrow 0$. Assume the system of functions p_k such that they satisfy all the requirements of (4–5) with $\beta=\beta_k$, respectively and the additional requirement

$$p_k\equiv 0\text{ on }[-\infty,0],\ \ k\in\mathbb N.\tag{29}$$

[1]The authors are deeply indebted to Jiří Outrata for indicating them this result.

Let $\boldsymbol{u}_k$ be respective solutions of the problem (12) with $p = p_k$. It is not difficult to prove that the estimates (18–22) hold for the sequence $\{\boldsymbol{u}_k\}$ independently of $k \in \mathbb{N}$. Hence there is $\boldsymbol{u}$ such that the convergences in (23) to it are satisfied for an appropriate subsequence of the sequence $\{\boldsymbol{u}_k\}$ with $\theta \geq 0$ in the dual sense. Moreover, since $u_{k\nu} < g + \beta_k$ a.e. on S_3, the limit $\boldsymbol{u}$ belongs to the cone $\mathscr{K}_g$ defined as

$$\mathscr{K}_g := \{\boldsymbol{v} \in \boldsymbol{H}_0^{1/2,1}(Q); v_\nu \leq g \text{ on } S_3\}. \tag{30}$$

Observe that for $\boldsymbol{v} \in \mathscr{K}_g$ the identity $p_k(v_\nu - g) = 0$ holds for any $k \in \mathbb{N}$. Then for $\Theta \equiv \lim_{k\to+\infty} \langle p_k(u_{k\nu}), u_{k\nu}\rangle_{S_3}$ the monotonicity of p_k yields that $\Theta \geq \langle \theta, u_\nu\rangle_{S_3}$. If we pass to the limit in (12) with the test function $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}_k$ in the problem for $\beta = \beta_k$ and put $\boldsymbol{v} = \boldsymbol{u}$ in the limit variational equation, we get the oposite inequality, hence $\Theta = \langle \theta, u_\nu\rangle_{S_3}$. This yields that for any $\boldsymbol{v} \in \mathscr{K}_g$ the variational inequality

$$\begin{aligned}\langle \dot{\boldsymbol{u}}(T), \boldsymbol{v}(T) - \boldsymbol{u}(T)\rangle_\Omega - \langle \boldsymbol{u}_1, \boldsymbol{v}(0) - \boldsymbol{u}_0\rangle_\Omega - \langle \dot{\boldsymbol{u}}, \dot{\boldsymbol{v}} - \dot{\boldsymbol{u}}\rangle_Q \\ + \langle \boldsymbol{\sigma}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u})\rangle_Q \geq \langle \boldsymbol{\ell}, \boldsymbol{v} - \boldsymbol{u}\rangle_Q.\end{aligned} \tag{31}$$

holds, hence the limit $\boldsymbol{u}$ is a solution of the appropriate unilateral contact problem without interpenetration (the Signorini contact problem). We had proved

Theorem 3 *Let us have a sequence of problems* (12) *satisfying all assumption of Theorem 2 such that the corresponding limits of interpenetration* $\beta_k \searrow 0$ *and the requirement* (29) *holds. Then there is a subsequence of the respective solutions tending to a solution of the appropriate Signorini contact problem.*

Observe that no uniqueness of solutions of the problem (31) can be expected due to the well-known lack of condition ensuring the energy conservation. The uniqueness of solutions to our problem (12) remains open.

6 Conclusion

The existence of a solution of a rational contact model with limited interpenetration in viscoelastodynamics has been proved here. We hope that this result will draw attention both of numerical analysts to study it and of engineers to apply it.

7 Appendix - decomposition of the space $\boldsymbol{H}^1(\Omega)^*$

Let $\mathscr{A}$ be a fixed linear Lamé operator (e.g. our operator $\mathscr{A}^{(0)}$). Define the operator $\mathscr{B} : \boldsymbol{H}^{1/2}(\Gamma) \to \boldsymbol{H}^1(\Omega)$ as $\mathscr{B} : \boldsymbol{w} \mapsto \boldsymbol{v}$, where the equation

$$\begin{aligned}\mathscr{A}\boldsymbol{v} = 0 \text{ on } \Omega, \\ \boldsymbol{v} = \boldsymbol{w} \text{ on } \Gamma\end{aligned} \tag{32}$$

is satisfied. Let $\mathscr{T} : \boldsymbol{H}^1(\Omega) \to \boldsymbol{H}^{1/2}(\Gamma)$ be the trace operator. Let $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ is arbitrary. We decompose it as $\mathscr{B} \circ \mathscr{T}\boldsymbol{v} + \tilde{\boldsymbol{v}}$, where $\tilde{\boldsymbol{v}} \equiv (\boldsymbol{v} - \mathscr{B} \circ \mathscr{T}\boldsymbol{v}) \in \mathring{\boldsymbol{H}}^1(\Omega)$. Let $\boldsymbol{\varphi} \in \boldsymbol{H}^1(\Omega)^*$, let $\boldsymbol{\varphi}_0 \in \boldsymbol{H}^{-1}(\Omega)$ is such that $\boldsymbol{\varphi}_0 = \boldsymbol{\varphi}$ on $\mathring{\boldsymbol{H}}^1(\Omega)$. Then we have the decomposition $\boldsymbol{\varphi} = [\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_{|\operatorname{Im}\mathscr{B}}]$, $\boldsymbol{\varphi}_{|\operatorname{Im}\mathscr{B}} \circ \mathscr{B} \in \boldsymbol{H}^{-1/2}(\Gamma)$. With an abuse of notation we denote $\boldsymbol{\varphi}_{|\Gamma} = 0$ if for any $\boldsymbol{v} \in \boldsymbol{H}^{1/2}(\Gamma)$ $\langle \boldsymbol{\varphi}_{|\operatorname{Im}\mathscr{B}} \circ \mathscr{B}, \boldsymbol{v}\rangle_\Gamma = 0$. The definition does not depend on the choice of $\mathscr{A}$.

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