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On joint numerical radius II

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ON JOINT NUMERICAL RADIUS II

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Abstract. Let $T_1, \dots, T_n$ be operators on a Hilbert space H. We continue the study of the question whether it is possible to find a unit vector $x \in H$ such that $|\langle T_j x, x\rangle|$ is large for all j. Thus we are looking for a generalization of the well-known inequality $w(T) \ge \frac{\|T\|}{2}$ for the numerical radius $w(T)$ of a single operator T.

1. Introduction

Let H be a complex Hilbert space. Denote by $B(H)$ the set of all bounded linear operators on H. The numerical range of an operator $T \in B(H)$ is defined by

$$W(T) = \bigl\{\langle Tx, x\rangle : x \in H, \|x\| = 1\bigr\}$$

and the numerical radius by

$$w(T) = \sup\bigl\{|\langle Tx, x\rangle| : x \in H, \|x\| = 1\bigr\} = \sup\bigl\{|\lambda| : \lambda \in W(T)\bigr\}.$$

It is well known that the numerical range $W(T)$ is a convex subset of the complex plane and $w(T) \ge \frac{\|T\|}{2}$ for all T. In other words, given a non-zero operator $T \in B(H)$ and a number $\varepsilon > 0$, there exists a unit vector $x \in H$ such that $|\langle Tx, x\rangle| > (1-\varepsilon)\frac{\|T\|}{2}$.

If $\dim H < \infty$ then the numerical range $W(T)$ is compact, and so there exists a unit vector $x \in H$ such that $|\langle Tx, x\rangle| \ge \frac{\|T\|}{2}$.

In [M2] the following question was studied: Given $T_1, \dots, T_n \in B(H)$, does there exist a unit vector $x \in H$ such that $|\langle T_j x, x\rangle|$ is "large" for all $j = 1, \dots, n$?

It is easy to see that it is possible to assume that $\dim H < \infty$. Moreover, considering the real and imaginary parts of each operator T_j it is possible to reduce the question (at least up to a constant) to the case of n-tuples of selfadjoint operators.

In [M2] there were obtained sharp estimates in cases $n = 2, 3$. If T_1, T_2 is a pair of selfadjoint operators on a finite-dimensional Hilbert space, then there exists a unit vector x such that $|\langle T_j x, x\rangle| \ge \frac{1}{3}\|T_j\| \quad (j = 1, 2)$. For triples of selfadjoint operators the corresponding best estimate is $|\langle T_j x, x\rangle| \ge \frac{1}{5}\|T_j\|$.

For $n \ge 4$ the question is essentially more difficult, among other reasons because the joint numerical range $W(T_1, \dots, T_n)$ is no longer a convex set. In [M2] there were obtained only some estimates: if $T_1, \dots, T_n \in B(H)$ then there exists a unit vector $x \in H$ such that $|\langle T_j x, x\rangle| \ge \frac{\text{const}}{n^3} \cdot \|T_j\|$ for all j.

If $T_1, \dots, T_n$ are commuting selfadjoint operators, then there exists a unit vector $x \in H$ with $|\langle T_j x, x\rangle| \ge \frac{const}{n^2} \cdot \|T_j\|$ for all j.

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The purpose of this note is to improve the above estimates. We improve the estimate in the general case to $|\langle T_j x, x\rangle| \geq \frac{\text{const}}{n^2} \cdot \|T_j\|$ and in the case of commuting selfadjoint operators to $|\langle T_j x, x\rangle| \geq \frac{\text{const}}{n\sqrt{n}} \cdot \|T_j\|$. Note that in [M2] it is conjectured that the best lower estimates are $\frac{1}{2n-1}\|T_j\|$. So it is still a gap between these two lower estimates.

Similar estimates can be obtained also for other types of numerical ranges — for the essential numerical range and the algebraic numerical range.

At the end of the paper we also give a short proof of the inequality between the norm and the joint numerical radius of an n-tuple of operators. This estimate was given in [P], where the joint numerical radius is called the Euclidean operator radius.

2. General case

Let H be a Hilbert space and $T_1, \dots, T_n \in B(H)$. Recall that the joint numerical range $W(T_1, \dots, T_n)$ is defined by

$$W(T_1, \dots, T_n) = \Big\{ \big(\langle T_1 x, x\rangle, \dots, \langle T_n x, x\rangle\big) : x \in H, \|x\| = 1 \Big\}.$$

We study first the situation in 2-dimensional spaces. The estimates will be then used in general case.

Let S be the unit sphere in $\mathbb{C}^2$, $S = \big\{(\lambda, \mu) \in \mathbb{C}^2 : |\lambda|^2 + |\mu|^2 = 1\big\}$. Let m be the Lebesgue measure on S. Recall that $m(S) = 2\pi^2$.

Lemma 1. Let $a \in [-1, 1]$ and $\varepsilon > 0$. Let $L_{a,\varepsilon} = \big\{(\lambda, \mu) \in S : -\varepsilon < |\lambda|^2 + a|\mu|^2 < \varepsilon\big\}$. Then $m(L_{a,\varepsilon}) < 4\pi^2\varepsilon$.

Proof. Note that for $a > 0$ we have $L_{a,\varepsilon} \subset L_{0,\varepsilon}$, so it is enough to consider the case when $a \leq 0$.

Write $\lambda = r\cos\alpha + ir\sin\alpha$, $\mu = \sqrt{1-r^2}\cos\beta + i\sqrt{1-r^2}\sin\beta$ with $0 \leq \alpha < 2\pi$, $0 \leq \beta < 2\pi$, $0 \leq r \leq 1$. An elementary calculation gives $dm = r dr\, d\alpha\, d\beta$.

We distinguish two cases:

A. $-\varepsilon \leq a \leq 0$.

Then $(\lambda, \mu) \in L_{a,\varepsilon}$ if and only if $0 \leq r^2 < \frac{\varepsilon - a}{1-a}$. So

$$m(L_{a,\varepsilon}) = \int_0^{\sqrt{\frac{\varepsilon-a}{1-a}}} r dr \int_0^{2\pi} d\alpha \int_0^{2\pi} d\beta = 4\pi^2 \Big[\frac{r^2}{2}\Big]_0^{\sqrt{\frac{\varepsilon-a}{1-a}}} = 2\pi^2 \frac{\varepsilon - a}{1-a} < 4\pi^2\varepsilon.$$

B. $-1 \leq a < -\varepsilon$.

Then $(\lambda, \mu) \in L_{a,\varepsilon}$ if and only if $\frac{-\varepsilon-a}{1-a} < r^2 < \frac{\varepsilon-a}{1-a}$. So

$$m(L_{a,\varepsilon}) = \int_{\sqrt{\frac{-\varepsilon-a}{1-a}}}^{\sqrt{\frac{\varepsilon-a}{1-a}}} r dr \int_0^{2\pi} d\alpha \int_0^{2\pi} d\beta = 2\pi^2 \Big(\frac{\varepsilon - a}{1-a} - \frac{-\varepsilon - a}{1-a}\Big) = 2\pi^2 \frac{2\varepsilon}{1-a} < 4\pi^2\varepsilon.$$

So $m(L_{a,\varepsilon}) < 4\pi^2\varepsilon$ for all $a \in [-1, 1]$ and $\varepsilon > 0$. □

Proposition 2. Let $\dim H = 2$, let $T_1, T_2, \dots \in B(H)$ be a sequence of selfadjoint operators satisfying $\|T_j\| = 1 \quad (j = 1, 2, \dots)$. Let $\alpha_j \geq 0 \quad (j \in \mathbb{N})$ satisfy $\sum_{j=1}^\infty \alpha_j = 1$. Then there exists a unit vector $x \in H$ such that

$$|\langle T_j x, x\rangle| \geq \frac{\alpha_j}{2} \qquad (j = 1, 2, \dots).$$

Proof. Let S_H be the unit sphere in H. We may assume that $\alpha_j > 0$ for all j. For each j there exist an orthonormal basis $\{e_j, f_j\}$ in H and $a_j \in [-1,1]$ such that $T_j e_j = \pm e_j$ and $T_j f_j = a_j f_j$. If $T_j e_j = e_j$ and $x = \alpha e_j + \beta f_j \in S_H$ we have $\bigl|\langle T_j x, x\rangle\bigr| = \bigl||\alpha|^2 + a_j|\beta|^2\bigr|$. So $|\langle T_j x, x\rangle| < \alpha_j/2$ if and only if $(\lambda,\mu) \in L_{a_j,\alpha_j/2}$ (we use the notation from the previous lemma). Similarly, if $T_j e_j = -e_j$ then $\bigl|\langle T_j x, x\rangle\bigr| = \bigl|-|\alpha|^2 + a_j|\beta|^2\bigr|$ and $|\langle T_j x, x\rangle| < \alpha_j/2$ if and only if $(\lambda,\mu) \in L_{-a_j,\alpha_j/2}$. By Lemma 1, we have in both cases $m\Bigl(\bigl\{x \in S_H : |\langle T_j x, x\rangle| < \alpha_j/2\bigr\}\Bigr) < 2\pi^2\alpha_j$. Thus

$$\sum_{j=1}^{\infty} m\Bigl(\bigl\{x \in S_H : |\langle T_j x, x\rangle| < \alpha_j/2\bigr\}\Bigr) < \sum_{j=1}^{\infty} 2\pi^2\alpha_j = 2\pi^2 = m(S_H).$$

So there exists a unit vector $x \in S_H \setminus \bigcup_{j=1}^{\infty}\bigl\{x \in S_H : |\langle T_j x, x\rangle| < \alpha_j/2\bigr\}$. Clearly this x satisfies

$$|\langle T_j x, x\rangle| \geq \frac{\alpha_j}{2} \qquad (j = 1, 2, \dots).$$

□

Theorem 3. Let H be a Hilbert space, let $T_1, T_2, \dots \in B(H)$. Let $\alpha_j \geq 0$ satisfy $\sum_{j=1}^{\infty} \alpha_j^{1/2} < 1$. Then there exist a unit vector $x \in H$ such that

$$|\langle T_j x, x\rangle| \geq \frac{\alpha_j}{4}\|T_j\| \qquad (j = 1, 2, \dots).$$

If the operators $T_1, T_2, \dots$ are selfadjoint, then there exist a unit vector $x \in H$ such that

$$|\langle T_j x, x\rangle| \geq \frac{\alpha_j}{2}\|T_j\| \qquad (j = 1, 2, \dots).$$

Proof. We prove first the second statement. Let $T_j^* = T_j$ for all j and $\sum_{j=1}^{\infty} \alpha_j^{1/2} < 1$. By [M1], Theorem 39.8, there exist vectors $u, v \in H$ such that

$$|\langle T_j u, v\rangle| \geq \alpha_j^{1/2}\|T_j\| \qquad (j = 1, 2, \dots).$$

Moreover, it is clear from the proof (cf. [M1], Theorem 37.17) that the vectors u, v can be taken of norm 1.

If the vectors u, v are linearly dependent then

$$|\langle T_j u, u\rangle| = |\langle T_j u, v\rangle| \geq \alpha_j^{1/2}\|T_j\| \geq \alpha_j\|T_j\|.$$

Suppose that u, v are linearly independent. Let H_0 be the 2-dimensional subspace generated by u, v and let P be the orthogonal projection onto H_0. Then the operators $PT_j|H_0 \in B(H_0)$ are selfadjoint and $\|PT_j|H_0\| \geq |\langle T_j u, v\rangle| \geq \alpha_j^{1/2}\|T_j\|$. By Proposition 2, there exists a unit vector $x \in H_0 \subset H$ such that

$$|\langle T_j x, x\rangle| = |\langle PT_j x, x\rangle| \geq \frac{\alpha_j^{1/2}}{2}\|PT_j|H_0\| \geq \frac{\alpha_j}{2}\|T_j\|$$

for all j.

Let $T_1, T_2, \dots \in B(H)$ be now general operators. We may assume that $T_j \neq 0$ for all j. Choose numbers $\alpha'_j > \alpha_j$ such that $\sum_{j=1}^\infty {\alpha'_j}^{1/2} < 1$. Since the numerical radius of T_j satisfies $w(T_j) \geq \frac{\|T_j\|}{2}$, there exists $\lambda_j \in W(T_j)$ with $|\lambda_j| \geq \frac{\alpha_j \|T_j\|}{2\alpha'_j}$. Consider the selfadjoint operators

$$S_j = \operatorname{Re} \frac{T_j}{\lambda_j} = \frac{1}{2}\Bigl(\frac{T_j}{\lambda_j} + \frac{T_j^*}{\bar{\lambda}_j}\Bigr).$$

Then $1 \in W(S_j)$ and so $\|S_j\| \geq 1$. By the previous statement there exists a unit vector $x \in H$ such that

$$|\langle S_j x, x\rangle| \geq \frac{\alpha'_j}{2}\|S_j\|$$

for all j. Then

$$|\langle T_j x, x\rangle| \geq |\lambda_j| \cdot |\operatorname{Re}\langle \lambda_j^{-1} T_j x, x\rangle| = |\lambda_j| \cdot |\langle S_j, x, x\rangle| > \frac{\alpha_j}{2\alpha'_j}\|T_j\| \cdot \frac{\alpha'_j}{2}\|S_j\| \geq \frac{\alpha_j}{4}\|T_j\|$$

for all $j \in \mathbb{N}$. □

Corollary 4. Let $\dim H < \infty$, $n \in \mathbb{N}$, let $T_1, \dots, T_n \in B(H)$. Then there exists a unit vector $x \in H$ such that

$$|\langle T_j x, x\rangle| \geq \frac{\|T_j\|}{4n^2} \qquad (j = 1, \dots, n).$$

If the operators $T_1, \dots, T_n \in B(H)$ are selfadjoint, then there exists a unit vector $x \in H$ such that

$$|\langle T_j x, x\rangle| \geq \frac{\|T_j\|}{2n^2} \qquad (j = 1, \dots, n).$$

Proof. It follows from the previous Theorem and the compactness of the unit sphere in H. □

3. Convex case

The following lemma is an improvement of Lemma 13 of [M2].

Lemma 5. Let $n \in \mathbb{N}$ and let $K \subset [-1,1]^n$ be a convex set. Let $u_j = (u_{j1}, \dots, u_{jn}) \in K$ satisfy $u_{jj} = 1 \quad (j = 1, \dots, n)$. Then there exists $v = (v_1, \dots, v_n) \in K$ such that

$$|v_j| \geq \frac{1}{2n\sqrt{n}} \quad (j = 1, \dots, n).$$

Proof. Let $M = \Bigl\{(m_1, \dots, m_n) \in [0,1]^n : \sum_{j=1}^n m_j \leq 1\Bigr\}$. Clearly M is a compact convex set.

Define the width of M by

$$\operatorname{width}(M) = \inf\Bigl\{\sup_{v \in M}\langle v, f\rangle - \inf_{v \in M}\langle v, f\rangle : f \in \mathbb{R}^n, \|f\| = 1\Bigr\}.$$

Then $\operatorname{width}(M) = \frac{1}{\sqrt{n}}$. Indeed, for $f = (\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}) \in \mathbb{R}^n$ we have $\|f\| = 1$ and $\sup_{v \in M}\langle v, f\rangle = \frac{1}{\sqrt{n}}$, $\inf_{v \in M}\langle v, f\rangle = 0$. So $\operatorname{width}(M) \leq \frac{1}{\sqrt{n}}$.

On the other hand, let $f = (f_1, \dots, f_n) \in \mathbb{R}^n, \|f\| = \Bigl(\sum_{j=1}^n f_j^2\Bigr)^{1/2} = 1$. Let $J_1 = \bigl\{j \in \{1, \dots, n\} : f_j \geq 0\bigr\}$ and $J_2 = \{1, \dots, n\} \setminus J_1$.

Then

$$\sup_{v\in M}\langle v,f\rangle=\sup_{v\in M}\sum_{j\in J_1} v_jf_j=\max_{j\in J_1} f_j$$

and

$$\inf_{v\in M}\langle v,f\rangle=\inf_{v\in M}\sum_{j\in J_2} v_jf_j=\min_{j\in J_2} f_j,$$

and so

$$\sup_{v\in M}\langle v,f\rangle-\inf_{v\in M}\langle v,f\rangle=\max_{j\in J_1} f_j+\max_{j\in J_2}(-f_j)\geq \max_j |f_j|\geq \frac{1}{\sqrt{n}}.$$

Hence width $(M)=\frac{1}{\sqrt{n}}$.

For $j=1,\dots,n$ let $L_j=\Big\{(t_1,\dots,t_n)\in\mathbb{R}^n:\Big|\sum_{k=1}^n t_ku_{kj}\Big|<\frac{1}{2n\sqrt{n}}\Big\}$. Since $\Big(\sum_{k=1}^n |u_{kj}|^2\Big)^{1/2}\geq |u_{jj}|=1$, we have width $(L_j)\leq\frac{1}{n\sqrt{n}}$. For each $\varepsilon>0$ we have

$$\sum_{j=1}^n \text{width}\,\big((1-\varepsilon)L_j\big)<\text{width}\,(M),$$

so by the plank theorem [B] there exists $t^{(\varepsilon)}=(t_1^{(\varepsilon)},\dots,t_n^{(\varepsilon)})\in M\setminus\bigcup_{j=1}^n(1-\varepsilon)L_j$. By a compactness argument, there exists $t=(t_1,\dots,t_n)\in M\setminus\bigcup_{j=1}^n L_j$, i.e.,

$$\sum_{k=1}^n |t_ku_{kj}|\geq\frac{1}{2n\sqrt{n}}$$

for all $j=1,\dots,n$.

Let $s=\frac{t}{\sum_{j=1}^n t_j}$. Then $\sum_{k=1}^n s_k=1$ and for each $j=1,\dots,n$ we have

$$\Big|\sum_{k=1}^n s_ku_{kj}\Big|=\frac{\big|\sum_{k=1}^n t_ku_{kj}\big|}{\sum_{k=1}^n t_k}\geq\frac{1}{2n\sqrt{n}}.$$

So $v=\sum_{k=1}^n s_ku_k\in K$ and

$$|v_j|\geq\frac{1}{2n\sqrt{n}}\qquad(j=1,\dots,n).$$

□

Corollary 6. Let $\dim H<\infty$ and let $T_1,\dots,T_n\in B(H)$ be commuting selfadjoint operators. Then there exists a unit vector $x\in H$ such that

$$|\langle T_jx,x\rangle|\geq\frac{\|T_j\|}{2n\sqrt{n}}\qquad(j=1,\dots,n).$$

Proof. Without loss of generality we may assume that $\|T_j\|=1$ and $1\in\sigma(T_j)$ for all j. The joint numerical range $W(T_1,\dots,T_n)=\text{conv}\,\sigma(T_1,\dots,T_n)$ is a closed convex subset of $[-1,1]^n$. For each $j=1,\dots,n$ there exists a unit vector $x_j\in H$ with $\langle T_jx_j,x_j\rangle=1$, so there exists $\lambda_j=(\lambda_{j1},\dots,\lambda_{jn})\in W(T_1,\dots,T_n)$ with $|\lambda_{jj}|=1$.

By Lemma 5, there exists $v\in W(T_1,\dots,T_n)$ with $|v_j|\geq\frac{\|T_j\|}{2n\sqrt{n}}\qquad(j=1,\dots,n)$. □

Lemma 5 can be also applied for other types of convex numerical ranges.

Let H be an infinite-dimensional Hilbert space and let $T_1,\dots,T_n\in B(H)$. Recall that the joint essential numerical range $W_e(T_1,\dots,T_n)$ is the set of all $\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n$ such that there exists an orthonormal sequence $(x_k)\subset H$ with

$$\lambda_j=\lim_{k\to\infty}\langle T_jx_k,x_k\rangle.$$

The joint essential numerical range is always a closed convex set, see [LP].

For a single selfadjoint operator $S\in B(H)$ we have $\sup\{|\mu|:\mu\in W_e(S)\}=\|S\|_e$, the essential norm of S. So an easy application of Lemma 5 gives

Theorem 7. Let H be an infinite-dimensional Hilbert space, let $T_1,\dots,T_n\in B(H)$. Then there exists an orthonormal sequence $(x_k)\subset H$ such that $a_j:=\lim_{k\to\infty}\langle T_jx_k,x_k\rangle$ exists and

$$|a_j|\ge\frac{\|T_j\|_e}{4n\sqrt{n}}$$

for all $j=1,\dots,n$.

If the operators T_j are selfadjoint then there exists an orthonormal sequence $(x_k)\subset H$ with

$$|a_j|\ge\frac{\|T_j\|_e}{2n\sqrt{n}}$$

for all $j=1,\dots,n$.

Proof. We prove first the second statement. Let $T_j^*=T_j$ for all j. Without loss of generality we may assume that $\|T_j\|_e=1$ for all j and $1\in W_e(T_j)$. Since the set $W_e(T_1,\dots,T_n)$ is convex, by Lemma 5 there exists an element $\lambda=(\lambda_1,\dots,\lambda_n)\in W_e(T_1,\dots,T_n)$ satisfying $|\lambda_j|\ge\frac{1}{2n\sqrt{n}}$ for all $j=1,\dots,n$.

Let now $T_1,\dots,T_n\in B(H)$ be arbitrary operators; we may assume that $\|T_j\|_e\ne0$ for all j. For each j there exists $\lambda_j\in W_e(T_j)$ with $|\lambda_j|\ge\frac{\|T_j\|_e}{2}$. Let $S_j=\operatorname{Re}\frac{T_j}{\lambda_j}=\frac12\Bigl(\frac{T_j}{\lambda_j}+\frac{T_j^*}{\bar\lambda_j}\Bigr)$. Then $S_j^*=S_j$ and $1\in W_e(S_j)$. By the previous statement, there exists an orthonormal sequence $(x_k)\subset H$ with $\lim_{k\to\infty}|\langle S_jx_k,x_k\rangle|\ge\frac{1}{2n\sqrt{n}}\|S_j\|_e\ge\frac{1}{2n\sqrt{n}}$ for all j. Hence

$$\liminf_{k\to\infty}|\langle T_jx_k,x_k\rangle|\ge|\lambda_j|\cdot\liminf_{k\to\infty}|\operatorname{Re}\langle\lambda_j^{-1}T_jx_k,x_k\rangle|=|\lambda_j|\cdot\lim_{k\to\infty}|\langle S_jx_k,x_k\rangle|\ge\frac{\|T_j\|_e}{4n\sqrt{n}}.$$

Taking a subsequence of (x_k) if necessary we can assume that all the sequences in the above formula converge.

□

Another situation where the results can be applied is the algebraic numerical range.

Let $\mathcal{A}$ be a unital Banach algebra, let $a_1,\dots,a_n\in\mathcal{A}$. The algebraic numerical range is defined by

$$V(a_1,\dots,a_n,\mathcal{A})=\bigl\{(f(a_1),\dots,f(a_n)):f\in\mathcal{A}^*,\|f\|=1=f(1_{\mathcal{A}})\bigr\},$$

where $1_{\mathcal{A}}$ denotes the unit in $\mathcal{A}$.

It is well known that $V(a_1,\dots,a_n,\mathcal{A})$ is always a closed convex subset of $\mathbb{C}^n$. For a single element $a_1\in\mathcal{A}$ we have

$$\sup\bigl\{|\mu|:\mu\in V(a_1,\mathcal{A})\bigr\}\ge\frac{\|a_1\|}{e}$$

(where $e=2.71...$), see [BD], p. 34.

Corollary 8. Let $\mathcal{A}$ be a unital Banach algebra, let $a_1, \dots, a_n \in \mathcal{A}$. Then there exists $f \in \mathcal{A}^*$, $\|f\| = 1 = f(1_\mathcal{A})$ such that

$$|f(a_j)| \geq \frac{\|a_j\|}{2en\sqrt{n}} \qquad (j = 1, \dots, n).$$

Proof. For $j = 1, \dots, n$ there exists $f_j \in \mathcal{A}^*$ with $\|f_j\| = 1 = f_j(1_\mathcal{A})$, $|f_j(a_j)| \geq \frac{\|a_j\|}{e}$. Let α_j be the complex unit such that $f_j(\alpha_j a_j) \geq \frac{\|a_j\|}{e}$. The numerical range $V(\alpha_1 a_1, \dots \alpha_n a_n, \mathcal{A})$ is a convex set, and so is the set $K := \{(\operatorname{Re}\lambda_1, \dots, \operatorname{Re}\lambda_n) : (\lambda_1, \dots, \lambda_n) \in V(\alpha_1 a_1, \dots, \alpha_n a_n, \mathcal{A})\}$. By Lemma 5, there exists $\mu \in K \subset \mathbb{R}^n$ with $|\mu_j| \geq \frac{\|a_j\|}{2en\sqrt{n}}$ for all j. So there exists $\lambda \in V(a_1, \dots, a_n, \mathcal{A})$ with $|\lambda_j| \geq \frac{\|a_j\|}{2en\sqrt{n}}$ $(j = 1, \dots, n)$. □

4. Joint numerical radius

Let $T = (T_1, \dots, T_n) \in B(H)^n$. The norm of T is defined as

$$\|T\| = \sup\Big\{ \Big(\sum_{i=1}^n \|T_i x\|^2\Big)^{1/2} : x \in H, \|x\| = 1 \Big\}.$$

The joint numerical radius of T is defined by

$$w(T) = \sup\Big\{ \Big(\sum_{i=1}^n |\lambda_i|^2\Big)^{1/2} : (\lambda_1, \dots, \lambda_n) \in W(T) \Big\}.$$

The latter unitary invariant of T is called the Euclidean operator radius (and denoted $w_e(T)$) by Popescu [P]. We provide a short proof of the following theorem given in [P], Proposition 1.21.

Theorem 9. Let $T = (T_1, \dots, T_n) \in B(H)^n$. Then

$$w(T) \geq \frac{\|T\|}{2\sqrt{n}}.$$

Moreover, the estimate is sharp.

Proof. Without loss of generality we may assume that $\|T\| = 1$. For each $\varepsilon > 0$ there exists a unit vector $x \in H$ such that $\left(\sum_{i=1}^n \|T_i x\|^2\right)^{1/2} > 1 - \varepsilon$. So there exists $j_0 \in \{1, \dots, n\}$ such that $\|T_{j_0} x\|^2 > \frac{(1-\varepsilon)^2}{n}$. It follows that

$$w(T_{j_0}) \geq \frac{1}{2}\|T_{j_0}\| \geq \frac{1}{2}\|T_{j_0} x\| > \frac{1-\varepsilon}{2\sqrt{n}}.$$

Consequently, $w(T) > \frac{1-\varepsilon}{2\sqrt{n}}$. Since $\varepsilon > 0$ was arbitrary, we have $w(T) \geq \frac{1}{2\sqrt{n}}$.

To show that the estimate is sharp, let H be the $(n+1)$-dimensional Hilbert space with an orthonormal basis $e_0, e_1, \dots, e_n$. Define $T = (T_1, \dots, T_n) \in B(H)^n$ by

$$T_j e_0 = \frac{e_j}{\sqrt{n}}, \quad T_j e_i = 0 \qquad (i, j = 1, \dots, n).$$

Then $\|T\| \geq \left(\sum_{j=1}^n \|T_j e_0\|^2\right)^{1/2} = 1$ (in fact it is easy to show that $\|T\| = 1$).

Let $x=\sum_{i=0}^n \alpha_i e_i \in H$ be a unit vector. So $\sum_{i=0}^n |\alpha_i|^2=1$. We have

$$\sum_{j=1}^n |\langle T_j x,x\rangle|^2=\sum_{j=1}^n \Bigl(\frac{|\alpha_0\alpha_j|}{\sqrt{n}}\Bigr)^2=\frac{|\alpha_0|^2(1-|\alpha_0|^2)}{n}.$$

So

$$\sup\Bigl\{\sum_{i=1}^n |\lambda_i|^2:(\lambda_1,\dots,\lambda_n)\in W(T)\Bigr\}=\frac{1}{n}\sup\bigl\{|\alpha_0|^2(1-|\alpha_0|^2):\alpha_0\in\mathbb{C},|\alpha_0|\le 1\bigr\}=\frac{1}{4n}.$$

Hence $w(T)=\frac{1}{2\sqrt{n}}$. □

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