Euler system: Well vs. ill posedness

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Nelder Lecture Series, Imperial College, London
20 April - 21 April 2022

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Perfect fluids - Euler system

prefect $=$ inviscid, non(heat) conducting

ϱ .... mass density
$\mathbf{m} = \varrho \mathbf{u}$ .... momentum
p .... pressure
E .... energy

Leonhard Paul Euler 1707–1783

Euler system of gas dynamics

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p = 0$$

$$\partial_t E + \mathrm{div}_x \left[(E + p) \frac{\mathbf{m}}{\varrho} \right] = 0$$

$$E = \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + \varrho e, \ e \text{ internal energy}$$

(Incomplete) equation of state (gases)

$$p = (\gamma - 1) \varrho e, \ \gamma \text{ - adiabatic coefficient}$$

Isentropic (barotropic) Euler system

Gibbs' relation

ϑ .... (absolute) temperature

s .... entropy

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right)$$

$$s = \bar{s} - \text{constant} \ \Rightarrow \ p = p(\varrho) = a\varrho^{\gamma}, \ a > 0, \ \gamma > 1$$

Isentropic (barotropic) Euler system

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho}\right) + \nabla_x p(\varrho) = 0$$

Boundary conditions

periodic: $x \in \Omega = \mathbb{T}^d, \ d = 2, 3$

impermeable boundary: $\mathbf{m} \cdot \mathbf{n}|_{\partial\Omega} = 0$

First and Second law – energy

Energy

$$\mathcal{E} = \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho),\ \ P'(\varrho)\varrho - P(\varrho) = p(\varrho)$$

$$p' \geq 0 \Rightarrow [\varrho, \mathbf{m}] \mapsto \left\{ \begin{array}{l} \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \text{ if } \varrho > 0 \\ P(\varrho) \text{ if } |\mathbf{m}| = 0 \\ \infty \text{ if } \varrho = 0, |\mathbf{m}| \neq 0 \end{array} \right. \quad \text{is convex l.s.c}$$

Energy balance (conservation)

$$\partial_t \mathcal{E} + \mathrm{div}_x \left(\mathcal{E} \frac{\mathbf{m}}{\varrho} \right) + \mathrm{div}_x \left(p \frac{\mathbf{m}}{\varrho} \right) = 0$$

Energy dissipation

$$\partial_t \mathcal{E} + \mathrm{div}_x (\mathcal{E} \mathbf{u}) + \mathrm{div}_x (p \mathbf{u}) \boxed{\leq} 0$$

$$E = \int_\Omega \mathcal{E} \ \mathrm{d}x,\ \ \partial_t E \leq 0,\ \ E(0+) = \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \ \mathrm{d}x$$

Weak solutions

Field equations

$$\int_0^\infty \int_\Omega [\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi] \ \mathrm{d}x \mathrm{d}t = - \int_\Omega \varrho_0 \varphi(0,\cdot) \ \mathrm{d}x, \ \varphi \in C^1_c([0,\infty) \times \overline{\Omega})$$

$$\int_0^\infty \int_\Omega \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x \mathrm{d}t$$

$$= - \int_\Omega \mathbf{m}_0 \cdot \boldsymbol{\varphi}(0,\cdot) \ \mathrm{d}x, \ \boldsymbol{\varphi} \in C^1_c([0,T) \times \overline{\Omega}; R^N), \ \boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

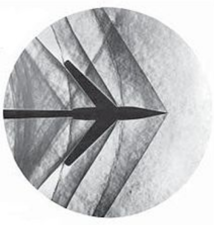
Admissible weak solutions

$$\int_0^\infty \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] \ \mathrm{d}x \ \partial_t \psi \ \mathrm{d}t \geq 0$$

$$\psi \in C^1_c(0,\infty), \ \psi \geq 0$$

Known properties of the Euler system

- **Local existence.** Classical solutions exist locally in time as long as the initial data are regular and the initial density strictly positive
- **Finite time blow–up.** Classical solutions develop singularity (become discontinuous) in a *finite* time for a fairly generic class of initial data
- **Non–uniqueness.** Weak solutions are, in general, not uniquely determined by the data
- **Well–posedness of admissible solutions.** Admissible solutions are, in certain sense, uniquely determined by the data if $d = 1$



Mythology concerning Euler equations in several dimensions

- **Existence.** The long time existence of (possibly weak) solutions is not known
  [addressed in Lecture I]
- **Uniqueness.** The is no (known) selection criterion to identify a unique solution (semiflow)
  [addressed in Lecture II]
- **Turbulence.** Euler or even stochastically driven Euler are relevant in the description of flows in turbulent regime
  [addressed in Lecture III]
- **Computation.** Oscillatory solutions cannot be visualized by numerical simulation (weak convergence)
  [addressed in Lecture IV]

"Typical" convex integration results(ignoring Riemann problem)

Result A: (De Lellis-Székelyhidy, Chiodaroli)
For any smooth initial data there exist infinitely many solutions satisfying the energy inequality on the open interval $(0, T)$ but experiencing initial energy "jump"

Result B: (De Lellis-Székelyhidy, Chiodaroli, Xin et al., EF)
For any smooth initial density ϱ_0 there exists $\mathbf{m}_0$ (not enecessarily regular) such that there are infinitely many weak solutions satisfying the energy inequality on the open interval $(0, T)$ and with the energy continous at $t = 0$

Result C: (Giri and Kwon)
There is a set of smooth initial densities ϱ_0 and Hölder $\mathbf{m}_0$ such that there are infinitely many solutions satisfying the energy equation on the open interval $(0, T)$ (with the energy continous at $t = 0$)

Problem of continuity in time

Weak continuity

$$\mathbf{U} \in C_{\rm weak}([0, T]; L^p(\Omega; R^d)), \ t \mapsto \int_\Omega \mathbf{U} \cdot \boldsymbol{\varphi} \ {\rm d}x \in C[0, T]$$

$$\boldsymbol{\varphi} \in L^{p'}(\Omega; R^d)$$

Strong continuity

$$\tau \in [0, T], \ \|\mathbf{U}(t, \cdot) - \mathbf{U}(\tau, \cdot)\|_{L^p(\Omega; R^d)} \text{ whenever } t \to \tau$$

Strong vs. weak

$$\text{strong } \Rightarrow \text{weak}, \ \text{weak} \not\Rightarrow \text{strong}$$

Class of Riemann integrable functions

Class $\mathcal{R}$

The complement of the points of continuity of **U** is of zero Lebesgue measure in a domain Q

Riemann integrability

A function **U** is Riemann integrable in Q only if **U** belongs to the class $\mathcal{R}$

Oscillations

$$\mathrm{osc}[v](y) = \lim_{s \searrow 0} \left[\sup_{B((y),s)\cap\overline{Q}} v - \inf_{B((y),s)\cap\overline{Q}} v \right],$$

$$A_\eta = \left\{ (y) \in \overline{Q} \;\middle|\; \mathrm{osc}[v](y) \geq \eta \right\} \text{ is closed and of zero content}$$

$$A_\eta \subset \cup_{i \in \mathrm{fin}} Q_i, \; \sum_i |Q_i| < \delta \text{ for any } \delta > 0, \; Q_i \text{ - a box}$$

Main result

Theorem

Let $d = 2, 3$. Let ϱ_0, $\mathbf{m}_0$, and E be given such that

$$\varrho_0 \in \mathcal{R}(\Omega),\ 0 \le \underline{\varrho} \le \varrho_0 \le \overline{\varrho},$$

$$\mathbf{m}_0 \in \mathcal{R}(\Omega; R^d),\ \mathrm{div}_x \mathbf{m}_0 \in \mathcal{R}(\Omega),\ \mathbf{m}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

$$0 \le E \le \overline{E},\ E \in \mathcal{R}(0, T).$$

Then there exists a positive constant E_∞ (large) such that the Euler problem admits infinitely many weak solutions with the energy profile

$$\int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] (t, \cdot)\ \mathrm{d}x = E_\infty + E(t) \text{ for a.a. } t \in (0, T)$$

Strongly discontinuous solutions, I

Let $d = 2, 3$. Let ϱ_0, $\mathbf{m}_0$ be given such that

$$\varrho_0 \in \mathcal{R}(\Omega),\ 0 \leq \underline{\varrho} \leq \varrho_0 \leq \overline{\varrho},$$

$$\mathbf{m}_0 \in \mathcal{R}(\Omega; R^d),\ \mathrm{div}_x \mathbf{m}_0 \in \mathcal{R}(\Omega),\ \mathbf{m}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Let $\{\tau_i\}_{i=1}^{\infty} \subset (0, T)$ be an arbitrary (countable dense) set of times.

Then the Euler problem admits infinitely many weak solutions ϱ, $\mathbf{m}$ with a strictly decreasing total energy profile such that

$$\varrho \in C_{\mathrm{weak}}([0, T]; L^{\gamma}(\Omega)),\ \mathbf{m} \in C_{\mathrm{weak}}([0, T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d))$$

but

$$t \mapsto [\varrho(t, \cdot), \mathbf{m}(t, \cdot)]\ \boxed{\text{is not}}\ \text{strongly continuous at any } \tau_i,\ i = 1, 2, \dots .$$

Strongly discontinuous solutions, II

Let $d = 2, 3$. Let ϱ_0,

$$\varrho_0 \in C^\infty(\overline{\Omega}),\ 0 \leq \underline{\varrho} \leq \varrho_0 \leq \overline{\varrho},$$

be given, together with an F_σ subset G of Ω, $|G| = 0$, and an arbitrary (countable dense) set of times $\{\tau_i\}_{i=1}^\infty \subset (0, T)$

Then there exists

$$\mathbf{m}_0 \in \mathcal{R}(\Omega; R^d),\ \mathrm{div}_x \mathbf{m}_0 \in \mathcal{R}(\Omega),\ \mathbf{m}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0$$

such that the Euler problem admits infinitely many weak solution ϱ, $\mathbf{m}$ with a strictly decreasing total energy profile such that ϱ is not continuous at any point

$$t > 0,\ x \in G,$$

and

$$\varrho \in C_{\rm weak}([0, T]; L^\gamma(\Omega)),\ \mathbf{m} \in C_{\rm weak}([0, T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d))$$

with

$$t \mapsto [\varrho(t, \cdot), \mathbf{m}(t, \cdot)] \text{ not strongly continuous at any } \tau_i,\ i = 1, 2, \dots.$$

Strongly discontinuous solutions, III

Let $d = 2, 3$. Let ϱ_0,

$$\varrho_0 \in C^\infty(\overline{\Omega}),\ 0 \leq \underline{\varrho} \leq \varrho_0 \leq \overline{\varrho},$$

be given, together with an F_σ subset G of Ω, $|G| = 0$, an arbitrary (countable dense) set of times $\{\tau_i\}_{i=1}^\infty \subset (0, T)$, and a number $\delta > 0$.

Then there exists

$$\mathbf{m}_0 \in L^\infty(\Omega; R^d),\ \mathrm{div}_x \mathbf{m}_0 \in \mathcal{R}(\Omega),\ \mathbf{m}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0$$

such that the Euler problem admits infinitely many weak solution ϱ, $\mathbf{m}$ with a strictly decreasing total energy profile *continuous* at $t = 0$ such that ϱ is not continuous at any point

$$t > \delta,\ x \in G,$$

$$\varrho \in C_{\rm weak}([0, T]; L^\gamma(\Omega)),\ \mathbf{m} \in C_{\rm weak}([0, T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d))$$

with

$t \mapsto [\varrho(t, \cdot), \mathbf{m}(t, \cdot)]$ not strongly continuous at any τ_i, $i = 1, 2, \dots,\ \tau_i > \delta$

Convex integration ansatz

Helmholtz decomposition of the initial data

$$\mathbf{m}_0 = \mathbf{v}_0 + \nabla_x \Phi_0,\ \mathrm{div}_x \mathbf{v}_0 = 0,\ \Delta_x \Phi_0 = \mathrm{div}_x \mathbf{m}_0,\ (\nabla_x \Phi_0 - \mathbf{m}_0) \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Convex integration ansatz

$$\varrho(t,x) = \varrho_0 + h(t)\Delta_x \Phi_0,\ h(0) = 0,\ h'(0) = -1$$

$$\mathbf{m}(t,x) = \mathbf{v} - h'(t)\nabla_x \Phi_0,\ \mathrm{div}_x \mathbf{v} = 0,$$

$$\mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0,\ \mathbf{v}(0,\cdot) = \mathbf{v}_0$$

“Overdetermined” Euler system

Given quantities

$$h, \ \Phi_0, \ \varrho$$

Balance of momentum

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} - h'(t)\nabla_x \Phi_0) \otimes (\mathbf{v} - h'(t)\nabla_x \Phi_0)}{\varrho} - \frac{1}{d} \frac{|\mathbf{v} - h'(t)\nabla_x \Phi_0|^2}{\varrho} \mathbb{I} \right)$$
$$= 0$$

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \mathbf{v}(0, \cdot) = \mathbf{v}_0$$

Energy

$$\frac{1}{2} \frac{|\mathbf{v} - h'(t)\nabla_x \Phi_0|^2}{\varrho} = \Lambda(t) - \frac{d}{2} p(\varrho) + \frac{d}{2} h''(t) \Phi_0$$

Subsolutions

Energy profile

$$e = e(t,x) = \frac{E(t)}{|\Omega|} + \Lambda_0(t) - \frac{d}{2} p(\varrho) + \frac{d}{2} h''(t)\Phi_0, \ \ e \in \mathcal{R}([0,T] \times \overline{\Omega}).$$

Field equations

$$\mathrm{div}_x \mathbf{v} = 0, \ \ \partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0, \ \ \mathbf{v}(0,\cdot) = \mathbf{v}_0, \ \ \mathbb{U}(t,x) \in R^{d\times d}_{\mathrm{sym},0}$$

Convex constraint

$$\frac{d}{2} \lambda_{\max} \left[\frac{(\mathbf{v} - h'(t)\nabla_x \Phi_0) \otimes (\mathbf{v} - h'(t)\nabla_x \Phi_0)}{\varrho} - \mathbb{U} \right] \boxed{<} e$$

Algebraic inequality

$$\frac{1}{2} \frac{|\mathbf{v} - h'(t)\nabla_x \Phi_0|^2}{\varrho} \leq \frac{d}{2} \lambda_{\max} \left[\frac{(\mathbf{v} - h'(t)\nabla_x \Phi_0) \otimes (\mathbf{v} - h'(t)\nabla_x \Phi_0)}{\varrho} - \mathbb{U} \right]$$

Closure of the space of subsolutions

X the set of subsolutions $\subset L^\infty$

topology of $C_{\rm weak}([0,T]; L^2(\Omega; R^d))$

Limit equality

$$\frac{1}{2}\frac{|\mathbf{v} - h'(t)\nabla_x \Phi_0|^2}{\varrho}$$

$$= \frac{d}{2}\lambda_{\rm max}\left[\frac{(\mathbf{v} - h'(t)\nabla_x \Phi_0)\otimes(\mathbf{v} - h'(t)\nabla_x \Phi_0)}{\varrho} - \mathbb{U}\right] = e$$

$$\Rightarrow$$

$$\mathbb{U} = \left(\frac{(\mathbf{v} - h'(t)\nabla_x \Phi_0)\otimes(\mathbf{v} - h'(t)\nabla_x \Phi_0)}{\varrho} - \frac{1}{d}\frac{|\mathbf{v} - h'(t)\nabla_x \Phi_0|^2}{\varrho}\mathbb{I}\right)$$

Critical points (De Lellis- Székelyhidi)

Convex functional

$$I[\mathbf{v}] = \int_0^T \int_\Omega \left(\frac{1}{2} \frac{|\mathbf{v} - h'(t)\nabla_x \Phi_0|^2}{\varrho} - e \right) \, \mathrm{d}x\mathrm{d}t \text{ for } \mathbf{v} \in X.$$

Zero points

$$I[\mathbf{v}] = 0 \ \Rightarrow \ \mathbf{v} \text{ is a weak solution of the problem}$$

Points of continuity

$\mathbf{v}$ – a point of continuity of I on X $\Rightarrow I[\mathbf{v}] = 0$

Baire category argument

I convex l.s.c. on the (complete metric space) of subsolutions

$\Rightarrow$

points of continuity are dense

Oscillatory Lemma (De Lellis, Székelyhidi)

Oscillatory Lemma, basic constant coefficients form

Let $Q = (0,1) \times (0,1)^d$, $d = 2,3$. Suppose that $\mathbf{v} \in R^d$, $\mathbb{U} \in R^{d \times d}_{0,\mathrm{sym}}$, $e \leq \overline{e}$ are given constant quantities such that

$$\frac{d}{2} \lambda_{\max} \left[\mathbf{v} \otimes \mathbf{v} - \mathbb{U} \right] < e.$$

Then there is a constant $c = c(d, \overline{e})$ and sequences of vector functions $\{\mathbf{w}_n\}_{n=1}^{\infty}$, $\{\mathbb{V}_n\}_{n=1}^{\infty}$,

$$\mathbf{w}_n \in C^\infty_c(Q; R^d),\ \mathbb{V}_n \in C^\infty_c(Q; R^{d \times d}_{0,\mathrm{sym}})$$

satisfying

$$\partial_t \mathbf{w}_n + \mathrm{div}_x \mathbb{V}_n = 0,\ \mathrm{div}_x \mathbf{w}_n = 0 \text{ in } Q,$$

$$\frac{d}{2} \lambda_{\max} \left[(\mathbf{v} + \mathbf{w}_n) \otimes (\mathbf{v} + \mathbf{w}_n) - (\mathbb{U} + \mathbb{V}_n) \right] < e \text{ in } Q \text{ for all } n = 1, 2, \dots,$$

$$\mathbf{w}_n \to 0 \text{ in } C_{\mathrm{weak}}([0,1]; L^2((0,1)^d; R^d)) \text{ as } n \to \infty,$$

$$\liminf_{n \to \infty} \int_Q |\mathbf{w}_n|^2 \mathrm{d}x \mathrm{d}t \geq c(d, \overline{e}) \int_Q \left(e - \frac{1}{2} |\mathbf{v}|^2 \right)^2 \mathrm{d}x \mathrm{d}t$$

Oscillatory Lemma, continuous form

$$\mathbf{v} \in C(\overline{Q}; R^d),\ \mathbb{U} \in C(\overline{Q}; R^{d\times d}_{0,\mathrm{sym}}),\ e \in C(\overline{Q}),\ r \in \mathcal{C}(\overline{Q}),\ Q = (0,T) \times \Omega$$

$$0 < \underline{r} \le r(t,x) \le \overline{r},\ e(t,x) \le \overline{e} \text{ for all } (t,x) \in \overline{Q},$$

$$\frac{d}{2} \sup_{\overline{Q}} \lambda_{\max} \left[\frac{\mathbf{v} \otimes \mathbf{v}}{r} - \mathbb{U} \right] < \inf_{\overline{Q}} e.$$

Then there is a constant $c = c(d, \overline{e})$ and sequences $\{\mathbf{w}_n\}_{n=1}^{\infty}$, $\{\mathbb{V}_n\}_{n=1}^{\infty}$,

$$\mathbf{w}_n \in C^\infty_c(Q; R^d),\ \mathbb{V}_n \in C^\infty_c(Q; R^{d\times d}_{0,\mathrm{sym}})$$

satisfying

$$\partial_t \mathbf{w}_n + \mathrm{div}_x \mathbb{V}_n = 0,\ \mathrm{div}_x \mathbf{w}_n = 0 \text{ in } Q,$$

$$\frac{d}{2} \sup_{\overline{Q}} \lambda_{\max} \left[\frac{(\mathbf{v} + \mathbf{w}_n) \otimes (\mathbf{v} + \mathbf{w}_n)}{r} - (\mathbb{U} + \mathbb{V}_n) \right] < \inf_{\overline{Q}} e,$$

$$\mathbf{w}_n \to 0 \text{ in } C_{\mathrm{weak}}([0,T]; \Omega; R^d)) \text{ as } n \to \infty,$$

$$\liminf_{n\to\infty} \int_Q \frac{|\mathbf{w}_n|^2}{r} \mathrm{d}x\mathrm{d}t \ge c(d, \overline{e}) \int_Q \left(e - \frac{1}{2} \frac{|\mathbf{v}|^2}{r} \right)^2 \mathrm{d}x\mathrm{d}t$$

Oscillatory Lemma, proof via decomposition

- Domain decomposition

$$Q = \cup_{i \in \mathrm{fin}} Q_i, \;\; Q_i \text{ boxes}$$

- Replace the functions by constants (integral means) on each Q_i. The difference is small if the functions are continuous and $\mathrm{diam}[Q_i]$ is small so that all relevant inequalities remain valid
- Use the fact that the constant version of oscillatory lemma is invariant under scaling and apply it on each Q_i
- Sum up the results

Oscillatory Lemma, "Riemann" form

$$\mathbf{v} \in \mathcal{R}(\overline{Q}; R^d),\ \mathbb{U} \in \mathcal{R}(\overline{Q}; R^{d\times d}_{0,\mathrm{sym}}),\ e \in \mathcal{R}(\overline{Q}),\ r \in \mathcal{R}(\overline{Q}),\ Q = (0,T)\times\Omega$$

$$0 < \underline{r} \leq r(t,x) \leq \overline{r},\ \ e(t,x) \leq \overline{e} \text{ for all } (t,x) \in \overline{Q},$$

$$\frac{d}{2} \sup_{\overline{Q}} \lambda_{\max}\left[\frac{\mathbf{v}\otimes\mathbf{v}}{r} - \mathbb{U}\right] < \inf_{\overline{Q}} e.$$

Then there is a constant $c = c(d, \overline{e})$ and sequences $\{\mathbf{w}_n\}_{n=1}^{\infty}$, $\{\mathbb{V}_n\}_{n=1}^{\infty}$,

$$\mathbf{w}_n \in C^\infty_c(Q; R^d),\ \mathbb{V}_n \in C^\infty_c(Q; R^{d\times d}_{0,\mathrm{sym}})$$

satisfying

$$\partial_t \mathbf{w}_n + \mathrm{div}_x \mathbb{V}_n = 0,\ \mathrm{div}_x \mathbf{w}_n = 0 \text{ in } Q,$$

$$\frac{d}{2} \sup_{\overline{Q}} \lambda_{\max}\left[\frac{(\mathbf{v}+\mathbf{w}_n)\otimes(\mathbf{v}+\mathbf{w}_n)}{r} - (\mathbb{U}+\mathbb{V}_n)\right] < \inf_{\overline{Q}} e,$$

$$\mathbf{w}_n \to 0 \text{ in } C_{\mathrm{weak}}([0,T];\Omega;R^d)) \text{ as } n\to\infty,$$

$$\liminf_{n\to\infty} \int_Q \frac{|\mathbf{w}_n|^2}{r} \mathrm{d}x\mathrm{d}t \geq c(d,\overline{e}) \int_Q \left(e - \frac{1}{2}\frac{|\mathbf{v}|^2}{r}\right)^2 \mathrm{d}x\mathrm{d}t$$

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Dissipative solutions, stability, weak–strong uniqueness

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Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague

Nelder Lecture Series, Imperial College, London
20 April - 21 April 2022

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Navier–Stokes system

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{u})$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = 2\mu \left(\mathbb{D}_x \mathbf{u} - \frac{1}{d} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \lambda \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mathbb{D}_x \mathbf{u} \equiv \frac{1}{2} (\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^t)$$

Periodic conditions

$$\Omega = \mathbb{T}^d, \ d = 2, 3$$

Energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right) \ \mathrm{d}x + \int_\Omega \mathbb{S}(\mathbb{D}_x \mathbf{u}) : \mathbb{D}_x \mathbf{u} \ \mathrm{d}x \leq 0$$

$$P'(\varrho)\varrho - P(\varrho) = p(\varrho), \ p \text{ increasing}, \ \boxed{\textit{convex}}$$

Vanishing viscosity limit – dissipive solutions

$$p(\varrho) = a\varrho^\gamma$$

$$\mu = \mu_\varepsilon \to 0,\ \lambda = \lambda_\varepsilon \to 0$$

$$\varrho_\varepsilon \to \varrho \text{ in } C_{\rm weak}([0,T]; L^\gamma(\Omega))$$

$$\mathbf{m}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon \to \mathbf{m} \text{ in } C_{\rm weak}([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d))$$

Equation of continuity

$$\partial_t \varrho + {\rm div}_x \mathbf{m} = 0$$

Energy inequality

$$\int_\Omega \overline{\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)}(\tau,\cdot)\ {\rm d}x \le \int_\Omega \frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0)\ {\rm d}x$$

$$\overline{\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)} = \text{weak-(*) limit (in measures) of } \frac{1}{2}\frac{|\mathbf{m}_\varepsilon|^2}{\varrho_\varepsilon} + P(\varrho_\varepsilon)$$

Energy defect and Reynolds stress

Weak lower semi–continuity of convex functionals

$$\Rightarrow$$

$$\mathfrak{E} = \overline{\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)} - \left(\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right) \geq 0$$

Energy inequality revisited

$$\int_\Omega \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)(\tau, \cdot) \ \mathrm{d}x + \int_\Omega \mathfrak{E} \leq \int_\Omega \frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \ \mathrm{d}x$$

Momentum equation

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = -\mathrm{div}_x \mathfrak{R}$$

Reynolds stress:

$$\mathfrak{R} = \text{weak-(*) limit in measures of } \left(\frac{\mathbf{m}_\varepsilon \otimes \mathbf{m}_\varepsilon}{\varrho_\varepsilon} + p(\varrho_\varepsilon)\mathbb{I} \right) - \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} + p(\varrho)\mathbb{I} \right)$$

Dissipative solutions - Energy vs. Reynolds defect

Convexity revisited

$$(\varrho, \mathbf{m}) \mapsto \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} + p(\varrho)\mathbb{I} \right) : \xi \otimes \xi = \frac{|\mathbf{m} \cdot \xi|^2}{\varrho} + p(\varrho)|\xi|^2$$

$$\Rightarrow$$

$$\mathfrak{R} \geq 0, \ \mathrm{trace}[\mathfrak{R}] \leq c(\gamma)\mathfrak{E}$$

Dissipative solutions

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

$$\partial_t \varrho + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = -\mathrm{div}_x \mathfrak{R}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\int_\Omega \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)(\tau, \cdot) \ \mathrm{d}x + \int_\Omega \mathfrak{E} \right] \leq 0$$

$$\mathfrak{R} \geq 0, \ \mathrm{trace}[\mathfrak{R}] \leq c(\gamma)\mathfrak{E}$$

Dissipative solutions - basic properties

Existence

Dissipative solutions can be constructed as limits of energy dissipating numerical schemes (Lax–Friedrichs and similar). They appear as zero viscosity limit for the Navier–Stokes system

Dissipative–strong uniqueness

A dissipative solution coincides with a strong solution starting from the same initial data on the life–span of the latter

Uniqueness - semigroup selection

For each initial data, one can select a global in time dissipative solution so that the resulting system forms a semigroup. The selected solutions maximize the energy dissipation

Relative energy

Relative energy

$$\mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, r, \mathbf{U}\right) \equiv \frac{1}{2}\varrho\left|\frac{\mathbf{m}}{\varrho} - \mathbf{U}\right|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r) + \mathfrak{E}$$

$$P(\varrho) = \frac{a}{\gamma - 1}\varrho^\gamma$$

Relative energy decomposition

$$\int_\Omega \mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, r, \mathbf{U}\right) \, \mathrm{d}x$$

$$= \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right] \, \mathrm{d}x$$

$$- \int_\Omega \mathbf{m} \cdot \mathbf{U} \, \mathrm{d}x + \int_\Omega \varrho \left[\frac{1}{2}|\mathbf{U}|^2 - P'(r)\right] \, \mathrm{d}x$$

$$+ \int_\Omega \left[P'(r)r - P(r)\right] \, \mathrm{d}x$$

Relative energy inequality

$$
\begin{aligned}
&\int_\Omega \mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, r, \mathbf{U}\right)(\tau, \cdot)\ \mathrm{d}x - \int_\Omega \mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, r, \mathbf{U}\right)(0, \cdot)\ \mathrm{d}x \\
&\leq - \int_0^\tau \int_\Omega \nabla_x \mathbf{U} : \varrho \left(\mathbf{U} - \frac{\mathbf{m}}{\varrho} \right) \otimes \left(\mathbf{U} - \frac{\mathbf{m}}{\varrho} \right)\ \mathrm{d}x\mathrm{d}t \\
&- \int_0^\tau \int_\Omega \Big(p(\varrho) - p'(r)(\varrho - r) - p(r) \Big) \mathrm{div}_x \mathbf{U}\ \mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \Big[\partial_t (r\mathbf{U}) + \mathrm{div}_x \left(r\mathbf{U} \otimes \mathbf{U} \right) + \nabla_x p(r) \Big] \cdot \frac{1}{r} \Big(\varrho \mathbf{U} - \mathbf{m} \Big)\ \mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \Big[\partial_t r + \mathrm{div}_x (r\mathbf{U}) \Big] \left[\left(1 - \frac{\varrho}{r} \right) p'(r) + \frac{1}{r} \mathbf{U} \cdot (\mathbf{m} - \varrho \mathbf{U}) \right]\ \mathrm{d}x\mathrm{d}t \\
&- \int_0^\tau \left(\int_\Omega \nabla_x \mathbf{U} : \mathrm{d}\mathfrak{R}(t) \right) \mathrm{d}t
\end{aligned}
$$

Dispersive velocity weak solutions

Besov spaces

$$v \in B_p^{\alpha,\infty}(Q) \ \Leftrightarrow \ \|v\|_{L^p(Q)} + \sup_\xi \frac{\|v(\cdot + \xi) - v(\cdot)\|_{L^p(Q\cap(Q-\xi))}}{|\xi|^\alpha} < \infty.$$

Class $\mathcal{D}$

$$\varrho \in C([0,T];L^1(\Omega)),\ \mathbf{u} \in C([0,T];L^1(\Omega;R^d))$$

$$0 < \underline{\varrho} \leq \varrho \leq \overline{\varrho},\ |\mathbf{u}| \leq \overline{\mathbf{u}} \text{ a.a. in } (0,T)\times\Omega$$

$$\varrho \in B_p^{\alpha,\infty}([\delta,T]\times\Omega),\ \mathbf{u} \in B_p^{\alpha,\infty}([\delta,T]\times\Omega;R^d)$$

$$\text{for any } 0 < \delta < T,\ \alpha > \frac{1}{2},\ p \geq \frac{4\gamma}{\gamma-1}$$

$$\int_\Omega \left[-\xi \cdot \mathbf{u}(\tau,\cdot)(\xi\cdot\nabla_x)\varphi + D(\tau)|\xi|^2\varphi \right] \ \mathrm{d}x \geq 0 \text{ for a.a. } \tau \in (0,T)$$

$$\text{for any } \xi \in R^d \text{ and any } \varphi \in C^1(\Omega),\ \varphi \geq 0,\ \text{where } D \in L^1(0,T)$$

Weak (dissipative) – weak uniqueness

Theorem

Let ϱ, $\mathbf{m} = \varrho\mathbf{u}$ be a weak solution of the Euler system belonging to class $\mathcal{D}$, and let $\widetilde{\varrho}$, $\widetilde{\mathbf{m}}$ be a dissipative solution of the same problem starting from the same initial data.
Then

$$\varrho = \widetilde{\varrho},\ \mathbf{m} = \widetilde{\mathbf{m}}.$$

Semigroup (semiflow) selection

Set of data

$$\mathcal{D} = \left\{ \varrho, \mathbf{m}, E \ \Big| \ \int_\Omega \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \ {\rm d}x \leq E \right\}$$

Set of trajectories

$$\mathcal{T} = \left\{ \varrho(t, \cdot), \mathbf{m}(t, \cdot), E(t-, \cdot) \Big| t \in (0, \infty) \right\}$$

Solution set

$$\mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] = \Big\{ [\varrho, \mathbf{m}, E] \ \Big| [\varrho, \mathbf{m}, E] \text{ dissipative solution}$$

$$\varrho(0, \cdot) = \varrho_0, \ \mathbf{m}(0, \cdot) = \mathbf{m}_0, \ E(0+) \leq E_0 \Big\}$$

Semiflow selection – semigroup

$$U[\varrho_0, \mathbf{m}_0, E_0] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0], \ [\varrho_0, \mathbf{m}_0, E_0] \in \mathcal{D}$$

$$U(t_1 + t_2)[\varrho_0, \mathbf{m}_0, E_0] = U(t_1) \circ \Big[U(t_2)[\varrho_0, \mathbf{m}_0, E_0] \Big], \ t_1, t_2 > 0$$

Andrej Markov (1856–1933)

N. V. Krylov

Abstract setting

Phase space

$$(\varrho, \mathbf{m}, E) \in X = W^{-\ell,2}(Q) \times W^{-\ell,2}(Q; R^N) \times R$$

Data space

$$D = \left\{ [\varrho_0, \mathbf{m}_0, E_0] \in X \;\middle|\; \varrho_0 \geq 0,\ \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + \frac{a}{\gamma - 1} \varrho_0^\gamma \right] \, \mathrm{d}x \leq E_0 \right\}.$$

Trajectory space

$$\Omega = C_{\rm loc}([0,\infty); W^{-\ell,2}(Q)) \times C_{\rm loc}([0,\infty); W^{-\ell,2}(Q; R^N)) \times L^1_{\rm loc}(0,\infty)$$

Method by Krylov adapted by Cardona and Kapitanski

Multi–valued solution mapping

$$\mathcal{U} : [\varrho_0, \mathbf{m}_0, E_0] \mapsto [\varrho, \mathbf{m}, E] \in 2^{\Omega}$$

Time shift

$$S_T \circ \xi,\ \ S_T \circ \xi(t) = \xi(T + t),\ \ t \geq 0.$$

Continuation

$$\xi_1 \cup_T \xi_2(\tau) = \begin{cases} \xi_1(\tau) \text{ for } 0 \leq \tau \leq T, \\ \\ \xi_2(\tau - T) \text{ for } \tau > T. \end{cases}$$

Basic axioms

(A1) Compactness: For any $[\varrho_0, \mathbf{m}_0, E_0] \in D$, the set $\mathcal{U}[\varrho_0, \mathbf{m}_0, E_0]$ is a non–empty compact subset of Ω
(A2) The mapping

$$D \ni [\varrho_0, \mathbf{m}_0, E_0] \mapsto \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] \in 2^{\Omega}$$

is **Borel measurable**, where the range of $\mathcal{U}$ is endowed with the Hausdorff metric on the subspace of compact sets in 2^{Ω}
(A3) Shift invariance: For any

$$[\varrho, \mathbf{m}, E] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0],$$

we have

$$S_T \circ [\varrho, \mathbf{m}, E] \in \mathcal{U}[\varrho(T), \mathbf{m}(t), E(T-)] \text{ for any } T > 0.$$

(A4) Continuation: If $T > 0$, and

$$[\varrho^1, \mathbf{m}^1, E^1] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0],\ [\varrho^2, \mathbf{m}^2, E^2] \in \mathcal{U}[\varrho^1(T), \mathbf{m}^1(T), E^1(T-)],$$

then

$$[\varrho^1, \mathbf{m}^1, E^1] \cup_T [\varrho^2, \mathbf{m}^2, E^2] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0].$$

Induction argument

System of functionals

$$I_{\lambda,F}[\varrho, \mathbf{m}, E] = \int_0^\infty \exp(-\lambda t) F(\varrho, \mathbf{m}, E) \, \mathrm{d}t, \ \lambda > 0$$

where

$$F : X = W^{-\ell,2}(Q) \times W^{-\ell,2}(Q; R^N) \times R \to R$$

is a bounded and continuous functional

Semiflow reduction

$$\begin{aligned} & I_{\lambda,F} \circ \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] \\ & = \Big\{ [\varrho, \mathbf{m}, E] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] \ \Big| \\ & I_{\lambda,F}[\varrho, \mathbf{m}, E] \leq I_{\lambda,F}[\widetilde{\varrho}, \widetilde{\mathbf{m}}, \widetilde{E}] \text{ for all } [\widetilde{\varrho}, \widetilde{\mathbf{m}}, \widetilde{E}] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] \Big\} \end{aligned}$$

Induction argument

$$\mathcal{U} \text{ satisfies (A1) - (A4)} \ \Rightarrow I_{\lambda,F} \circ \mathcal{U} \text{ satisfies (A1) - (A4)}$$

Maximal dissipation

Comparison of energy dissipation

$$[\varrho_1, \mathbf{m}_1, E_1] \prec [\varrho_2, \mathbf{m}_2, E_2] \ \Leftrightarrow \ E_1(t\pm) \leq E_2(t\pm) \text{ for any } t$$

Admissible solutions
Dissipative solution is admissible if it is minimal with respect to $\prec$

Admissibility of semigroup selection

The choice of the testinf functionals can be arranged in the way that the chosen solution is admissible

Asymptotic behavior of admissible solutions
If $(\varrho, \mathbf{m}, E)$ is admissible, then

$$\int_\Omega \mathfrak{E}(t, \cdot) \to 0 \text{ as } t \to \infty$$

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Stochastically driven Euler system its relevance to turbulence

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Nelder Lecture Series, Imperial College, London
20 April - 21 April 2022

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Prologue

Incompressible Euler system

$$\mathrm{div}_x \mathbf{v} = 0,\ \ \partial_t \mathbf{v} + \mathrm{div}_x(\mathbf{v} \otimes \mathbf{v}) + \nabla_x \Pi = 0$$

Result of Greengard and Thomann [1988]
There exists a sequence $\{\mathbf{v}_n\}_{n=1}^{\infty}$ of compactly supported (in the space variable R^3) of solutions to the incompressible Euler system converging *weakly* to zero.

Conclusion

Incompessible Euler system admits sequences of oscillatory spatially localized solutions converging weakly to another (weak) solution of the same problem

Obstacle problem

Fluid domain and obstacle

$$Q = R^d \setminus B,\ d = 2, 3$$

B compact, convex

Navier–Stokes system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

$$p(\varrho) \approx a\varrho^\gamma,\ \gamma > 1,\ \mathbb{S} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \lambda \mathrm{div}_x \mathbf{u} \mathbb{I},$$

Boundary and far field conditions

$$\mathbf{u}|_{\partial Q} = 0,\ \varrho \to \varrho_\infty,\ \mathbf{u} \to \mathbf{u}_\infty \text{ as } |x| \to \infty$$

High Reynolds number (vanishing viscosity) limit

Vanishing viscosity

$$\varepsilon_n \searrow 0, \ \mu_n = \varepsilon_n \mu, \mu > 0, \ \lambda_n = \varepsilon_n \lambda, \lambda \geq 0$$

Questions

- Identify the limit of the corresponding solutions $(\varrho_n, \mathbf{u}_n)$ as $n \to \infty$ in the fluid domain Q
- **Yakhot and Orszak [1986]:** *"The effect of the boundary in the turbulence regime can be modeled in a statistically equivalent way by fluid equations driven by stochastic forcing"*

  Clarify the meaning of "statistically equivalent way"

  Is the (compressible) Euler system driven by a general cylindrical white noise force adequate to describe the limit of $(\varrho_n, \mathbf{u}_n)$?

Bounded energy solutions

(Relative) energy

$$E\left(\varrho, \mathbf{u} \,\middle|\, \varrho_\infty, \mathbf{u}_\infty\right) = \frac{1}{2}\varrho|\mathbf{u} - \mathbf{u}_\infty|^2 + P(\varrho) - P'(\varrho_\infty)(\varrho - \varrho_\infty) - P(\varrho_\infty)$$

$$P(\varrho) = \frac{a}{\gamma - 1}\varrho^\gamma, \ \mathbf{u}_\infty = 0 \text{ for } |x| < R_1, \ \mathbf{u}_\infty = \mathbf{u}_\infty \text{ for } |x| > R_2$$

Energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_Q E\left(\varrho, \mathbf{u}\middle|\varrho_\infty, \mathbf{u}_\infty\right) \ \mathrm{d}x + \int_Q \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x$$

$$\leq - \int_Q \left(\varrho\mathbf{u} \otimes \mathbf{u} + p(\varrho)\mathbb{I}\right) : \nabla_x \mathbf{u}_\infty \ \mathrm{d}x + \frac{1}{2}\int_Q \varrho\mathbf{u} \cdot \nabla_x |\mathbf{u}_\infty|^2 \ \mathrm{d}x$$

$$+ \int_Q \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}_\infty \ \mathrm{d}x.$$

Statistical limit

Energy bounds

$$\mathbf{m} \equiv \varrho \mathbf{u}$$

$$\frac{1}{N} \sum_{n=1}^{N} \left[\sup_{0 \leq \tau \leq T} \int_Q E \left(\varrho_n, \mathbf{m}_n \Big| \varrho_\infty, \mathbf{u}_\infty \right) (\tau, \cdot) \ \mathrm{d}x + \varepsilon_n \int_0^T \int_Q \mathbb{S}(\nabla_x \mathbf{u}_n) : \nabla_x \mathbf{u}_n \ \mathrm{d}x \mathrm{d}t \right] \leq \overline{\mathcal{E}}$$

uniformly for $N \to \infty$

Trajectory space

$$(\varrho_n, \mathbf{m}_n) \in \mathcal{T} \equiv C_{\mathrm{weak}}([0, T]; L^\gamma_{\mathrm{loc}}(Q) \times L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{loc}}(Q; R^d))$$

Statistical limit

$$\mathcal{V}_N = \frac{1}{N} \sum_{n=1}^{N} \delta_{(\varrho_n, \mathbf{m}_n)}, \ \ \mathbf{m}_n = \varrho_n \mathbf{u}_n$$

Prokhorov theorem $\Rightarrow$ $\mathcal{V}_N \to \mathcal{V}$ narrowly in $\mathfrak{P}[\mathcal{T}]$

$(\varrho, \mathbf{m}) \approx \mathcal{V}$ a random process with paths in $\mathcal{T}$

Limit problem

Statistical dissipative solutions to the Euler system

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = -\mathrm{div}_x \mathfrak{R}$$

$\mathcal{V}$ a.s.

Reynolds stress

$$\mathfrak{R} \in L^\infty_{\mathrm{weak}-(*)}(0, T; \mathcal{M}^+(Q; R^{d \times d}_{\mathrm{sym}}))$$

$$\mathfrak{R} : (\xi \otimes \xi) \geq 0, \ \xi \in R^d$$

$$\mathbb{E} \left[\int_0^T \psi \int_Q \varphi \ \mathrm{d} \ \mathrm{trace}[\mathfrak{R}] \mathrm{d}t \right] \leq c \overline{\mathcal{E}} \|\psi\|_{L^1(0,T)} \|\varphi\|_{BC(Q)}$$

Reynolds stress

Skorokhod–Jakubowski representation theorem

$$\varrho_N \approx \widetilde{\varrho}_N,\ \mathbf{m}_N \approx \widetilde{\mathbf{m}}_N \text{ (equivalence in law)}$$

a.s. weak convergence

$$(\widetilde{\varrho}_N, \widetilde{\mathbf{m}}_N) \to (\varrho, \mathbf{m}) \text{ in } C_{\rm weak}([0,T]; L^{\gamma}_{\rm loc}(Q) \times L^{\frac{2\gamma}{\gamma+1}}_{\rm loc}(Q;R^d))$$

$$\frac{\widetilde{\mathbf{m}}_N \otimes \widetilde{\mathbf{m}}_N}{\widetilde{\varrho}_N} + p(\varrho_N)\mathbb{I} \to \overline{\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} + p(\varrho)\mathbb{I}}$$

$$\text{weakly-(*) in } L^\infty_{{\rm weak}-(*)}(0,T;\mathcal{M}(Q;R^{d\times d}_{\rm sym}))$$

Reynolds stress

$$\mathfrak{R} \equiv \overline{\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} + p(\varrho)} - \left(\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} + p(\varrho)\mathbb{I}\right)$$

$$\text{convexity of } (\varrho, \mathbf{m}) \mapsto \left(\frac{|\mathbf{m}\cdot\xi|^2}{\varrho} + p(\varrho)|\xi|^2\right) \ \Rightarrow\ \mathfrak{R} : (\xi\otimes\xi) \geq 0$$

Stochastic Euler system

Euler system with stochastic forcing

$$\mathrm{d}\widetilde{\varrho} + \mathrm{div}_x \widetilde{\mathbf{m}} \mathrm{d}t = 0$$

$$\mathrm{d}\widetilde{\mathbf{m}} + \mathrm{div}_x \left(\frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\widetilde{\varrho}} \right) \mathrm{d}t + \nabla_x p(\widetilde{\varrho}) \mathrm{d}t = \mathbf{F} \mathrm{d}W$$

$W = (W_k)_{k \geq 1}$ cylindrical Wiener process

$\mathbf{F} = (\mathbf{F}_k)_{k \geq 1}$ – diffusion coefficient

$$\mathbb{E} \left[\int_0^T \sum_{k \geq 1} \| \mathbf{F}_k \|^2_{W^{-\ell,2}(Q;R^d)} \mathrm{d}t \right] < \infty$$

we allow $\mathbf{F} = \mathbf{F}(\varrho, \mathbf{m})$

Statistical equivalence

statistical equivalence $\Leftrightarrow$ identity in expectation of some quantities

$(\varrho, \mathbf{m})$ statistically equivalent to $(\widetilde{\varrho}, \widetilde{\mathbf{m}})$

$$\Leftrightarrow$$

- **density and momentum**

$$\mathbb{E}\left[\int_D \varrho\right] = \mathbb{E}\left[\int_D \widetilde{\varrho}\right], \ \mathbb{E}\left[\int_D \mathbf{m}\right] = \mathbb{E}\left[\int_D \widetilde{\mathbf{m}}\right]$$

- **kinetic and internal energy**

$$\mathbb{E}\left[\int_D \frac{|\mathbf{m}|^2}{\varrho}\right] = \mathbb{E}\left[\int_D \frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}}\right], \ \mathbb{E}\left[\int_D p(\varrho)\right] = \mathbb{E}\left[\int_D p(\widetilde{\varrho})\right]$$

- **angular energy**

$$\mathbb{E}\left[\int_D \frac{1}{\varrho}(\mathbb{J}_{x_0} \cdot \mathbf{m}) \cdot \mathbf{m}\right] = \mathbb{E}\left[\int_D \frac{1}{\widetilde{\varrho}}(\mathbb{J}_{x_0} \cdot \widetilde{\mathbf{m}}) \cdot \widetilde{\mathbf{m}}\right]$$

$$D \subset (0,T) \times Q, \ x_0 \in R^d, \ \mathbb{J}_{x_0}(x) \equiv |x - x_0|^2 \mathbb{I} - (x - x_0) \otimes (x - x_0)$$

Results

Hypothesis:

$(\varrho, \mathbf{m})$ statistically equivalent to a solution of the stochastic Euler system $(\widetilde{\varrho}, \widetilde{\mathbf{m}})$

Conclusion:

- **Noise inactive**
  $\mathfrak{R} = 0$, $(\varrho, \mathbf{m})$ is a statistical solution to a deterministic Euler system
- **S-convergence (up to a subsequence) to the limit system**

$$\frac{1}{N} \sum_{n=1}^{N} b(\varrho_n, \mathbf{m}_n) \to \mathbb{E}\left[b(\varrho, \mathbf{m}) \right] \text{ strongly in } L^1_{\rm loc}((0,T) \times Q)$$

  for any $b \in C_c(R^{d+1})$, $\varphi \in C_c^\infty((0,T) \times Q)$
- **Conditional statistical convergence**

$$\text{barycenter } (\overline{\varrho}, \overline{\mathbf{m}}) \equiv \mathbb{E}\left[(\varrho, \mathbf{m}) \right] \text{ solves the Euler system}$$

$$\Rightarrow$$

$$\frac{1}{N} \# \left\{ n \leq N \middle| \| \varrho_n - \overline{\varrho} \|_{L^\gamma(K)} + \| \mathbf{m}_n - \overline{\mathbf{m}} \|_{L^{\frac{2\gamma}{\gamma+1}}(K;R^d)} > \varepsilon \right\} \to 0$$

  as $N \to \infty$ for any $\varepsilon > 0$, and any compact $K \subset [0,T] \times Q$

Main ideas

- Use statistical equivalence of $(\varrho, \mathbf{m})$ to $(\widetilde{\varrho}, \widetilde{\mathbf{m}})$ and the fact that the Itô integral is a martingale to obtain the identity

$$\mathbb{E}\left[\mathrm{div}_x \mathfrak{R}\right] = \mathbb{E}\left[\mathrm{div}_x \left(\frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\widetilde{\varrho}} - \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right)\right] \tag{1}$$

  in $\mathcal{D}'((0,T) \times Q)$

- Show that if Q is exterior to a ball and $(\varrho, \mathbf{m})$ statistically equivalent to $(\widetilde{\varrho}, \widetilde{\mathbf{m}})$, then

$$\mathfrak{R} = 0 \text{ a.s.}$$

  Hint: Use test functions of the form

$$\phi_L(x) = \chi\left(\frac{|x|}{L}\right) \nabla_x F(|x|^2),\ \phi \in C^1_c(Q),\ L \geq 1$$

$$\chi \in C^\infty_c[0,\infty),\ \chi(Z) = 1 \text{ for } Z \leq 1,\ \chi(Z) = 0 \text{ for } Z \geq 2$$

$$F \text{ convex},\ F(Z) = 0 \text{ for } 0 \leq Z \leq R^2,\ 0 < F'(Z) \leq \overline{F} \text{ for } R^2 < Z < R^2 + 1$$

$$F'(Z) = \overline{F} \text{ if } Z \geq R^2 + 1,$$

  and let $L \to \infty$ to conclude $\mathbb{E}\left[\int_0^T \int_Q \mathrm{tr}[\mathfrak{R}]\right] = 0$

- Extend the result to $Q = R^d \setminus B$, B compact, convex.

Stratonovich drift

Stochastic Euler system

$$\mathrm{d}\widetilde{\varrho} + \mathrm{div}_x \widetilde{\mathbf{m}} \mathrm{d}t = 0$$

$$\mathrm{d}\widetilde{\mathbf{m}} + \mathrm{div}_x \left(\frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\widetilde{\varrho}} \right) \mathrm{d}t + \nabla_x p(\widetilde{\varrho}) \mathrm{d}t = \boxed{(\sigma \cdot \nabla_x) \widetilde{\mathbf{m}} \circ \mathrm{d}W_1} + \mathbf{F}\ \mathrm{d}W_2$$

Additional hypotheses

- $Q = R^d$
- If $d = 2$, we need $\varrho_\infty = 0$; if $d = 3$, we need $\varrho_\infty = 0$, $\mathbf{u}_\infty = 0$, and $1 < \gamma \leq 3$

Similar type of noise used recently by Flandoli et al to produce a regularizing effect in the incompressible Navier–Stokes system

Conclusion

- Stochastically driven Euler system irrelevant in the description of compressible turbulence (slightly extrapolated statement)

Possible scenarios:

- **Oscillatory limit.** The sequence $(\varrho_n, \mathbf{m}_n)$ generates a Young measure. Its barycenter (weak limit of $(\varrho_n, \mathbf{m}_n)$) is not a weak solution of the Euler system. Statistically, however, the limit is a single object. This scenario is compatible with the hypothesis that the limit is independent of the choice of $\varepsilon_n \searrow 0 \Rightarrow$ computable numerically.
- **Statistical limit.** The limit is a statistical solution of the Euler system. In agreement with Kolmogorov hypothesis concerning turbulent flow advocated in the compressible setting by Chen and Glimm. This scenario is not compatible with the hypothesis that the limit is independent of $\varepsilon_n \searrow 0$ ($\Rightarrow$ numerically problematic) unless the limit is a monoatomic measure in which case the convergence must be strong.

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(S) – convergence, computing oscillatory solutions

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Example I: Strong Law of Large numbers

Strong Law of Large Numbers

$$\{U_n\}_{n=1}^{\infty} \ \boxed{\textit{independent}} \text{ random variables, } E(U_n) = \mu$$

$$\Rightarrow$$

$$\frac{1}{N}\sum_{n=1}^{N} U_n \to \mu \text{ as } N \to \infty \text{ a.s.}$$

Subsequence principle: Komlos (Banach–Saks) theorem

$$\int_{\Omega} |U_n| \, \mathrm{d}x \leq c \text{ uniformly for } n \to \infty$$

$$\Rightarrow$$

there is a subsequence $\{U_{n_k}\}_{k=1}^{\infty}$ such that

$$\frac{1}{N}\sum_{l=1}^{N} U_{n_l} \to U \in L^1(\Omega) \text{ as } N \to \infty \text{ a.a. in } \Omega$$

for any subsequence $\{n_l\} \subset \{n_k\}$

Example II: Ergodic hypothesis

Asymptotic behavior of dynamical systems

$$t \in [0, \infty) \mapsto \mathbf{U}(t) \in X,$$

ω-limit set

$$\omega[\mathbf{U}] = \left\{ \mathbf{u} \in X \;\middle|\; \text{there exists } t_n \to \infty \ \mathbf{U}(t_n) \to \mathbf{u} \right\}$$

Ergodic hypothesis

$$\frac{1}{T}\int_0^T F(\mathbf{U}(t))\mathrm{d}t \to \overline{F} \text{ as } T \to \infty \text{ for any Borel } F \in \mathcal{B}(X; R)$$

Birkhoff–Khinchin ergodic theorem

$$\mathbf{U}(t) : R \to X \text{ stationary process } \Rightarrow \ \frac{1}{T}\int_0^T F(\mathbf{U}(t))\mathrm{d}t \to \overline{F} \text{ a.s.}$$

(S) - convergence, basic idea

Trivial example of oscillatory sequence

$$U_n = \begin{cases} 1 \text{ for } n \text{ odd} \\ -1 \text{ for } n \text{ even} \end{cases}$$

Convergence via Young measure approach

Convergence up to a *subsequence*

$$U_n \approx \delta_{U_n},\ U_{n_k} \to \begin{cases} \delta_1 \text{ as } k \to \infty,\ n_k \text{ odd} \\ \delta_{-1} \text{ as } k \to \infty,\ n_k \text{ even} \end{cases}$$

Convergence via averaging

$$U_n \approx \delta_{U_n},\ \frac{1}{N}\sum_{n=1}^{N} U_n \to \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$$

$$\frac{1}{w_N}\sum_{n=1}^{N} w\left(\frac{n}{N}\right) U_n \to \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1,\ w_N \equiv \sum_{n=1}^{N} w\left(\frac{n}{N}\right)$$

Euler system of gas dynamics

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

Momentum balance – Newton's Second Law

$$\partial_t \mathbf{m} + \mathrm{div}_x \left[\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right] + \nabla_x p = 0$$

Energy balance – First Law of Thermodynamics

$$\partial_t E + \mathrm{div}_x \left[(E + p) \frac{\mathbf{m}}{\varrho} \right] = 0$$

Boundary conditions

$$\mathbf{m} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Constitutive relations – Second Law of Thermodynamics

Pressure, internal energy, entropy

$$E = \underbrace{\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}}_{\text{kinetic energy}} + \underbrace{\varrho e}_{\text{internal energy}}, \quad \underbrace{p = (\gamma - 1)\varrho e}_{\text{EOS (incomplete)}}$$

Entropy

$$s = \mathcal{S}\left(\frac{p}{\varrho^\gamma}\right), \quad \underbrace{S = \varrho s}_{\text{total entropy}}$$

Boyle–Mariotte Law

$$p = \varrho\vartheta, \; e = \frac{1}{\gamma - 1}\vartheta, \; s = \log\left(\frac{\vartheta^{\frac{1}{\gamma-1}}}{\varrho}\right)$$

Entropy balance (inequality) – Second Law of Thermodynamics

$$\partial_t S + \mathrm{div}_x\left[S\frac{\mathbf{m}}{\varrho}\right] = (\geq) 0$$

Thermodynamic stability

Conservative–entropy variables

$$\text{density } \varrho, \ \text{momentum } \mathbf{m}, \ \text{total entropy } S, \ [\varrho, \mathbf{m}, S]$$

Thermodynamic stability – energy

$$E = E(\varrho, \mathbf{m}, S) : R^{d+2} \to [0, \infty]$$

$$E = (\varrho, \mathbf{m}, S) = \infty \text{ if } \varrho < 0, \ E(0, \mathbf{m}, S) = \lim_{\varrho \to 0+} E(\varrho, \mathbf{m}, S)$$

convex, lower semi–continuous on R^{d+2}

Thermodynamic stability in standard variables

$$\text{positive compressibility } \frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0$$

$$\text{positive specific heat at constant volume } \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

Known facts about solvability of Euler system

Classical solutions
Smooth initial state gives rise to smooth solution on a (generically) finite time interval $T_{\max}$, singularities (shocks) develop after $T_{\max}$

Weak solutions
Admissible (weak + entropy inequality) weak solutions exist globally in time. There is a "vast" class of initial data for which the problem admits infinitely many admissible weak solutions, **the system is ill–posed in the class of admissible weak solutions**

Generalized - oscillatory solutions
There are various concepts of generalized solutions: measure–valued solutions, dissipative measure–valued solutions, etc. They can be seen as limits of *consistent* approximations. They are **inseparable from the process** how they were obtained.

Consistent approximation

Approximate field equations (in the distributional sense)

$$\partial_t \varrho_n + \mathrm{div}_x \mathbf{m}_n = e_n^1$$

$$\partial_t \mathbf{m}_n + \mathrm{div}_x \left[\frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} \right] + \nabla_x p(\varrho_n, S_n) = e_n^2$$

$$\partial_t E(\varrho_n, \mathbf{m}_n, S_n) + \mathrm{div}_x \left[(E + p)(\varrho_n, \mathbf{m}_n, S_n) \frac{\mathbf{m}_n}{\varrho_n} \right] = e_n^3$$

$$\partial_t S_n + \mathrm{div}_x \left[S_n \frac{\mathbf{m}_n}{\varrho_n} \right] \geq e_n^4$$

Vanishing consistency errors

$$e_n^1,\ e_n^2,\ e_n^4 \to 0 \text{ in the distributional sense}$$

$$\int_\Omega e_n^3 \ \mathrm{d}x \to 0 \text{ uniformly in time}$$

Stability

$$\int_\Omega E(\varrho_n, \mathbf{m}_n, S_n) \ \mathrm{d}x \leq c,\ s_n = \frac{S_n}{\varrho_n} \geq -c \ \text{ uniformly in time}$$

Consistent approximation - basic properties

Examples of consistent approximations

- **Vanishing dissipation limit** from the Navier–Stokes–Fourier system to the Euler system
- **Limits of entropy (energy) preserving numerical schemes,** Lax–Friedrichs scheme, Rusanov scheme, Brenner model based scheme (EF, M.Lukáčová, H. Mizerová)

Convergence of consistent approximation

- $$\varrho_{n_k} \to \varrho, \ S_{n_k} \to S \text{ weakly-(*) in } L^\infty(0, T; L^\gamma(\Omega))$$
  $$\mathbf{m}_{n_k} \to \mathbf{m} \text{ weakly-(*) in } L^\infty(0, T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d))$$
- the limit $[\varrho, \mathbf{m}, S]$ is a generalized (dissipative) solution of the Euler system
- $\{\varrho_{n_k}, \mathbf{m}_{n_k}, S_{n_k}\} \approx \{\delta_{\varrho_{n_k}, \mathbf{m}_{n_k}, S_{n_k}}\}$ generates a Young measure

up to a suitable subsequence!

Convergence of consistent approximation

Strong convergence

- **Strong convergence to strong solution (uncoditional)**
  Euler system admits a smooth solution $\Rightarrow [\varrho, \mathbf{m}, S]$ is the unique smooth solution and convergence is strong and unconditional (no need for subsequence) in L^1
- **Strong convergence to smooth limit (unconditional)**
  The limit $[\varrho, \mathbf{m}, S]$ is of class $C^1 \Rightarrow$ the limit is the unique strong solution of the Euler system and convergence is strong and unconditional (no need for subsequence) in L^1
- **Strong convergence to weak solution (up to a subsequence)**
  EF, M.Hofmanová (2019):

  The limit $[\varrho, \mathbf{m}, S]$ is a weak solution of the Euler system $\Rightarrow$ convergence is strong in L^1

Weak convergence of consistent approximation

Weak convergence
If consistent approximation DOES NOT converge strongly, the following must be satisfied:

- the limit Euler system does not admit a strong solution
- the limit $[\varrho, S, \mathbf{m}]$ is not C^1 smooth
- the limit $[\varrho, S, \mathbf{m}]$ IS NOT a weak solution of the Euler system

Visualization of weak convergence?

- **Oscillations.** Weakly converging sequence may develop oscillations. Example:

$$\sin(nx) \to 0 \text{ weakly as } n \to \infty$$

- **Concentrations.**

$$n\theta(nx) \to \delta_0 \text{ weakly-(*) in } \mathcal{M}(R)$$

if

$$\theta \in C_c^\infty(R),\ \theta \geq 0, \int_R \theta = 1$$

Statistical description of oscillations – Young measures

Young measure

$$b(\varrho_n, \mathbf{m}_n, S_n) \to \overline{b(\varrho, \mathbf{m}, S)} \text{ weakly-(*) in } L^\infty((0,T)\times\Omega)$$

(up to a subsequence) for any $b \in C_c(R^{d+2})$

Young measure $\mathcal{V}$ – a parametrized family of probability measures $\{\mathcal{V}_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$ on the phase space R^{d+2}:

$$\overline{b(\varrho, \mathbf{m}, S)}(t,x) = \left\langle \mathcal{V}_{t,x}; b(\widetilde{\varrho}, \widetilde{\mathbf{m}}, \widetilde{S}) \right\rangle \text{ for a.a. } (t,x)$$

Visualizing Young measure

$$\text{visualizing Young measure} \Leftrightarrow \text{computing } \overline{b(\varrho, \mathbf{m}, S)}$$

Problems

- $b(\varrho_n, \mathbf{m}_n, S_n)$ converge only weakly
- extracting subsequences
- only statistical properties relevant $\Rightarrow$ knowledge of the "tail" of the sequence of approximate solutions absolutely necessary

(S)–convergence

(S)–convergent approximate sequence

An approximate sequence $\{\mathbf{U}_n\}_{n=1}^{\infty}$ is $\boxed{(S) - \mathit{convergent}}$ if for any $b \in C_c(R^D)$:

- **Correlation limit**

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_Q b(\mathbf{U}_n) b(\mathbf{U}_m) \mathrm{d}y \text{ exists for any fixed } m$$

- **Correlation disintegration**

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{n,m=1}^{N} \int_Q b(\mathbf{U}_n) b(\mathbf{U}_m) \ \mathrm{d}y$$
$$= \lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{N} \left(\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_Q b(\mathbf{U}_n) b(\mathbf{U}_m) \ \mathrm{d}y \right)$$

Basic properties of (S)–convergence, I

Equivalence to convergence of ergodic (Cesàro means)

$$\{\mathbf{U}_n\}_{n=1}^{\infty} \text{ (S)–convergent } \Leftrightarrow \frac{1}{N}\sum_{n=1}^{N} b(\mathbf{U}_n) \to \overline{b(\mathbf{U})} \ \boxed{\textit{strongly}} \text{ in } L^1(Q)$$

(S)– limit (parametrized measure)

$$\mathbf{U}_n \overset{(S)}{\to} \mathcal{V},\ \{\mathcal{V}_y\}_{y\in Q},\ \mathcal{V}_y \in \mathfrak{P}(R^D),\ \left\langle \mathcal{V}_y; b(\widetilde{U}) \right\rangle = \overline{b(\mathbf{U})}(y)$$

Convergence in Wasserstein distance

$$\int_Q |\mathbf{U}_n|^p \ \mathrm{d}y \leq c \text{ uniformly for } n = 1, 2, \ldots,\ p > 1$$

$$\mathbf{U}_n \overset{(S)}{\to} \mathcal{V} \ \Rightarrow \ \int_Q \left| d_{W_s} \left[\frac{1}{N} \sum_{n=1}^{N} \delta_{\mathbf{U}_n(y)}; \mathcal{V}_y \right] \right|^s \ \mathrm{d}y \to 0 \text{ as } N \to \infty,\ s < p$$

Basic properties of (S)–convergence, II

Statistically equivalent sequences

$$\{\mathbf{U}_n\}_{n=1}^{\infty} \overset{(S)}{\approx} \{\mathbf{V}_n\}_{n=1}^{\infty},$$

$$\Leftrightarrow \text{ for any } \varepsilon > 0$$

$$\frac{\#\left\{k \leq N \;\middle|\; \int_Q |\mathbf{U}_n - \mathbf{V}_n| \, \mathrm{d}y > \varepsilon\right\}}{N} \to 0 \text{ as } N \to \infty.$$

Robustness

$$\{\mathbf{U}_n\}_{n=1}^{\infty} \overset{(S)}{\approx} \{\mathbf{V}_n\}_{n=1}^{\infty} \;\Rightarrow\; \mathbf{U}_n \overset{(S)}{\to} \mathcal{V} \;\Leftrightarrow \mathbf{V}_n \overset{(S)}{\to} \mathcal{V}$$

Corollary

$$\mathbf{U}_n \to \mathbf{U} \text{ in } L^1(Q) \;\Rightarrow\; \mathbf{U}_n \overset{(S)}{\to} \delta_{\mathbf{U}(y)}$$

Basic properties of (S)–convergence III

Stationarity

$$\int_Q B(\mathbf{U}_{k_1}, \ldots, \mathbf{U}_{k_j})\mathrm{d}y = \int_Q B(\mathbf{U}_{k_1+n}, \ldots, \mathbf{U}_{k_j+n})\mathrm{d}y$$

Birkhoff–Khinchin Theorem

$$\{\mathbf{U}_n\}_{n=1}^{\infty} \text{ stationary, } b \in \mathcal{B}(R^d) \text{ Borel measurable } \int_Q b(\mathbf{U}_0) < \infty$$

$$\Rightarrow$$

$$\frac{1}{N}\sum_{n=1}^{N} b(\mathbf{U}_n) \text{ converges for a.a. } y \in Q$$

$$\Rightarrow$$

$\mathbf{U}_n$ is (S)–convergent

Asymptotically stationary consistent approximation

Asymptotically stationary sequence

$\{\mathbf{U}_n\}_{n=1}^{\infty}$ is *asymptotically stationary* if for any $b \in BC(R^D)$ there holds:

- **Correlation limit**

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_Q b(\mathbf{U}_n) b(\mathbf{U}_m)\ \mathrm{d}y \text{ exists}$$

  for any fixed m

- **Asymptotic correlation stationarity**

$$\left| \int_Q \Big[b(\mathbf{U}_{k_1}) b(\mathbf{U}_{k_2}) - b(\mathbf{U}_{k_1+n}) b(\mathbf{U}_{k_2+n}) \Big] \mathrm{d}y \right| \le \omega(b, k)$$

  for any $1 \le k \le k_1 \le k_2$, and any $n \ge 0$

$$\omega(b, k) \to 0 \text{ as } k \to \infty$$

Sufficient conditions for (S)–convergence

Asymptotically stationary sequence

$\{\mathbf{U}_n\}_{n=1}^{\infty}$ *asymptotically stationary* $\Rightarrow$ $\{\mathbf{U}_n\}_{n=1}^{\infty}$ (S)–convergent

Subsequence principle [Balder]

$$\int_Q F(|\mathbf{U}_n|)\, \mathrm{d}y \leq 1 \text{ uniformly for } n \to \infty,$$

$$F : [0, \infty) \to [0, \infty) \text{ continuous}, \ \lim_{r \to \infty} F(r) = \infty$$

$$\Rightarrow$$

there is an (S)–convergent subsequence $\{\mathbf{U}_{n_k}\}_{k=1}^{\infty}$

Application to consistent approximation of the Euler system

(S)–convergent consistent approximation

$$\mathbf{U}_n = [\varrho_n, \mathbf{m}_n, S_n] \;\; Q = (0, T) \times \Omega$$

$$\mathbf{U}_n \stackrel{(S)}{\to} \mathcal{V}$$

DMV solution

$\mathcal{V}$ is a dissipative measure valued solutions of the Euler system

Convergence in Wasserstein distance

$$\int_0^T \int_\Omega \left| d_{W_s} \left[\frac{1}{N} \sum_{n=1}^N \delta_{\mathbf{U}_n(y)}; \mathcal{V}_y \right] \right|^s \, \mathrm{d}x \, \mathrm{d}t \to 0 \text{ as } N \to \infty$$

$$1 \leq s < \frac{2\gamma}{\gamma + 1}$$

Deterministic convergence

Strong solution

Euler system admits strong solution $\Rightarrow \mathcal{V}_{(t,x)} = \delta_{[\varrho,\mathbf{m},S](t,x)}$

Regular limit

$$\left[\varrho = \langle \mathcal{V}; \widetilde{\varrho} \rangle, \ \mathbf{m} = \langle \mathcal{V}; \widetilde{\mathbf{m}} \rangle, \ S = \left\langle \mathcal{V}; \widetilde{S} \right\rangle \right] \in C^1$$

$$\Rightarrow$$

$$[\varrho, \mathbf{m}, S] \text{ strong solution of Euler, } \ \mathcal{V}_{(t,x)} = \delta_{[\varrho,\mathbf{m},S](t,x)}$$

Convergence to weak solution

$$\left[\varrho = \langle \mathcal{V}; \widetilde{\varrho} \rangle, \ \mathbf{m} = \langle \mathcal{V}; \widetilde{\mathbf{m}} \rangle, \ S = \left\langle \mathcal{V}; \widetilde{S} \right\rangle \right] \text{ weak solution to Euler system}$$

$$\Rightarrow$$

$$\mathcal{V}_{(t,x)} = \delta_{[\varrho,\mathbf{m},S](t,x)}$$

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