Long-time behaviour of open fluid systems and turbulence

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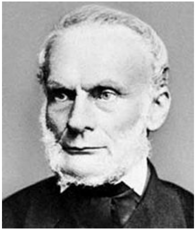
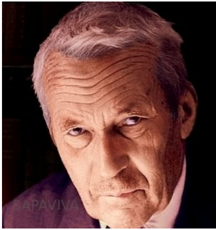
Motto



Rudolf Clasius
1822–1888

Basic principles of thermodynamics of closed systems

Die Energie der Welt ist constant. Die Entropie der Welt strebt einem Maximum zu.



Andrey Nikolaevich Kolmogorov
1903–1987

Turbulence - ergodic hypothesis

Time averages along trajectories of the flow converge, for large enough times, to an ensemble average given by a certain probability measure

Navier–Stokes–Fourier system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Newton's Second law (momentum balance)

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S} + \varrho \mathbf{g}$$

Second law of thermodynamics (entropy balance)

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \frac{1}{\vartheta} \left(\mathbb{S} : \mathbb{D}_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Newton's rheological law

$$\mathbb{S}(\vartheta, \mathbb{D}_x \mathbf{u}) = \mu(\vartheta) \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta(\vartheta) \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Fourier's law

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta) \nabla_x \vartheta$$

Boundary conditions

Closed systems

$$\textbf{impermeability:}\quad \mathbf{u}\cdot\mathbf{n}|_{\partial\Omega}=0,\ \textbf{no–slip:}\quad \mathbf{u}\times\mathbf{n}|_{\partial\Omega}=0$$

$$\textbf{thermal insulation:}\quad \mathbf{q}\cdot\mathbf{n}|_{\partial\Omega}=0$$

Open systems

$$\mathbf{u}|_{\partial\Omega}=\mathbf{u}_B,\ \textbf{inflow}\ \Gamma_{\rm in}:\mathbf{u}_B\cdot\mathbf{n}<0,\ \textbf{outflow}\ \Gamma_{\rm out}:\mathbf{u}_B\cdot\mathbf{n}>0$$

$$\varrho|_{\Gamma_{\rm in}}=\varrho_B$$

Boundary temperature (Dirichlet boundary conditions: $\vartheta=\vartheta_B$ on $\partial\Omega$

Problem with conservative boundary conditions

Conservative boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \ \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega=0}$$

Driving force

$$\varrho \mathbf{f}, \ \mathbf{f} = \mathbf{f}(x)$$

- $$\mathbf{f} = \nabla_x G, \ G = G(x) \ \Rightarrow \ \varrho \to \varrho_S, \ \vartheta \to \vartheta_S, \ \mathbf{m} = \varrho \mathbf{u} \to 0 \text{ as } t \to \infty$$

  $$\vartheta_S \text{ - a positive constant, } \nabla_x p(\varrho_S, \vartheta_S) = \varrho_S \nabla_x G$$

- $$\mathbf{f} \neq \nabla_x G \ \Rightarrow \ \int_\Omega \left[\frac{1}{\varrho} |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] \ \mathrm{d}x \to \infty \text{ as } t \to \infty$$

Rayleigh–Benard problem

Navier–Stokes–Fourier system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0,$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \varrho \nabla_x G,$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \frac{1}{\vartheta} \left(\mathbb{S} : \mathbb{D}_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Boundary conditions

$$\Omega = \mathbb{T}^2 \times (0,1)$$

$$\mathbf{u}|_{x_3=0} = \mathbf{u}|_{x_3=1} = 0,$$

$$\vartheta|_{x_3=0} = \Theta_B, \ \vartheta|_{x_3=1} = \Theta_U.$$

$$\mathbb{S}(\vartheta, \mathbb{D}_x \mathbf{u}) = \mu(\vartheta) \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta(\vartheta) \mathrm{div}_x \mathbf{u} \mathbb{I}$$

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta) \nabla_x \vartheta$$

Abstract theory

Global existence: The problem admits global–in–time solutions defined for all $t \geq t_0$ for any admissible data

- **Levinson dissipativity or bounded absorbing set.** Any global–in–time weak solution to the Navier–Stokes–Fourier system in a domain with impermeable boundary endowed with the Dirichlet boundary conditions for the temperature enters eventually a bounded absorbing set.
- **Asymptotic compactness.** Any bounded family of global solutions is precompact in a suitable topology of the trajectory space, whereas any of its accumulation points represents a weak solution of the same problem.

Equilibrium states – static solutions

$$\mathbf{u}_S = 0$$

$$\mathbf{u}_S = 0 \ \Rightarrow \ \mathrm{div}_x(\kappa(\vartheta_S)\nabla_x \vartheta_S) = 0, \ \vartheta_S|_{\partial\Omega} = \vartheta_B$$

$$\nabla_x p(\varrho_S, \vartheta_S) = \varrho_S \nabla_x G \ \Rightarrow \ \mathbf{curl}_x(\varrho_S \nabla_x G) = \nabla_x \varrho_S \times \nabla_x G = 0$$

$$\partial_\varrho p(\varrho_S, \vartheta_S)\nabla_x \varrho_S + \partial_\vartheta p(\varrho, \vartheta)\nabla_x \vartheta_S = \varrho_S \nabla_x G$$

$$\Rightarrow$$

$$\nabla_x \vartheta_S \times \nabla_x G = 0$$

Abstract setting

George Roger Sell
1937–2015

Space of entire trajectories

$$\mathcal{T} = C_{\rm loc}(R; X), \ \ t \in (-\infty, \infty)$$

ω**–limit set**

$$\omega[\mathbf{U}(\cdot, X_0)] \subset \mathcal{T}$$

$$\omega[\mathbf{U}(\cdot, X_0)] = \left\{ \mathbf{V} \in \mathcal{T} \ \middle| \ \mathbf{U}(\cdot + t_n, X_0) \to \mathbf{V} \text{ in } \mathcal{T} \text{ as } t_n \to \infty \right\}$$

Necessary ingredients

- **Dissipativity** – ultimate boundedness of trajectories
- **Compactness** – in appropriate spaces

Bounded absorbing set

Bounded absorbing set [EF - A. Świerczewska-Gwiazda]
For any global–in–time weak solution $(\varrho, \vartheta, \mathbf{u})$ defined on a time interval (T, ∞), there exists a constant $\mathcal{E}_\infty$ that depends only on the boundary data and the total mass of the fluid

$$M = \int_\Omega \varrho \ \mathrm{d}x,$$

such that

$$\mathrm{ess} \limsup_{t \to \infty} \int_\Omega E(\varrho, \vartheta, \mathbf{u})(t, \cdot) \ \mathrm{d}x \leq \mathcal{E}_\infty, \ E(\varrho, \vartheta, \mathbf{u}) \equiv \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta)$$

If, moreover,

$$\mathrm{ess} \limsup_{t \to T+} \int_\Omega E(\varrho, \vartheta, \mathbf{u})(t, \cdot) \ \mathrm{d}x \leq \mathcal{E}_0 < \infty,$$

then the convergence is uniform in $\mathcal{E}_0$. Specifically, for any $\varepsilon > 0$, there exists a time $T(\varepsilon, \mathcal{E}_0)$ such that

$$\mathrm{ess} \sup_{t > T(\varepsilon, \mathcal{E}_0)} \int_\Omega E(\varrho, \vartheta, \mathbf{u})(t, \cdot) \ \mathrm{d}x \leq \mathcal{E}_\infty + \varepsilon.$$

Asymptotic compactness

Asymptotic compactness [EF - A. Świerczewska-Gwiazda 2021]
Let $(\varrho_n, \vartheta_n, \mathbf{u}_n)_{n=1}^{\infty}$ be a sequence of weak solutions defined on the time intervals

$$(T_n, \infty),\ \ T_n \geq -\infty,\ \ T_n \to -\infty \text{ as } n \to \infty,$$

such that

$$\operatorname{ess}\sup_{t \to T_n} \int_{\Omega} E(\varrho_n, \vartheta_n, \mathbf{u}_n)(t, \cdot)\ \mathrm{d}x \leq \mathcal{E}_0.\ \int_{\Omega} \varrho\ \mathrm{d}x = M > 0,$$

Then there is a subsequence (not relabelled) such that

$$\varrho_n \to \varrho \text{ in } C_{\rm weak}([-M, M]; L^{\frac{5}{3}}(\Omega)) \cap C([-M, M]; L^1(\Omega)),$$
$$\vartheta_n \to \vartheta \text{ in } L^q((-M, M); L^4(\Omega)) \text{ for any } 1 \leq q < \infty,$$
$$\mathbf{u}_n \to \mathbf{u} \text{ weakly in } L^2((-M, M); W^{1,2}(\Omega; R^3))$$

for any $M > 0$, where the limit $(\varrho, \vartheta, \mathbf{u})$ is an entire weak solution defined for all $t \in R$ and satisfying

$$\int_{\Omega} E(\varrho, \vartheta, \mathbf{u})(t, \cdot)\ \mathrm{d}x \leq \mathcal{E}_{\infty} \text{ for a.a. } t \in R.$$

Trajectory space, attractor

Trajectory space

$$\mathcal{T} = \cup_{L=1}^{\infty} \mathcal{T}_L,$$

where

$$\mathcal{T}_L = \Big\{ (\varrho, S, \mathbf{m}) \ \Big| \ \varrho \in L^\infty(R; W^{-k,2}(\Omega)),\ \langle \varrho; \phi_n \rangle \in C(R),\ n = 1, 2, \dots,$$
$$\sup_{t \in R} \|\varrho(t, \cdot)\|_{W^{-k,2}(\Omega)} \leq L,$$
$$\mathbf{m} \in L^\infty(R; W^{-k,2}(\Omega; R^3)),\ \langle \mathbf{m}; \boldsymbol{\varphi}_n \rangle \in C(R),\ n = 1, 2, \dots,$$
$$\sup_{t \in R} \|\mathbf{m}(t, \cdot)\|_{W^{-k,2}(\Omega; R^3)} \leq L,$$
$$S \in L^\infty(R; W^{-k,2}(\Omega)),\ \langle S; \phi_n \rangle \ \text{càglàd in } R,\ n = 1, 2, \dots,$$
$$\sup_{t \in R} \|S(t, \cdot)\|_{W^{-k,2}(\Omega)} \leq L \Big\}.$$

Attractor

$$\mathcal{A} = \Big\{ (\varrho, S, \mathbf{m}) \ \Big| \ (\varrho, S, \mathbf{m}) \text{ a weak solution of the Navier–Stokes–Fourier system}$$
$$\text{on the time interval } t \in R \Big\},$$

Attractor

Trajectory attractor [EF - A. Świerczewska-Gwiazda 2021]
Let $M > 0$, $\mathcal{E}_0$ be given. Let $\mathcal{F}[M, \mathcal{E}_0]$ be a family of weak solutions to the Rayleigh–Bénard problem for the Navier–Stokes–Fourier system on the time interval $(0, \infty)$ satisfying

$$\int_\Omega \varrho \ \mathrm{d}x = M, \ \mathrm{ess} \limsup_{\tau \to 0+} \int_\Omega E(\varrho, S, \mathbf{m})(\tau, \cdot) \ \mathrm{d}x \leq \mathcal{E}_0.$$

We identify the set $\mathcal{F}[M, \mathcal{E}_0]$ with a subset of the trajectory space $\mathcal{T}$ extending them by constant values for $\tau < 0$.

Then for any $\varepsilon > 0$, there exists a time $T(\varepsilon)$ such that

$d_{\mathcal{T}}[(\varrho, S, \mathbf{m})(\cdot + T); \mathcal{A}] < \varepsilon$ for any $(\varrho, S, \mathbf{m}) \in \mathcal{F}[M, \mathcal{E}_0]$ and any $T > T(\varepsilon)$.

Statistical solutions

$$\mathbf{U}: \ t \in R \mapsto [\varrho(t,\cdot), \mathbf{m}(t,\cdot), S(t,\cdot)] \in X$$

Statistical solution

- **U** random process with continuous (càglàd) trajectories
- **U** solves the Rayleigh–Benard problem a.s.

equivalently (canonical representation)

$\mathcal{V}$ -probability measure on the trajectory space $\mathcal{T}$

$\mathrm{supp}[\mathcal{V}] \subset$ solutions of Rayleigh–Benard problem

- probability space $(\mathcal{T}, \mathcal{V})$
- $t \mapsto \mathbf{U}(t)$, $\mathbf{U} \in \mathcal{T}$

Stationary statistical solutions

Stationary statistical solutions [EF - A. Świerczewska-Gwiazda 2021]
Let $\mathcal{U} \subset \mathcal{A}$ be a non-empty time–shift invariant set, meaning

$$(\varrho, S, \mathbf{m}) \in \mathcal{U} \ \Rightarrow \ (\varrho, S, \mathbf{m})(\cdot + T) \in \mathcal{U} \text{ for any } T \in R.$$

Then there exists a stationary statistical solution $\mathcal{V}$ supported by $\overline{\mathcal{U}}$:

- $\mathcal{V}$ is a Borel probability measure, $\mathcal{V} \in \mathfrak{P}(\overline{\mathcal{U}})$;
- $\text{supp}\mathcal{V} \subset \overline{\mathcal{U}}$, where the closure of a $\mathcal{U}$ is a compact invariant set;
- $\mathcal{V}$ is shift invariant, i.e., $\mathcal{V}[\mathfrak{B}] = \mathcal{V}[\mathfrak{B}(\cdot + T)]$ for any Borel set $\mathfrak{B} \subset \mathcal{T}$ and any $T \in R$.

Statistical stationary solutions

Application of Krylov – Bogolyubov method

$$\frac{1}{T_n}\int_0^{T_n} \delta_{\varrho(\cdot+t,\cdot),\mathbf{m}(\cdot+t,\cdot),S(\cdot+t,\cdot)}\ \mathrm{d}t \to \mu \in \mathcal{P}[\mathcal{T}] \text{ narrowly}$$

$[\mathcal{T},\mu]$ (canonical representation) – statististical stationary solution

$\mu(t)|_X$ (marginal) independent of $t \in R$

Application of Birkhoff – Khinchin ergodic theorem

$$\frac{1}{T}\int_0^{T} F(\varrho(t,\cdot),\mathbf{m}(t,\cdot),S(t\cdot))\mathrm{d}t \to \overline{F} \text{ as } T \to \infty$$

F bounded Borel measurable on X for $\mu -$ a.a. $(\varrho,\mathbf{m}) \in \omega$

Strong and weak ergodic hypothesis

Krylov – Bogolyubov construction

$$T \mapsto \frac{1}{T}\int_0^T \delta_{\mathbf{U}(\cdot+t,X_0)}\mathrm{d}t \text{ -- a family of probability measures on } \mathcal{T}$$

$$\text{tightness in } \mathcal{T} \ \Rightarrow \ T_n \mapsto \frac{1}{T_n}\int_0^{T_n} \delta_{\mathbf{U}(\cdot+t,X_0)}\mathrm{d}t \to \mu \in \mathcal{P}[\mathcal{T}]$$

$[\mathcal{T}, \mu]$ stationary statistical solution

Ergodic hypothesis $\Leftrightarrow$ μ is unique $\Rightarrow$ $T \mapsto \frac{1}{T}\int_0^T \delta_{\mathbf{U}(\cdot+t,X_0)}\mathrm{d}t \to \mu$

unique $\approx$ unique on $\omega[\mathbf{U}(\cdot, X_0)]$

Weak ergodic hypothesis

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \delta_{\mathbf{U}(\cdot+t,X_0)}\mathrm{d}t = \mu \text{ exists in the narrow sense in } \mathcal{P}[\mathcal{T}]$$

$[\mathcal{T}, \mu]$ stationary statistical solution

Ergodic means

Phase space

$$H = W^{-k,2}(\Omega) \times W^{-k,2}(\Omega) \times W^{-k,2}(\Omega; R^3).$$

Convergence of ergodic means [application of Birkhoff–Khinchin ergodic theorem]
Let $\mathcal{V}$ be a stationary statistical solution and $(\varrho, S, \mathbf{m})$ the associated stationary process. Let $F : H \to R$ be a Borel measurable function such that

$$\int_{\mathcal{T}} |F(\varrho(0,\cdot), S(0,\cdot), \mathbf{m}(0,\cdot)| \ \mathrm{d}\mathcal{V} < \infty.$$

Then there exists a measurable function $\overline{F}$,

$$\overline{F} : (\mathcal{T}, \mathcal{V}) \to R$$

such that

$$\frac{1}{T} \int_0^T F(\varrho(t,\cdot), S(t,\cdot), \mathbf{m}(t,\cdot)) \mathrm{d}t \to \overline{F} \text{ as } T \to \infty$$

$\mathcal{V}-$a.s. and in $L^1(\mathcal{T}, \mathcal{V})$.