(S)-convergence and approximation of oscillatory solutions in fluid dynamics

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Example I: Strong Law of Large numbers

Strong Law of Large Numbers

$$\{U_n\}_{n=1}^{\infty} \ \boxed{\textit{independent}} \text{ random variables, } \ E(U_n) = \mu$$

$$\Rightarrow$$

$$\frac{1}{N}\sum_{n=1}^{N} U_n \to \mu \text{ as } N \to \infty \text{a.s.}$$

Subsequence principle: Komlos (Banach–Saks) theorem

$$\int_{\Omega} |U_n| \, \mathrm{d}x \leq c \text{ uniformly for } n \to \infty$$

$$\Rightarrow$$

there is a subsequence $\{U_{n_k}\}_{k=1}^{\infty}$ such that

$$\frac{1}{N}\sum_{l=1}^{N} U_{n_l} \to U \in L^1(\Omega) \text{ as } N \to \infty \text{ a.a. in } \Omega$$

for any subsequence $\{n_l\} \subset \{n_k\}$

Example II: Ergodic hypothesis

Asymptotic behavior of dynamical systems

$$t \in [0,\infty) \mapsto \mathbf{U}(t) \in X,$$

ω-limit set

$$\omega[\mathbf{U}] = \left\{ \mathbf{u} \in X \;\middle|\; \text{there exists } t_n \to \infty \ \mathbf{U}(t_n) \to \mathbf{u} \right\}$$

Ergodic hypothesis

$$\frac{1}{T}\int_0^T F(\mathbf{U}(t))\mathrm{d}t \to \overline{F} \text{ as } T \to \infty \text{ for any Borel } F \in \mathcal{B}(X;R)$$

Birkhoff–Khinchin ergodic theorem

$$\mathbf{U}(t) : R \to X \text{ stationary process } \Rightarrow \frac{1}{T}\int_0^T F(\mathbf{U}(t))\mathrm{d}t \to \overline{F} \text{ a.s.}$$

(S) - convergence, basic idea

Trivial example of oscillatory sequence

$$U_n = \begin{cases} 1 \text{ for } n \text{ odd} \\ -1 \text{ for } n \text{ even} \end{cases}$$

Convergence via Young measure approach

Convergence up to a *subsequence*

$$U_n \approx \delta_{U_n}, \ U_{n_k} \to \begin{cases} \delta_1 \text{ as } k \to \infty, \ n_k \text{ odd} \\ \delta_{-1} \text{ as } k \to \infty, \ n_k \text{ even} \end{cases}$$

Convergence via averaging

$$U_n \approx \delta_{U_n}, \ \frac{1}{N} \sum_{n=1}^{N} U_n \to \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$$

$$\frac{1}{w_N} \sum_{n=1}^{N} w\left(\frac{n}{N}\right) U_n \to \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1, \ w_N \equiv \sum_{n=1}^{N} w\left(\frac{n}{N}\right)$$

Euler system of gas dynamics

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

Momentum balance – Newton's Second Law

$$\partial_t \mathbf{m} + \mathrm{div}_x \left[\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right] + \nabla_x p = 0$$

Energy balance – First Law of Thermodynamics

$$\partial_t E + \mathrm{div}_x \left[(E + p) \frac{\mathbf{m}}{\varrho} \right] = 0$$

Boundary conditions

$$\mathbf{m} \cdot \mathbf{n}|_{\partial \Omega} = 0$$

Constitutive relations – Second Law of Thermodynamics

Pressure, internal energy, entropy

$$E = \underbrace{\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}}_{\text{kinetic energy}} + \underbrace{\varrho e}_{\text{internal energy}}, \quad \underbrace{p = (\gamma - 1)\varrho e}_{\text{EOS (incomplete)}}$$

Entropy

$$s = \mathcal{S}\left(\frac{p}{\varrho^{\gamma}}\right), \quad \underbrace{S = \varrho s}_{\text{total entropy}}$$

Boyle–Mariotte Law:

$$p = \varrho \vartheta, \ \ e = \frac{1}{\gamma - 1}\vartheta, \ \ s = \log\left(\frac{\vartheta^{\frac{1}{\gamma-1}}}{\varrho}\right)$$

Entropy balance (inequality) – Second Law of Thermodynamics

$$\partial_t S + \mathrm{div}_x\left[S\frac{\mathbf{m}}{\varrho}\right] = (\geq) 0$$

Thermodynamic stability

Conservative–entropy variables

density ϱ, momentum $\mathbf{m}$, total entropy S, $[\varrho, \mathbf{m}, S]$

Thermodynamic stability – energy

$$E = E(\varrho, \mathbf{m}, S) : R^{d+2} \to [0, \infty]$$

$$E = (\varrho, \mathbf{m}, S) = \infty \text{ if } \varrho < 0,\ E(0, \mathbf{m}, S) = \lim_{\varrho \to 0+} E(\varrho, \mathbf{m}, S)$$

convex, lower semi–continuous on R^{d+2}

Thermodynamic stability in standard variables

positive compressibility $\dfrac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0$

positive specific heat at constant volume $\dfrac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$

Known facts about solvability of Euler system

Classical solutions

Smooth initial state gives rise to smooth solution on a (generically) finite time interval $T_{\max}$, singularities (shocks) develop after $T_{\max}$

Weak solutions

Admissible (weak + entropy inequality) weak solutions exist globally in time. There is a "vast" class of initial data for which the problem admits infinitely many admissible weak solutions, **the system is ill–posed in the class of admissible weak solutions**

Generalized - oscillatory solutions

There are various concepts of generalized solutions: measure–valued solutions, dissipative measure–valued solutions, etc. They can be seen as limits of *consistent* approximations. They are **inseparable from the process** how they were obtained.

Consistent approximation

Approximate field equations (in the distributional sense)

$$\partial_t \varrho_n + \mathrm{div}_x \mathbf{m}_n = e_n^1$$

$$\partial_t \mathbf{m}_n + \mathrm{div}_x \left[\frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} \right] + \nabla_x p(\varrho_n, S_n) = e_n^2$$

$$\partial_t E(\varrho_n, \mathbf{m}_n, S_n) + \mathrm{div}_x \left[(E + p)(\varrho_n, \mathbf{m}_n, S_n) \frac{\mathbf{m}_n}{\varrho_n} \right] = e_n^3$$

$$\partial_t S_n + \mathrm{div}_x \left[S_n \frac{\mathbf{m}_n}{\varrho_n} \right] \geq e_n^4$$

Vanishing consistency errors

$$e_n^1,\ e_n^2,\ e_n^4 \to 0 \text{ in the distributional sense}$$

$$\int_\Omega e_n^3 \ \mathrm{d}x \to 0 \text{ uniformly in time}$$

Stability:

$$\int_\Omega E(\varrho_n, \mathbf{m}_n, S_n) \ \mathrm{d}x \leq c,\ s_n = \frac{S_n}{\varrho_n} \geq -c \ \text{ uniformly in time}$$

Consistent approximation - basic properties

Examples of consistent approximations

- **Vanishing dissipation limit** from the Navier–Stokes–Fourier system to the Euler system
- **Limits of entropy (energy) preserving numerical schemes,** Lax–Friedrichs scheme, Rusanov scheme, Brenner model based scheme (EF, M.Lukáčová, H. Mizerová)

Convergence of consistent approximation

- $$\varrho_{n_k} \to \varrho,\ S_{n_k} \to S \text{ weakly-(*) in } L^\infty(0,T;L^\gamma(\Omega))$$
  $$\mathbf{m}_{n_k} \to \mathbf{m} \text{ weakly-(*) in } L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega;R^d))$$
- the limit $[\varrho, \mathbf{m}, S]$ is a generalized (dissipative) solution of the Euler system
- $\{\varrho_{n_k}, \mathbf{m}_{n_k}, S_{n_k}\} \approx \{\delta_{\varrho_{n_k}, \mathbf{m}_{n_k}, S_{n_k}}\}$ generates a Young measure

up to a suitable subsequence!

Convergence of consistent approximation

Strong convergence

- **Strong convergence to strong solution (uncoditional)**
  Euler system admits a smooth solution $\Rightarrow [\varrho, \mathbf{m}, S]$ is the unique smooth solution and convergence is strong and unconditional (no need for subsequence) in L^1
- **Strong convergence to smooth limit (unconditional)**
  The limit $[\varrho, \mathbf{m}, S]$ is of class $C^1 \Rightarrow$ the limit is the unique strong solution of the Euler system and convergence is strong and unconditional (no need for subsequence) in L^1
- **Strong convergence to weak solution (up to a subsequence)**
  EF, M.Hofmanová (2019):

  The limit $[\varrho, \mathbf{m}, S]$ is a weak solution of the Euler system $\Rightarrow$ convergence is strong in L^1

Weak convergence of consistent approximation

Weak convergence

If consistent approximation DOES NOT converge strongly, the following must be satisfied:

- the limit Euler system does not admit a strong solution
- the limit $[\varrho, S, \mathbf{m}]$ is not C^1 smooth
- the limit $[\varrho, S, \mathbf{m}]$ IS NOT a weak solution of the Euler system

Visualization of weak convergence?

- **Oscillations.** Weakly converging sequence may develop oscillations. Example:

$$\sin(nx) \to 0 \text{ weakly as } n \to \infty$$

- **Concentrations.**

$$n\theta(nx) \to \delta_0 \text{ weakly-(*) in } \mathcal{M}(R)$$

if

$$\theta \in C_c^\infty(R),\ \theta \geq 0, \int_R \theta = 1$$

Statistical description of oscillations – Young measures

Young measure

$$b(\varrho_n, \mathbf{m}_n, S_n) \to \overline{b(\varrho, \mathbf{m}, S)} \text{ weakly-(*) in } L^\infty((0,T) \times \Omega)$$

(up to a subsequence) for any $b \in C_c(R^{d+2})$

Young measure $\mathcal{V}$ – a parametrized family of probability measures $\{\mathcal{V}_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$ on the phase space R^{d+2}:

$$\overline{b(\varrho, \mathbf{m}, S)}(t,x) = \left\langle \mathcal{V}_{t,x}; b(\widetilde{\varrho}, \widetilde{\mathbf{m}}, \widetilde{S}) \right\rangle \text{ for a.a. } (t,x)$$

Visualizing Young measure

visualizing Young measure $\Leftrightarrow$ computing $\overline{b(\varrho, \mathbf{m}, S)}$

Problems

- $b(\varrho_n, \mathbf{m}_n, S_n)$ converge only weakly
- extracting subsequences
- only statistical properties relevant $\Rightarrow$ knowledge of the "tail" of the sequence of approximate solutions absolutely necessary

(S)–convergence

(S)–convergent approximate sequence

An approximate sequence $\{\mathbf{U}_n\}_{n=1}^{\infty}$ is $\boxed{(S) - convergent}$ if for any $b \in C_c(R^D)$:

- **Correlation limit**

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_Q b(\mathbf{U}_n) b(\mathbf{U}_m) \mathrm{d}y \text{ exists for any fixed } m$$

- **Correlation disintegration**

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{n,m=1}^{N} \int_Q b(\mathbf{U}_n) b(\mathbf{U}_m)\ \mathrm{d}y$$
$$= \lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{N} \left(\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_Q b(\mathbf{U}_n) b(\mathbf{U}_m)\ \mathrm{d}y \right)$$

Basic properties of (S)–convergence, I

Equivalence to convergence of ergodic (Cesàro means)

$$\{\mathbf{U}_n\}_{n=1}^{\infty} \text{ (S)–convergent } \Leftrightarrow \frac{1}{N}\sum_{n=1}^{N} b(\mathbf{U}_n) \to \overline{b(\mathbf{U})} \ \boxed{\textit{strongly}} \text{ in } L^1(Q)$$

(S)– limit (parametrized measure)

$$\mathbf{U}_n \overset{(S)}{\to} \mathcal{V},\ \{\mathcal{V}_y\}_{y\in Q},\ \mathcal{V}_y \in \mathfrak{P}(R^D),\ \left\langle \mathcal{V}_y; b(\widetilde{U}) \right\rangle = \overline{b(\mathbf{U})}(y)$$

Convergence in Wasserstein distance

$$\int_Q |\mathbf{U}_n|^p \,\mathrm{d}y \le c \text{ uniformly for } n = 1, 2, \dots,\ p > 1$$

$$\mathbf{U}_n \overset{(S)}{\to} \mathcal{V} \ \Rightarrow \ \int_Q \left| d_{W_s}\left[\frac{1}{N}\sum_{n=1}^{N} \delta_{\mathbf{U}_n(y)}; \mathcal{V}_y \right] \right|^s \mathrm{d}y \to 0 \text{ as } N \to \infty,\ s < p$$

Basic properties of (S)–convergence, II

Statistically equivalent sequences

$$\{\mathbf{U}_n\}_{n=1}^{\infty} \overset{(S)}{\approx} \{\mathbf{V}_n\}_{n=1}^{\infty},$$

$$\Leftrightarrow \ \text{for any } \varepsilon > 0$$

$$\frac{\# \left\{ k \leq N \ \middle| \ \int_Q |\mathbf{U}_n - \mathbf{V}_n| \ \mathrm{d}y > \varepsilon \right\}}{N} \to 0 \text{ as } N \to \infty.$$

Robustness

$$\{\mathbf{U}_n\}_{n=1}^{\infty} \overset{(S)}{\approx} \{\mathbf{V}_n\}_{n=1}^{\infty} \ \Rightarrow \ \mathbf{U}_n \overset{(S)}{\to} \mathcal{V} \ \Leftrightarrow \mathbf{V}_n \overset{(S)}{\to} \mathcal{V}$$

Corollary

$$\mathbf{U}_n \to \mathbf{U} \text{ in } L^1(Q) \ \Rightarrow \ \mathbf{U}_n \overset{(S)}{\to} \delta_{\mathbf{U}(y)}$$

Basic properties of (S)–convergence III

Stationarity

$$\int_Q B(\mathbf{U}_{k_1}, \ldots, \mathbf{U}_{k_j})\mathrm{d}y = \int_Q B(\mathbf{U}_{k_1+n}, \ldots, \mathbf{U}_{k_j+n})\mathrm{d}y$$

Birkhoff–Khinchin Theorem

$$\{\mathbf{U}_n\}_{n=1}^{\infty} \text{ stationary, } b \in \mathcal{B}(R^d) \text{ Borel measurable } \int_Q b(\mathbf{U}_0) < \infty$$

$$\Rightarrow$$

$$\frac{1}{N}\sum_{n=1}^{N} b(\mathbf{U}_n) \text{ converges for a.a. } y \in Q$$

$$\Rightarrow$$

$\mathbf{U}_n$ is (S)–convergent

Asymptotically stationary consistent approximation

Asymptotically stationary sequence

$\{\mathbf{U}_n\}_{n=1}^{\infty}$ is *asymptotically stationary* if for any $b \in BC(R^D)$ there holds:

- **Correlation limit**

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_Q b(\mathbf{U}_n) b(\mathbf{U}_m) \, \mathrm{d}y \text{ exists}$$

  for any fixed m

- **Asymptotic correlation stationarity**

$$\left| \int_Q \left[b(\mathbf{U}_{k_1}) b(\mathbf{U}_{k_2}) - b(\mathbf{U}_{k_1+n}) b(\mathbf{U}_{k_2+n}) \right] \mathrm{d}y \right| \leq \omega(b, k)$$

  for any $1 \leq k \leq k_1 \leq k_2$, and any $n \geq 0$

$$\omega(b, k) \to 0 \text{ as } k \to \infty$$

Sufficient conditions for (S)–convergence

Asymptotically stationary sequence

$\{\mathbf{U}_n\}_{n=1}^{\infty}$ *asymptotically stationary* $\Rightarrow$ $\{\mathbf{U}_n\}_{n=1}^{\infty}$ (S)–convergent

Subsequence principle [Balder]

$$\int_Q F(|\mathbf{U}_n|)\,\mathrm{d}y \leq 1 \text{ uniformly for } n \to \infty,$$

$$F : [0,\infty) \to [0,\infty) \text{ continuous}, \ \lim_{r\to\infty} F(r) = \infty$$

$$\Rightarrow$$

there is an (S)–convergent subsequence $\{\mathbf{U}_{n_k}\}_{k=1}^{\infty}$

Application to consistent approximation of the Euler system

(S)–convergent consistent approximation

$$\mathbf{U}_n = [\varrho_n, \mathbf{m}_n, S_n] \;\; Q = (0, T) \times \Omega$$

$$\mathbf{U}_n \overset{(S)}{\to} \mathcal{V}$$

DMV solution

$\mathcal{V}$ is a dissipative measure valued solutions of the Euler system

Convergence in Wasserstein distance

$$\int_0^T \int_\Omega \left| d_{W_s} \left[\frac{1}{N} \sum_{n=1}^N \delta_{\mathbf{U}_n(y)}; \mathcal{V}_y \right] \right|^s \mathrm{d}x \; \mathrm{d}t \to 0 \text{ as } N \to \infty$$

$$1 \leq s < \frac{2\gamma}{\gamma + 1}$$

Deterministic convergence

Strong solution

Euler system admits strong solution $\Rightarrow \mathcal{V}_{(t,x)} = \delta_{[\varrho,\mathbf{m},S](t,x)}$

Regular limit

$$\left[\varrho = \langle \mathcal{V}; \widetilde{\varrho} \rangle\,,\ \mathbf{m} = \langle \mathcal{V}; \widetilde{\mathbf{m}} \rangle\,,\ S = \left\langle \mathcal{V}; \widetilde{S} \right\rangle \right] \in C^1$$

$$\Rightarrow$$

$$[\varrho, \mathbf{m}, S] \text{ strong solution of Euler},\ \mathcal{V}_{(t,x)} = \delta_{[\varrho,\mathbf{m},S](t,x)}$$

Convergence to weak solution

$$\left[\varrho = \langle \mathcal{V}; \widetilde{\varrho} \rangle\,,\ \mathbf{m} = \langle \mathcal{V}; \widetilde{\mathbf{m}} \rangle\,,\ S = \left\langle \mathcal{V}; \widetilde{S} \right\rangle \right] \text{ weak solution to Euler system}$$

$$\Rightarrow$$

$$\mathcal{V}_{(t,x)} = \delta_{[\varrho,\mathbf{m},S](t,x)}$$

Improving convergence criteria

Statistical significant set

$$\mathcal{S} \subset N \times N$$

$$\frac{\#\left\{ n, m \leq N \middle| \ (n,m) \in S \right\}}{N^2} \to 0 \text{ as } N \to \infty$$

Asymptotic stationarity

$$\|U_n\|_{L^\infty(Q)} \leq C$$

for any $\varepsilon > 0$ there is k, L such that

$$\left| \int_Q (U_{n+l} - U)(U_n - U) - (U_{k+l} - U)(U_k - U) \ \mathrm{d}y \right| < \varepsilon$$

for all $n \geq k, \ l \geq L, \ (n+l, n) \in \mathcal{S}$

$$\Rightarrow$$

$$\frac{1}{N} \sum_{n=1}^{N} U_n \to U \text{ in } L^1(Q) \text{ as } N \to \infty$$