

(S)-convergence and approximation of oscillatory solutions in fluid dynamics

Eduard Feireisl

based on joint work with M. Hofmanová (TU Bielefeld), M. Lukáčová (JGU Mainz), H. Mizerová (KU Bratislava), B. She (IM Praha)

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague
Technische Universität Berlin

Oberseminar Numerik, JGU Mainz, 14 October 2020



Einstein Stiftung Berlin
Einstein Foundation Berlin



Example I: Strong Law of Large numbers

Strong Law of Large Numbers

$\{U_n\}_{n=1}^{\infty}$ independent random variables, $E(U_n) = \mu$

\Rightarrow

$$\frac{1}{N} \sum_{n=1}^N U_n \rightarrow \mu \text{ as } N \rightarrow \infty \text{ a.s.}$$

Subsequence principle: Komlos (Banach–Saks) theorem

$$\int_{\Omega} |U_n| \, dx \leq c \text{ uniformly for } n \rightarrow \infty$$

\Rightarrow

there is a subsequence $\{U_{n_k}\}_{k=1}^{\infty}$ such that

$$\frac{1}{N} \sum_{l=1}^N U_{n_l} \rightarrow U \in L^1(\Omega) \text{ as } N \rightarrow \infty \text{ a.a. in } \Omega$$

for any subsequence $\{n_l\} \subset \{n_k\}$

Example II: Ergodic hypothesis

Asymptotic behavior of dynamical systems

$$t \in [0, \infty) \mapsto \mathbf{U}(t) \in X,$$

ω -limit set

$$\omega[\mathbf{U}] = \left\{ \mathbf{u} \in X \mid \text{there exists } t_n \rightarrow \infty \text{ } \mathbf{U}(t_n) \rightarrow \mathbf{u} \right\}$$

Ergodic hypothesis

$$\frac{1}{T} \int_0^T F(\mathbf{U}(t)) dt \rightarrow \bar{F} \text{ as } T \rightarrow \infty \text{ for any Borel } F \in \mathcal{B}(X; R)$$

Birkhoff–Khinchin ergodic theorem

$$\mathbf{U}(t) : R \rightarrow X \text{ stationary process} \Rightarrow \frac{1}{T} \int_0^T F(\mathbf{U}(t)) dt \rightarrow \bar{F} \text{ a.s.}$$

(S) - convergence, basic idea

Trivial example of oscillatory sequence

$$U_n = \begin{cases} 1 & \text{for } n \text{ odd} \\ -1 & \text{for } n \text{ even} \end{cases}$$

Convergence via Young measure approach

Convergence up to a subsequence

$$U_n \approx \delta_{U_n}, U_{n_k} \rightarrow \begin{cases} \delta_1 & \text{as } k \rightarrow \infty, n_k \text{ odd} \\ \delta_{-1} & \text{as } k \rightarrow \infty, n_k \text{ even} \end{cases}$$

Convergence via averaging

$$U_n \approx \delta_{U_n}, \frac{1}{N} \sum_{n=1}^N U_n \rightarrow \frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_1$$

$$\frac{1}{w_N} \sum_{n=1}^N w\left(\frac{n}{N}\right) U_n \rightarrow \frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_1, w_N \equiv \sum_{n=1}^N w\left(\frac{n}{N}\right)$$

Euler system of gas dynamics

Mass conservation

$$\partial_t \varrho + \operatorname{div}_x \mathbf{m} = 0$$

Momentum balance – Newton's Second Law

$$\partial_t \mathbf{m} + \operatorname{div}_x \left[\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right] + \nabla_x p = 0$$

Energy balance – First Law of Thermodynamics

$$\partial_t E + \operatorname{div}_x \left[(E + p) \frac{\mathbf{m}}{\varrho} \right] = 0$$

Boundary conditions

$$\mathbf{m} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Constitutive relations – Second Law of Thermodynamics

Pressure, internal energy, entropy

$$E = \underbrace{\frac{1}{2} \frac{|\mathbf{m}|^2}{\rho}}_{\text{kinetic energy}} + \underbrace{\rho e}_{\text{internal energy}}, \quad \underbrace{p = (\gamma - 1)\rho e}_{\text{EOS (incomplete)}}$$

Entropy

$$s = S \left(\frac{p}{\rho^\gamma} \right), \quad \underbrace{S = \rho s}_{\text{total entropy}}$$

Boyle–Mariotte Law:

$$p = \rho \vartheta, \quad e = \frac{1}{\gamma - 1} \vartheta, \quad s = \log \left(\frac{\vartheta^{\frac{1}{\gamma-1}}}{\rho} \right)$$

Entropy balance (inequality) – Second Law of Thermodynamics

$$\partial_t S + \operatorname{div}_x \left[S \frac{\mathbf{m}}{\rho} \right] = (\geq) 0$$

Thermodynamic stability

Conservative–entropy variables

density ϱ , momentum \mathbf{m} , total entropy S , $[\varrho, \mathbf{m}, S]$

Thermodynamic stability – energy

$$E = E(\varrho, \mathbf{m}, S) : R^{d+2} \rightarrow [0, \infty]$$

$$E(\varrho, \mathbf{m}, S) = \infty \text{ if } \varrho < 0, \quad E(0, \mathbf{m}, S) = \lim_{\varrho \rightarrow 0^+} E(\varrho, \mathbf{m}, S)$$

convex, lower semi–continuous on R^{d+2}

Thermodynamic stability in standard variables

$$\text{positive compressibility } \frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0$$

$$\text{positive specific heat at constant volume } \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

Known facts about solvability of Euler system

Classical solutions

Smooth initial state gives rise to smooth solution on a (generically) finite time interval T_{\max} , singularities (shocks) develop after T_{\max}

Weak solutions

Admissible (weak + entropy inequality) weak solutions exist globally in time. There is a “vast” class of initial data for which the problem admits infinitely many admissible weak solutions, **the system is ill-posed in the class of admissible weak solutions**

Generalized - oscillatory solutions

There are various concepts of generalized solutions: measure-valued solutions, dissipative measure-valued solutions, etc. They can be seen as limits of *consistent* approximations. They are **inseparable from the process** how they were obtained.

Consistent approximation

Approximate field equations (in the distributional sense)

$$\partial_t \varrho_n + \operatorname{div}_x \mathbf{m}_n = e_n^1$$

$$\partial_t \mathbf{m}_n + \operatorname{div}_x \left[\frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} \right] + \nabla_x p(\varrho_n, S_n) = e_n^2$$

$$\partial_t E(\varrho_n, \mathbf{m}_n, S_n) + \operatorname{div}_x \left[(E + p)(\varrho_n, \mathbf{m}_n, S_n) \frac{\mathbf{m}_n}{\varrho_n} \right] = e_n^3$$

$$\partial_t S_n + \operatorname{div}_x \left[S_n \frac{\mathbf{m}_n}{\varrho_n} \right] \geq e_n^4$$

Vanishing consistency errors

$e_n^1, e_n^2, e_n^4 \rightarrow 0$ in the distributional sense

$$\int_{\Omega} e_n^3 \, dx \rightarrow 0 \text{ uniformly in time}$$

Stability:

$$\int_{\Omega} E(\varrho_n, \mathbf{m}_n, S_n) \, dx \leq c, \quad s_n = \frac{S_n}{\varrho_n} \geq -c \text{ uniformly in time}$$

Consistent approximation - basic properties

Examples of consistent approximations

- **Vanishing dissipation limit** from the Navier–Stokes–Fourier system to the Euler system
- **Limits of entropy (energy) preserving numerical schemes**, Lax–Friedrichs scheme, Rusanov scheme, Brenner model based scheme (EF, M.Lukáčová, H. Mizerová)

Convergence of consistent approximation

- $$\varrho_{n_k} \rightarrow \varrho, S_{n_k} \rightarrow S \text{ weakly-} (*) \text{ in } L^\infty(0, T; L^\gamma(\Omega))$$
$$\mathbf{m}_{n_k} \rightarrow \mathbf{m} \text{ weakly-} (*) \text{ in } L^\infty(0, T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d))$$
- the limit $[\varrho, \mathbf{m}, S]$ is a generalized (dissipative) solution of the Euler system
- $\{\varrho_{n_k}, \mathbf{m}_{n_k}, S_{n_k}\} \approx \{\delta_{\varrho_{n_k}, \mathbf{m}_{n_k}, S_{n_k}}\}$ generates a Young measure
up to a suitable subsequence!

Convergence of consistent approximation

Strong convergence

- **Strong convergence to strong solution (unconditional)**

Euler system admits a smooth solution $\Rightarrow [\varrho, \mathbf{m}, S]$ is the unique smooth solution and convergence is strong and unconditional (no need for subsequence) in L^1

- **Strong convergence to smooth limit (unconditional)**

The limit $[\varrho, \mathbf{m}, S]$ is of class $C^1 \Rightarrow$ the limit is the unique strong solution of the Euler system and convergence is strong and unconditional (no need for subsequence) in L^1

- **Strong convergence to weak solution (up to a subsequence)**

EF, M.Hofmanová (2019):

The limit $[\varrho, \mathbf{m}, S]$ is a weak solution of the Euler system \Rightarrow convergence is strong in L^1

Weak convergence of consistent approximation

Weak convergence

If consistent approximation DOES NOT converge strongly, the following must be satisfied:

- the limit Euler system does not admit a strong solution
- the limit $[\varrho, S, \mathbf{m}]$ is not C^1 smooth
- the limit $[\varrho, S, \mathbf{m}]$ IS NOT a weak solution of the Euler system

Visualization of weak convergence?

- **Oscillations.** Weakly converging sequence may develop oscillations.
Example:

$$\sin(nx) \rightarrow 0 \text{ weakly as } n \rightarrow \infty$$

- **Concentrations.**

$$n\theta(nx) \rightarrow \delta_0 \text{ weakly-} (*) \text{ in } \mathcal{M}(R)$$

if

$$\theta \in C_c^\infty(R), \theta \geq 0, \int_R \theta = 1$$

Statistical description of oscillations – Young measures

Young measure

$$b(\varrho_n, \mathbf{m}_n, S_n) \rightarrow \overline{b(\varrho, \mathbf{m}, S)} \text{ weakly-} (*) \text{ in } L^\infty((0, T) \times \Omega)$$

(up to a subsequence) for any $b \in C_c(R^{d+2})$

Young measure \mathcal{V} – a parametrized family of probability measures $\{\mathcal{V}_{t,x}\}_{(t,x) \in (0,T) \times \Omega}$ on the phase space R^{d+2} :

$$\overline{b(\varrho, \mathbf{m}, S)}(t, x) = \left\langle \mathcal{V}_{t,x}; b(\tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{S}) \right\rangle \text{ for a.a. } (t, x)$$

Visualizing Young measure

visualizing Young measure \Leftrightarrow computing $\overline{b(\varrho, \mathbf{m}, S)}$

Problems

- $b(\varrho_n, \mathbf{m}_n, S_n)$ converge only weakly
- extracting subsequences
- only statistical properties relevant \Rightarrow knowledge of the “tail” of the sequence of approximate solutions absolutely necessary

(S)-convergence

(S)-convergent approximate sequence

An approximate sequence $\{\mathbf{U}_n\}_{n=1}^{\infty}$ is (S) - convergent if for any $b \in C_c(R^D)$:

■ Correlation limit

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \int_Q b(\mathbf{U}_n) b(\mathbf{U}_m) dy \text{ exists for any fixed } m$$

■ Correlation disintegration

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{n,m=1}^N \int_Q b(\mathbf{U}_n) b(\mathbf{U}_m) dy \\ &= \lim_{M \rightarrow \infty} \frac{1}{M} \sum_{m=1}^M \left(\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \int_Q b(\mathbf{U}_n) b(\mathbf{U}_m) dy \right) \end{aligned}$$

Basic properties of (S)-convergence, I

Equivalence to convergence of ergodic (Cesàro means)

$$\{\mathbf{U}_n\}_{n=1}^{\infty} \text{ (S)-convergent} \Leftrightarrow \frac{1}{N} \sum_{n=1}^N b(\mathbf{U}_n) \rightarrow \overline{b(\mathbf{U})} \text{ strongly in } L^1(Q)$$

(S)- limit (parametrized measure)

$$\mathbf{U}_n \xrightarrow{(S)} \mathcal{V}, \{\mathcal{V}_y\}_{y \in Q}, \mathcal{V}_y \in \mathfrak{P}(R^D), \langle \mathcal{V}_y; b(\tilde{U}) \rangle = \overline{b(\mathbf{U})}(y)$$

Convergence in Wasserstein distance

$$\int_Q |\mathbf{U}_n|^p dy \leq c \text{ uniformly for } n = 1, 2, \dots, p > 1$$

$$\mathbf{U}_n \xrightarrow{(S)} \mathcal{V} \Rightarrow \int_Q \left| d_{W_s} \left[\frac{1}{N} \sum_{n=1}^N \delta_{\mathbf{U}_n(y)}; \mathcal{V}_y \right] \right|^s dy \rightarrow 0 \text{ as } N \rightarrow \infty, s < p$$

Basic properties of (S)-convergence, II

Statistically equivalent sequences

$$\begin{aligned} \{\mathbf{U}_n\}_{n=1}^{\infty} &\stackrel{(S)}{\approx} \{\mathbf{V}_n\}_{n=1}^{\infty}, \\ &\Leftrightarrow \text{for any } \varepsilon > 0 \\ \frac{\#\left\{k \leq N \mid \int_Q |\mathbf{U}_n - \mathbf{V}_n| \, dy > \varepsilon\right\}}{N} &\rightarrow 0 \text{ as } N \rightarrow \infty. \end{aligned}$$

Robustness

$$\{\mathbf{U}_n\}_{n=1}^{\infty} \stackrel{(S)}{\approx} \{\mathbf{V}_n\}_{n=1}^{\infty} \Rightarrow \mathbf{U}_n \xrightarrow{(S)} \mathcal{V} \Leftrightarrow \mathbf{V}_n \xrightarrow{(S)} \mathcal{V}$$

Corollary

$$\mathbf{U}_n \rightarrow \mathbf{U} \text{ in } L^1(Q) \Rightarrow \mathbf{U}_n \xrightarrow{(S)} \delta_{\mathbf{U}(y)}$$

Basic properties of (S)–convergence III

Stationarity

$$\int_Q B(\mathbf{U}_{k_1}, \dots, \mathbf{U}_{k_j}) dy = \int_Q B(\mathbf{U}_{k_1+n}, \dots, \mathbf{U}_{k_j+n}) dy$$

Birkhoff–Khinchin Theorem

$\{\mathbf{U}_n\}_{n=1}^{\infty}$ stationary, $b \in \mathcal{B}(R^d)$ Borel measurable $\int_Q b(\mathbf{U}_0) < \infty$

\Rightarrow

$\frac{1}{N} \sum_{n=1}^N b(\mathbf{U}_n)$ converges for a.a. $y \in Q$

\Rightarrow

\mathbf{U}_n is (S)–convergent

Asymptotically stationary consistent approximation

Asymptotically stationary sequence

$\{\mathbf{U}_n\}_{n=1}^{\infty}$ is *asymptotically stationary* if for any $b \in BC(R^D)$ there holds:

■ Correlation limit

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \int_Q b(\mathbf{U}_n) b(\mathbf{U}_m) \, dy \text{ exists}$$

for any fixed m

■ Asymptotic correlation stationarity

$$\left| \int_Q \left[b(\mathbf{U}_{k_1}) b(\mathbf{U}_{k_2}) - b(\mathbf{U}_{k_1+n}) b(\mathbf{U}_{k_2+n}) \right] dy \right| \leq \omega(b, k)$$

for any $1 \leq k \leq k_1 \leq k_2$, and any $n \geq 0$

$$\omega(b, k) \rightarrow 0 \text{ as } k \rightarrow \infty$$

Sufficient conditions for (S)–convergence

Asymptotically stationary sequence

$\{\mathbf{U}_n\}_{n=1}^{\infty}$ asymptotically stationary $\Rightarrow \{\mathbf{U}_n\}_{n=1}^{\infty}$ (S)–convergent

Subsequence principle [Balder]

$$\int_Q F(|\mathbf{U}_n|) \, dy \leq 1 \text{ uniformly for } n \rightarrow \infty,$$

$$F : [0, \infty) \rightarrow [0, \infty) \text{ continuous, } \lim_{r \rightarrow \infty} F(r) = \infty$$

\Rightarrow

there is an (S)–convergent subsequence $\{\mathbf{U}_{n_k}\}_{k=1}^{\infty}$

Application to consistent approximation of the Euler system

(S)–convergent consistent approximation

$$\mathbf{U}_n = [\varrho_n, \mathbf{m}_n, S_n] \quad Q = (0, T) \times \Omega$$

$$\mathbf{U}_n \xrightarrow{(S)} \mathcal{V}$$

DMV solution

\mathcal{V} is a dissipative measure valued solutions of the Euler system

Convergence in Wasserstein distance

$$\int_0^T \int_{\Omega} \left| d_{W_s} \left[\frac{1}{N} \sum_{n=1}^N \delta_{\mathbf{u}_n(y)}; \mathcal{V}_y \right] \right|^s dx dt \rightarrow 0 \text{ as } N \rightarrow \infty$$
$$1 \leq s < \frac{2\gamma}{\gamma + 1}$$

Deterministic convergence

Strong solution

Euler system admits strong solution $\Rightarrow \mathcal{V}_{(t,x)} = \delta_{[\varrho, \mathbf{m}, S](t,x)}$

Regular limit

$$[\varrho = \langle \mathcal{V}; \tilde{\varrho} \rangle, \mathbf{m} = \langle \mathcal{V}; \tilde{\mathbf{m}} \rangle, S = \langle \mathcal{V}; \tilde{S} \rangle] \in C^1$$

\Rightarrow

$[\varrho, \mathbf{m}, S]$ strong solution of Euler, $\mathcal{V}_{(t,x)} = \delta_{[\varrho, \mathbf{m}, S](t,x)}$

Convergence to weak solution

$[\varrho = \langle \mathcal{V}; \tilde{\varrho} \rangle, \mathbf{m} = \langle \mathcal{V}; \tilde{\mathbf{m}} \rangle, S = \langle \mathcal{V}; \tilde{S} \rangle]$ weak solution to Euler system

\Rightarrow

$$\mathcal{V}_{(t,x)} = \delta_{[\varrho, \mathbf{m}, S](t,x)}$$

Improving convergence criteria

Statistical significant set

$$\mathcal{S} \subset N \times N$$
$$\frac{\#\{n, m \leq N \mid (n, m) \in \mathcal{S}\}}{N^2} \rightarrow 0 \text{ as } N \rightarrow \infty$$

Asymptotic stationarity

$$\|U_n\|_{L^\infty(Q)} \leq C$$

for any $\varepsilon > 0$ there is k, L such that

$$\left| \int_Q (U_{n+l} - U)(U_n - U) - (U_{k+l} - U)(U_k - U) \, dy \right| < \varepsilon$$

for all $n \geq k, l \geq L, (n+l, n) \in \mathcal{S}$

\Rightarrow

$$\frac{1}{N} \sum_{n=1}^N U_n \rightarrow U \text{ in } L^1(Q) \text{ as } N \rightarrow \infty$$