Sobolev embeddings and interpolations
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This is a second iteration of a text, which is intended to be an introduction into Sobolev embeddings and interpolations. The goal is to show the main ideas of the proofs, so that the reader can derive himself/herself particular formulas in cases that are not explicitly treated in textbooks. Hence, emphasis is put on methods rather than on a collection of results. Some additional information can also be found in [1, 2, 3, 4]. Note that in fact, the only tool behind all the estimates is the elementary Hölder inequality.

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1 Preliminaries

The word embedding is used in the situation of two Banach spaces $U$ and $V$, endowed with respective norms $\| \cdot \|_U$ and $\| \cdot \|_V$, and such that

\[
V \subset U, \quad \exists C > 0 \quad \forall v \in V : \|v\|_U \leq C\|v\|_V.
\] (1.1)

If (1.1) holds, then we say that $V$ is embedded in $U$.

The embedding is said to be compact, if every bounded set $A \subset V$ is precompact in $U$, that is,

\[
\forall \varepsilon > 0 \quad \exists a_1, \ldots, a_n \in A \quad \forall a \in A \quad \exists k \in \{1, \ldots, n\} : \|a - a_k\|_U < \varepsilon.
\] (1.2)

The following theorem represents a basic tool in the theory of compact embeddings in function spaces.
Theorem 1.1 (Arzelà-Ascoli) Let $X, Y$ be Banach spaces and let $A \subset X, B \subset Y$ be compact sets. Let $C(A; B)$ be the Banach space of all continuous mappings from $A$ into $B$. Let $K \subset C(A; B)$ be an equicontinuous set, that is,

$$\forall \varepsilon > 0 \exists \delta > 0 \forall f \in K \forall x, y \in A : \|x - y\| < \delta \Rightarrow \|f(x) - f(y)\|_Y < \varepsilon.$$ 

Then $K$ is compact in $C(A; B)$. Conversely, every relatively compact set in $C(A; B)$ is equicontinuous.

Proof. Let $K \subset C(A; B)$ be equicontinuous, and let $\varepsilon > 0$ be given. We find $\delta > 0$ such that for all $f \in K$ we have $\|f(x) - f(y)\|_Y < \varepsilon/4$ whenever $\|x - y\| < \delta$. Since $A$ is compact, there exist $x_1, \ldots, x_p \in A$ such that for every $x \in A$ there exists $i \in I := \{1, \ldots, p\}$ such that $\|x - x_i\| < \delta$. Furthermore, $B$ is compact, hence there exist $y_1, \ldots, y_q \in B$ such that for every $y \in B$ there exists $j \in J := \{1, \ldots, q\}$ such that $\|y - y_j\|_Y < \varepsilon/4$.

For $z \in J^p$, $z = \{z_1, \ldots, z_p\}$, we now denote

$$K_z = \left\{ f \in K ; \forall i \in I : \|f(x_i) - y_{z_i}\|_Y < \frac{\varepsilon}{4} \right\}.$$ 

Set $M := \{z \in J^p : K_z \neq \emptyset\}$. The set $M$ is indeed finite and we have $K = \bigcup_{z \in J^p} K_z$, hence we may fix one representative $f_z \in K_z$ for each $z \in J$. For any $f \in K_z$ and $x \in A$ we find $x_i$ such that $\|x - x_i\|_X < \delta$, and estimate

$$\|f(x) - f_z(x)\|_Y \leq \|f(x) - f(x_i)\|_Y + \|f(x_i) - y_{z_i}\|_Y + \|f_z(x_i) - y_{z_i}\|_Y + \|y - y_j\|_Y < \varepsilon,$$

which we wanted to prove. Since every finite set of mappings in $C(A; B)$ is equicontinuous, the fact that that relatively compact sets are equicontinuous follows easily. 

\[\square\]

2 Admissible domains

We fix an open connected bounded set $\Omega \subset \mathbb{R}^N$, where $N \in \mathbb{N}$ is an integer, and denote by $\bar{\Omega}$ its closure and by $\partial \Omega$ its boundary. We assume that the following condition holds (see Fig. 1)

(L) There exist $\delta > 0$ and $m \in \mathbb{N}$, and for each $k = 1, \ldots, m$ there exists an open convex sets $\Delta_k \subset \mathbb{R}^{N-1}$, a Lipschitz continuous function $a_k : \Delta_k \rightarrow \mathbb{R}$, and a rotation $A_k$ in $\mathbb{R}^N$ (represented by an $N \times N$ matrix, still denoted by $A_k$, such that $A_k^{-1} = A_k^T$ and $\det A_k = 1$), such that

(i) $\partial \Omega \subset \bigcup_{j=1}^m A_k(G_k^j)$,

(ii) $G_k = \{y \in \mathbb{R}^N ; y = (y', y_N), y' \in \Delta_k, y_N \in (a_k(y') - \delta, a_k(y') + \delta)\}$,

(iii) $G_k^+ = \{y \in G_k ; y_N \in (a_k(y') - \delta, a_k(y'))\}$,

(iv) $G_k^- = \{y \in G_k ; y_N = a_k(y')\}$.

(v) $\Omega \cap A_k(G_k) = A_k(G_k^-)$,

(vi) $\partial \Omega \cap A_k(G_k) = A_k(G_k^+)$,
If \((L)\) (i)–(vi) hold, then we say that \(\Omega\) has \textit{Lipschitzian boundary}.

As an example, consider the spaces \(C(\overline{\Omega})\) of continuous real functions defined on \(\overline{\Omega}\), endowed with the norm
\[
\|f\|_{C,0} = \sup\{|f(x)| ; x \in \overline{\Omega}\},
\]
and \(C^1(\overline{\Omega})\) of continuously differentiable real functions on \(\overline{\Omega}\), endowed with the norm
\[
\|f\|_{C,1} = \sup \left\{ |f(x)| + \sum_{k=1}^{N} \left| \frac{\partial f}{\partial x_i}(x) \right| ; x \in \overline{\Omega} \right\}.
\]

**Proposition 2.1** If \(\Omega\) has Lipschitzian boundary, then the space \(C^1(\overline{\Omega})\) is compactly embedded in \(C(\overline{\Omega})\).

\[
\text{Proof.}\quad \text{Condition (1.1) is automatically satisfied. Furthermore, let } K \subset C^1(\overline{\Omega}) \text{ be bounded. Hence, there exists } M > 0 \text{ such that}
\]
\[
\forall f \in K \quad \forall x \in \overline{\Omega} : |f(x)| + \sum_{k=1}^{N} \left| \frac{\partial f}{\partial x_i}(x) \right| \leq M.
\]

We are thus in the situation of Theorem 1.1 with \(X = \mathbb{R}^N\), \(Y = \mathbb{R}\), \(A = \overline{\Omega}\), \(B = [-M, M]\), provided we check that \(K\) is equicontinuous. Let \(x, y \in \overline{\Omega}\) be arbitrarily chosen. We find a Lipschitz continuous function \(\xi : [0,1] \to \Omega\) and a constant \(C > 0\) such that \(\xi(0) = x\), \(\xi(1) = y\), \(|\xi'(\sigma)| \leq C|x-y|\) a.e. (this is possible by the hypotheses on \(\Omega\)), and use the chain rule to estimate
\[
|f(x) - f(y)| = \left| \int_0^1 \frac{d}{d\sigma} f(\xi(\sigma)) \, d\sigma \right| \\
= \left| \int_0^1 \langle \nabla f(\xi(\sigma)), \xi'(\sigma) \rangle \, d\sigma \right| \\
\leq MC|x-y|,
\]
where we denote by \(\langle \cdot, \cdot \rangle\) the canonical scalar product in \(\mathbb{R}^N\). The relative compactness now follows from Theorem 1.1.
3 Spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$

Let $\Omega \subset \mathbb{R}^N$ be any open set. We denote as usual by $L^p(\Omega)$ the space of measurable functions $u : \Omega \to \mathbb{R}$, for which the norm $|u|_{p,\Omega}$ is finite, where

$$
|u|_{p,\Omega} = \begin{cases} 
\left( \int_{\Omega} |u(x)|^p \, dx \right)^{1/p} & \text{if } 1 \leq p < \infty, \\
\sup_{x \in \Omega} |u(x)| & \text{if } p = \infty.
\end{cases}
$$

The spaces $L^p(\Omega)$ with the above norms are Banach spaces. We say that $v \in L^p(\Omega)$ is a generalized partial derivative of $u \in L^p(\Omega)$ with respect to $x_i$, $i \in \{1, \ldots, N\}$, if for every Lipschitz continuous function $\varphi : \Omega \to \mathbb{R}$ with compact support in $\Omega$, that is,

$$
\exists K = \bar{K} \subset \Omega \ \forall x \in \Omega \setminus K : \varphi(x) = 0,
$$

we have

$$
\int_{\Omega} u(x) \frac{\partial \varphi}{\partial x_i}(x) \, dx = - \int_{\Omega} v(x) \varphi(x) \, dx.
$$

By [4, Chap. 2, Sect. 2.2], condition (3.3) is fulfilled if and only if $u$ is absolutely continuous along almost all lines parallel to the $x_i$-axis and $v$ coincides with $\partial u/\partial x_i$ almost everywhere.

The Sobolev space $W^{1,p}(\Omega)$ is defined as the subspace of $L^p(\Omega)$ of all functions $u$, which together with all generalized partial derivatives $\partial u/\partial x_i$ belong to $L^p(\Omega)$. With the norm

$$
\|u\|_{1;p,\Omega} = |u|_{p,\Omega} + \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|_{p,\Omega},
$$

$W^{1,p}(\Omega)$ is also a Banach space.

The following result is crucial for the proof of embedding theorems, and its proof can be found in [4, Chap. 2, Sect. 3.6]. We fix an open bounded connected set $\Omega \subset \mathbb{R}^N$ with Lipschitzian boundary, and an open ball $B \subset \mathbb{R}^N$ such that $\bar{\Omega} \subset B$. We define $W^{1,p}_B$ to be the subset of $W^{1,p}(\mathbb{R}^N)$ consisting of all functions vanishing outside $B$. The norms in $L^p(\mathbb{R}^N)$, $W^{1,p}(\mathbb{R}^N)$ will simply be denoted by $| \cdot |_p$, $\| \cdot \|_{1,p}$, respectively.

**Theorem 3.1** There exists a linear prolongation operator $E_p : W^{1,p}(\Omega) \to W^{1,p}_B$ such that for every $u \in W^{1,p}(\Omega)$ we have

(i) $E_p u(x) = u(x)$ for a.e. $x \in \Omega$;

(ii) There exists a constant $c_p > 0$ such that for every $u \in W^{1,p}(\Omega)$ we have

$$
\|E_p u\|_{1;p} \leq c_p \|u\|_{1;p,\Omega};
$$

(iii) For every $u \in W^{1,p}(\Omega)$ we have

$$
|E_p u|_p \leq c_p |u|_{p,\Omega}.
$$
4 Some inequalities

This section collects some auxiliary inequalities that are needed in the sequel.

Proposition 4.1 (Young’s inequality) Let \( f : [0, \infty) \rightarrow [0, \infty) \) be an absolutely continuous increasing function, \( f(0) = 0 \). Then for every \( x, y \geq 0 \) we have (see Fig. 2)

\[
xy \leq \int_0^x f(u) \, du + \int_0^y f^{-1}(v) \, dv ,
\]

(4.1)

where \( f^{-1} \) is the inverse function to \( f \).

Proof. Substituting \( v = f(u) \) we have, with the convention \( \int_{f^{-1}(y)}^x = - \int_y^{f^{-1}(y)} \) if \( f^{-1}(y) < x \), that

\[
\int_0^x f(u) \, du + \int_0^y f^{-1}(v) \, dv = \int_0^x (f(u) + uf'(u)) \, du + \int_x^{f^{-1}(y)} uf'(u) \, du
\]

\[
\geq xf(x) + x(y - f(x)) = xy .
\]

\[\blacksquare\]

For \( 1 < p < \infty \), we denote by \( p' \) the conjugate exponent

\[
p' = \frac{p}{p-1} .
\]

(4.2)

Reciprocally, \( p \) is the conjugate of \( p' \) and we have

\[
\frac{1}{p} + \frac{1}{p'} = 1 , \quad p' - 1 = \frac{1}{p-1} .
\]

(4.3)

As an immediate consequence of Proposition 4.1 we obtain, putting \( f(x) = x^{p-1} \), that

\[
xy \leq \frac{1}{p} x^p + \frac{1}{p'} y^{p'}
\]

(4.4)

for every \( x, y \geq 0 \) and \( 1 < p < \infty \).
Proposition 4.2 (Hölder’s inequality) Let $\Omega \subset \mathbb{R}^N$ be any open set and let $1 \leq p \leq \infty$ be arbitrary. Then for every $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ we have

$$\int_{\Omega} f(x) g(x) \, dx \leq |f|_{p,\Omega} |g|_{p',\Omega}, \quad (4.5)$$

with the convention $1' = \infty$, $\infty' = 1$.

Proof. The case $p = 1$ or $p = \infty$ is obvious. For $1 < p < \infty$ we set

$$F(x) = \frac{f(x)}{|f|_{p,\Omega}}, \quad G(x) = \frac{g(x)}{|g|_{p',\Omega}}.$$

By (4.4) we have

$$|F(x)||G(x)| \leq \frac{1}{p}|F(x)|^p + \frac{1}{p'}|G(x)|^{p'} = \frac{|f(x)|^p}{p|f|_{p,\Omega}^p} + \frac{|g(x)|^{p'}}{p'|g|_{p',\Omega}^{p'}},$$

hence

$$\int_{\Omega} F(x) G(x) \, dx \leq \int_{\Omega} |F(x)||G(x)| \, dx \leq \frac{1}{p} + \frac{1}{p'} = 1,$$

which we wanted to prove. $\blacksquare$

Proposition 4.3 (Minkowski’s inequality) Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ be open sets, and let $f : X \times Y \to [0, \infty)$ be a measurable function. Then for every $1 < p < \infty$ we have

$$\left( \int_Y \left( \int_X f(x, y) \, dx \right)^p \, dy \right)^{1/p} \leq \int_X \left( \int_Y f^p(x, y) \, dy \right)^{1/p} \, dx. \quad (4.6)$$

Proof. For $y \in Y$ and $R > 0$ set

$$F(y) = \int_X f_R(x, y) \, dx, \quad g(y) = F^{p-1}(y),$$

where

$$f_R(x, y) = \begin{cases} \min\{R, f(x, y)\} & \text{if } \max\{|x|, |y|\} < R, \\ 0 & \text{if } \max\{|x|, |y|\} \geq R. \end{cases}$$

Then

$$\int_Y F^p(y) \, dy \leq \int_Y F(y) g(y) \, dy = \int_X \left( \int_Y f_R(x, y) g(y) \, dy \right) \, dx \\ \overset{\text{Hölder}}{\leq} \int_X \left( \int_Y f_R^p(x, y) \, dy \right)^{1/p} \left( \int_Y g^{p'}(y) \, dy \right)^{1/p'} \, dx \\ = \int_X \left( \int_Y f_R^p(x, y) \, dy \right)^{1/p} \, dx \left( \int_Y F^p(y) \, dy \right)^{1/p'},$$
hence

\[ \left( \int_Y F^p(y) \, dy \right)^{1/p} \leq \int_X \left( \int_Y f_R^p(x, y) \, dy \right)^{1/p} \, dx \leq \int_X \left( \int_Y f^p(x, y) \, dy \right)^{1/p} \, dx, \]

and we obtain the result from Fatou’s Lemma letting \( R \) tend to \(+\infty\).

Note that replacing \( X \) by a finite set \( \{1, \ldots, n\} \), \( f(x, y) \) by \( f_k(y) \), \( k = 1, \ldots, n \), and \( \int_X dx \) by \( \sum_{k=1}^n \), the Minkowski inequality reads

\[
\left| \sum_{k=1}^n f_k \right|_{p,Y} \leq \sum_{k=1}^n |f_k|_{p,Y}, \tag{4.7}
\]

which is nothing but the triangle inequality for the norm \( |\cdot|_{p,Y} \).

The following example shows that the Minkowski inequality cannot be reversed.

**Example 4.4** Consider \( X = Y = (0, 1) \), and \( f(x, y) = ((x - y)^+)^{-1/p} \) for some \( p > 1 \). Then

\[
\left( \int_Y \left( \int_X f(x, y) \, dx \right)^p \, dy \right)^{1/p} = \left( \int_0^1 \left( \int_y^1 (x - y)^{-1/p} \, dx \right)^p \, dy \right)^{1/p} = \frac{1}{p-1} p^{1-1/p},
\]

\[
\int_X \left( \int_Y f^p(x, y) \, dy \right)^{1/p} \, dx = \int_0^1 \left( \int_0^x (x - y)^{-1} \, dx \right)^{1/p} \, dy = +\infty.
\]

**Remark 4.5** In the same way we prove that for every \( 1 \leq q < p < \infty \) we have

\[
\left( \int_Y \left( \int_X f^q(x, y) \, dx \right)^{p/q} \, dy \right)^{1/p} \leq \left( \int_X \left( \int_Y f^p(x, y) \, dy \right)^{q/p} \, dx \right)^{1/q}. \tag{4.8}
\]

We set in this case

\[
F(y) = \int_X f_R^p(x, y) \, dx, \quad g(y) = F^{(p/q)-1}(y),
\]

and estimate \( \int_Y F^{p/q}(y) \, dy \) similarly as in the proof of Proposition 4.3.

The proof of the Minkowski inequality is related to the so-called reverse Hölder inequality:

\[
\int_\Omega f(x) g(x) \, dx \leq C |g|_{p',\Omega} \forall g \in L^{p'}(\Omega) \implies |f|_{p,\Omega} \leq C. \tag{4.9}
\]

To prove this statement, it suffices to choose \( g(x) = \text{sign}(f_R(x)) |f_R(x)|^{p-1} \) with \( f_R \) defined analogously as in the proof of Proposition 4.3, use the fact that

\[
\int_\Omega |f_R(x)|^p \, dx \leq \int_\Omega f(x) g(x) \, dx \leq C |g|_{p',\Omega} = C |f_R|_{p,\Omega}^{p/p'},
\]

and let \( R \) tend to \( \infty \).

**Proposition 4.6** (Young’s inequality II for convolutions) Let \( 1 \leq p, q, r \leq \infty \) be given such that

\[
\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}. \tag{4.10}
\]
For \( u \in L^p(\mathbb{R}^N) \), \( v \in L^r(\mathbb{R}^N) \), and \( x \in \mathbb{R}^N \) set

\[
w(x) = \int_{\mathbb{R}^N} u(y) v(x - y) \, dy.
\]

Then \( w \in L^q(\mathbb{R}^N) \) and

\[
|w|_q \leq |u|_p |v|_r. \tag{4.11}
\]

**Proof.** The case \( q = \infty \) follows immediately from Hölder’s inequality. Hence, assume that \( q < \infty \), and set \( \alpha = \frac{r}{q} \in (0, 1] \). To make the use of the Minkowski inequality more transparent, we write \( \int_X dx \), \( \int_Y dy \) instead of \( \int_{\mathbb{R}^N} dx \), \( \int_{\mathbb{R}^N} dy \). Then, using the fact that \( 1 - \alpha = \frac{r}{p'} \) and that

\[
\int_Y |v(x - y)|^r \, dy = \int_Y |v(y)|^r \, dy
\]

for a.e. \( x \in X \), we obtain

\[
|w|_q = \left( \int_X \left( \int_Y |u(y)| v(x - y) \, dy \right)^q \, dx \right)^{1/q} \\
\leq \left( \int_X \left( \int_Y |u(y)| |v(x - y)|^\alpha |v(x - y)|^{1-\alpha} \, dy \right)^q \, dx \right)^{1/q} \\
\leq \left( \int_X \left( \int_Y |u(y)|^p |v(x - y)|^{p\alpha} \, dy \right)^{q/p} \, dx \right)^{1/p} \left( \int_Y |v(y)|^{p'(1-\alpha)} \, dy \right)^{1/p'} \\
\leq \left( \int_Y \left( \int_X |u(y)|^q |v(x - y)|^{q\alpha} \, dx \right)^{p/q} \, dy \right)^{1/p} \left( \int_X |v(x)|^{q\alpha} \, dx \right)^{1/q} |v|_{r}^{1-\alpha} \\
= \left( \int_Y |u(y)|^p \, dy \right)^{1/p} \left( \int_X |v(x)|^{q\alpha} \, dx \right)^{1/q} |v|_{r}^{1-\alpha} \\
= |u|_p |v|_r.
\]

We devote the next section to the **Hardy-Littlewood inequality**, the proof of which is quite involved and requires a certain number of auxiliary steps. The proof we give here is a modification of the one from [2].

## 5 Hardy-Littlewood inequality

We state the Hardy-Littlewood inequality in the following form.

**Proposition 5.1** Let \( 1 < p, q, r < \infty \) be such that \( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2 \). Then there exists a constant \( H_{pr} > 0 \) such that for every \( f \in L^p(\mathbb{R}) \), \( g \in L^q(\mathbb{R}) \) we have

\[
\int_{\mathbb{R}^2} f(x) g(y) |x - y|^{-1/r} \, dx \, dy \leq H_{pr} |f|_p |g|_q. \tag{5.1}
\]
An explicit estimate for $H_{pr}$ will be given in (5.18) below.

We first fix an even locally integrable function $h : \mathbb{R} \to [0, \infty)$, which is non-decreasing in $(-\infty, 0)$ and non-increasing in $(0, +\infty)$, and establish the following easy result.

**Lemma 5.2** For $a, b > 0$ and $r, s \in \mathbb{R}$ set

$$\varphi_{ab}(r, s) = \int_{-a+r}^{a+r} \int_{-b+s}^{b+s} h(x - y) \, dy \, dx. \quad (5.2)$$

Then for all $r, s \in \mathbb{R}$ we have $\partial \varphi_{ab}/\partial r \geq 0$, $\partial \varphi_{ab}/\partial s \leq 0$ for $r < s$, $\partial \varphi_{ab}/\partial r \leq 0$, $\partial \varphi_{ab}/\partial s \geq 0$ for $r > s$, $\varphi_{ab}(r, r) = \varphi_{ab}(0, 0)$.

**Proof.** We obviously have $\varphi_{ab}(r, s) = \varphi_{ab}(r - s, 0) = \varphi_{ab}(0, s - r)$ for all $r, s$, hence it suffices to prove that $\partial \varphi_{ab}/\partial r(r, 0) \leq 0$ for $r > 0$, $\partial \varphi_{ab}/\partial s(0, s) \leq 0$ for $s > 0$. We have

$$\frac{\partial \varphi_{ab}}{\partial r}(r, 0) = \int_{-b}^{b} (h(a + r - y) - h(-a + r - y)) \, dy$$

$$= \int_{-b}^{b} (h(a + r + y) - h(-a + r - y)) \, dy$$

$$= \int_{a-b}^{a+b} (h(r + z) - h(r - z)) \, dz$$

$$= \int_{|a-b|}^{a+b} (h(r + z) - h(r - z)) \, dz.$$

For a.e. $z > 0$ we have $h(r + z) \leq h(r - z)$, and the assertion follows. The argument for $\partial \varphi_{ab}/\partial s(0, s)$ is identical.

The idea of the proof of Proposition 5.1 is based on approximations of the functions $f$ and $g$ by step functions, and for each step function we use a rearrangement formula which will be proved by induction (see Fig. 3). The induction step is carried out in the following way.

**Lemma 5.3** Let $h$ be as in Lemma 5.2, and let $m, n \in \mathbb{N} \cup \{0\}$ be given. Let $a_0, \ldots, a_n$, $b_0, \ldots, b_m$, $r_0, \ldots, r_n$, $s_0, \ldots, s_m$ be sequences such that $a_i > 0$, $b_j > 0$ for all $i = 0, \ldots, n$, $j = 0, \ldots, m$, and $r_i - r_{i-1} \geq a_i + a_{i-1}$, $s_j - s_{j-1} \geq b_j + b_{j-1}$ for all $i = 1, \ldots, n$, $j = 1, \ldots, m$.

(i) If $r_{i_0} - r_{i_0-1} = a_{i_0} + a_{i_0-1}$ for some $i_0 \in \{1, \ldots, n\}$, then there exist $a^*_i > 0$ and $r^*_i \in \mathbb{R}$ for $i = 0, \ldots, n-1$ such that $r^*_i - r^*_{i-1} \geq a^*_i + a^*_{i-1}$ for all $i = 1, \ldots, n-1$, and

$$\sum_{i=0}^{n-1} a^*_i = \sum_{i=0}^{n} a_i, \quad \sum_{i=0}^{m-1} \sum_{j=0}^{n} \varphi_{a_i b_j}(r^*_i, s_j) = \sum_{i=0}^{n} \sum_{j=0}^{m} \varphi_{a_i b_j}(r, s^*_j). \quad (5.3)$$

(ii) If $s_{j_0} - s_{j_0-1} = b_{j_0} + b_{j_0-1}$ for some $j_0 \in \{1, \ldots, m\}$, then there exist $b^*_j > 0$ and $s^*_j \in \mathbb{R}$ for $j = 0, \ldots, m-1$ such that $s^*_j - s^*_{j-1} \geq b^*_j + b^*_{j-1}$ for all $j = 1, \ldots, m-1$, and

$$\sum_{j=0}^{m-1} b^*_j = \sum_{j=0}^{m} b_j, \quad \sum_{i=0}^{n-1} \sum_{j=0}^{m} \varphi_{a_i b_j}(r^*_i, s^*_j) = \sum_{i=0}^{n} \sum_{j=0}^{m} \varphi_{a_i b_j}(r, s^*_j). \quad (5.4)$$
Proof. We prove only part (i), the rest is similar. If \( r_{i_0} - r_{i_0-1} = a_{i_0} + a_{i_0-1} \), then \( r_{i_0-1} + a_{i_0-1} = r_{i_0} - a_{i_0} \), hence for every \( j \) we have
\[
\varphi_{a_{i_0-1}b_j}(r_{i_0-1}, s_j) + \varphi_{a_{i_0}b_j}(r_{i_0}, s_j) = \varphi_{a_{i_0-1}b_j}(r^*_0, s_j),
\]
where
\[
r^*_0 = \frac{1}{2}(a_{i_0} - a_{i_0-1} + r_{i_0} + r_{i_0-1}), \quad a^*_{i_0-1} = a_{i_0} + a_{i_0-1}.
\]
We now set
\[
r^*_i = r_i, \quad a^*_i = a_i \quad \text{for} \quad i = 0, \ldots, i_0 - 2, \quad r^*_i = r_{i-1}, \quad a^*_i = a_{i-1} \quad \text{for} \quad i = i_0, \ldots, n - 1.
\]
Then (5.3) is automatically fulfilled by construction. It remains to check that
\[
r^*_i - r^*_{i-1} - a^*_i - a^*_{i-1} = r_{i+1} - \frac{1}{2}(a_{i_0} - a_{i_0-1} + r_{i_0} + r_{i_0-1}) - a_{i_0+1} - a_{i_0} - a_{i_0-1}
\]
\[
= r_{i+1} - r_{i_0} - a_{i_0+1} - a_{i_0} + \frac{1}{2}(r_{i_0} - r_{i_0-1} - a_{i_0} - a_{i_0-1}) \geq 0,
\]
\[
r^*_{i-1} - r^*_{i-2} - a^*_{i-1} - a^*_{i-2} = \frac{1}{2}(a_{i_0} - a_{i_0-1} + r_{i_0} + r_{i_0-1}) - r_{i_0-2} - a_{i_0} - a_{i_0-1} - a_{i_0-2}
\]
\[
= r_{i_0} - r_{i_0-2} - a_{i_0} - a_{i_0-2} - \frac{1}{2}(r_{i_0} - r_{i_0-1} - a_{i_0} - a_{i_0-1})
\]
\[
\geq 0,
\]
and the proof is complete. \(\blacksquare\)

Lemma 5.4 Let \( h \) be as in Lemma 5.2, and let \( a_0, \ldots, a_n, b_0, \ldots, b_m, r_0, \ldots, r_n, s_0, \ldots, s_m \) be as in Lemma 5.3. Set
\[
A = \sum_{i=0}^{n} a_i, \quad B = \sum_{j=0}^{m} b_j.
\]
Then
\[
S := \sum_{i=0}^{n} \sum_{j=0}^{m} \varphi_{a_i b_j}(r_i, s_j) \leq \varphi_{AB}(0, 0).
\]

Proof. We proceed by induction over \( N = n + m \). For \( N = 0 \) we have \( \varphi_{a_0 b_0}(r_0, s_0) = \varphi_{a_0 b_0}(r_0 - s_0, 0) \leq \varphi_{a_0 b_0}(0, 0) \) by Lemma 5.2. Suppose now that the statement is proven for some \( N \geq 0 \), and consider \( n, m \) such that \( n + m = N + 1 \). We will assume for definiteness that \( r_n \geq s_m \) (the opposite case is fully analogous). We distinguish two cases:

(i) \( n = 0 \). Then we set \( \hat{s}_j = s_j + s_{m-1} - b_m - b_{m-1} \geq s_j \) for \( j = 0, \ldots, m-1, \hat{s}_m = s_m \). Then \( \hat{s}_j - \hat{s}_{j-1} = s_j - s_{j-1} \) for \( j = 1, \ldots, m-1, \hat{s}_m - \hat{s}_{m-1} = b_m + b_{m-1} \). By Lemma
5.2 we have $\varphi_{\alpha_{ij}}(r_0, s_j) \leq \varphi_{\alpha_{ij}}(\hat{r}_0, \hat{s}_j)$ for all $j = 0, \ldots, m$. By Lemma 5.3 there exist $s_0^*, \ldots, s_{m-1}$ and $b_0^*, \ldots, b_{m-1}^*$ such that

$$m \sum_{j=0}^{m-1} b_j^* = \sum_{j=0}^{m} b_j, \quad m \sum_{j=0}^{m-1} \varphi_{\alpha_{ij}}(r_0, s_j^*) = \sum_{j=0}^{m} \varphi_{\alpha_{ij}}(r_0, \hat{s}_j).$$

We have $m - 1 = N$, and the induction hypothesis yields

$$m \sum_{j=0}^{m} \varphi_{\alpha_{ij}}(r_0, s_j) \leq m \sum_{j=0}^{m-1} \varphi_{\alpha_{ij}}(r_0, s_j^*) \leq \varphi_{\alpha_{ij}}(0, 0).$$

(ii) $n > 0$. Set

$$\hat{r}_n = \max\{s_m, r_n + a_n + a_{n-1}\} \leq r_n \quad \hat{r}_i = r_i \quad \text{for} \quad i = 0, \ldots, n - 1.$$

By Lemma 5.2, we have

$$\hat{S} = \sum_{i=0}^{n-1} \sum_{j=0}^{m} \varphi_{\alpha_{ij}}(\hat{r}_i, s_j) \geq S.$$

If $\hat{r}_n = r_n + a_n + a_{n-1}$, then $\hat{S} \leq \varphi_{\alpha_{ij}}(0, 0)$ by Lemma 5.3 and by the induction hypothesis similarly as in case (i). If $\hat{r}_n = s_m > r_n + a_n + a_{n-1}$, then set

$$\bar{s}_m = \max\{s_{m-1} + b_m + b_{m-1}, r_n + a_n + a_{n-1}\} \leq s_m; \quad \bar{s}_j = s_j \quad \text{for} \quad j = 0, \ldots, m - 1; \quad \bar{r}_n = \bar{s}_m; \quad \bar{r}_i = r_i \quad \text{for} \quad i = 0, \ldots, n - 1,$$

with the convention $s_{-1} = -\infty, b_{-1} = b_0$ if $m = 0$. We have

$$\hat{S} \leq \sum_{i=0}^{n-1} \sum_{j=0}^{m} \varphi_{\alpha_{ij}}(\bar{r}_i, \bar{s}_j) + \sum_{i=0}^{n-1} \varphi_{\alpha_{ij}}(\bar{r}_i, s_m) + \sum_{j=0}^{m-1} \varphi_{\alpha_{ij}}(s_m, s_j) + \varphi_{\alpha_{ij}}(s_m, s_m)$$

Lemma 5.2

$$\leq \sum_{i=0}^{n-1} \sum_{j=0}^{m} \varphi_{\alpha_{ij}}(\bar{r}_i, \bar{s}_j) + \sum_{i=0}^{n-1} \varphi_{\alpha_{ij}}(\bar{r}_i, \bar{s}_m) + \sum_{j=0}^{m-1} \varphi_{\alpha_{ij}}(\bar{s}_m, s_j) + \varphi_{\alpha_{ij}}(\bar{s}_m, \bar{s}_m)$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{m} \varphi_{\alpha_{ij}}(\bar{r}_i, \bar{s}_j).$$

By construction, we have either $\bar{s}_m = s_{m-1} + b_m + b_{m-1}$ or $\bar{r}_n = \bar{r}_{n-1} + a_n + a_{n-1}$, and the assertion follows again from Lemma 5.3 and the induction hypothesis.

\[\blacksquare\]
Corollary 5.5 Let $h$ be as in Lemma 5.2, and let $\{(\alpha_i, \beta_i); i = 0, \ldots, n\}$, $\{(\gamma_j, \delta_j); j = 0, \ldots, m\}$, be two systems of intervals such that $(\alpha_{i_1}, \beta_{i_1}) \cap (\alpha_{i_2}, \beta_{i_2}) = \emptyset$, $(\gamma_{j_1}, \delta_{j_1}) \cap (\gamma_{j_2}, \delta_{j_2}) = \emptyset$ for all $i_1 \neq i_2 \in \{0, \ldots, n\}$, $j_1 \neq j_2 \in \{0, \ldots, m\}$. Set

$$A = \frac{1}{2} \sum_{i=0}^n (\beta_i - \alpha_i), \quad B = \frac{1}{2} \sum_{j=0}^m (\delta_j - \gamma_j).$$

Then

$$\sum_{i=0}^n \sum_{j=0}^m \int_{\alpha_i}^{\beta_i} \int_{\gamma_j}^{\delta_j} h(x - y) \, dy \, dx \leq \varphi_{AB}(0, 0).$$

**Proof.** We change the ordering of the intervals $(\alpha_i, \beta_i)$, $(\gamma_j, \delta_j)$ in such a way that $\beta_{i-1} \leq \alpha_i$, $\delta_{j-1} \leq \gamma_j$ for all $i = 1, \ldots, n$, $j = 1, \ldots, m$, and set

$$r_i = \frac{1}{2} (\alpha_i + \beta_i), \quad a_i = \frac{1}{2} (\beta_i - \alpha_i),$$

$$s_j = \frac{1}{2} (\gamma_j + \delta_j), \quad b_j = \frac{1}{2} (\delta_j - \gamma_j)$$

for $i = 0, \ldots, n$, $j = 0, \ldots, m$. We have $r_i - r_{i-1} - a_i - a_{i-1} = \alpha_i - \beta_{i-1} \geq 0$, $s_j - s_{j-1} - b_j - b_{j-1} = \gamma_j - \delta_{j-1} \geq 0$, $\alpha_i = -a_i + r_i$, $\beta_i = a_i + r_i$, $\gamma_j = -b_j + s_j$, $\delta_j = b_j + s_j$, and Lemma 5.4 completes the proof.

The next step consists in a rearrangement formula we summarize in Lemma 5.6 below. We fix $K, L \in \mathbb{N}$ and for sequences

$$-\infty < a_0 < a_1 < \cdots < a_k < +\infty, \quad 0 = f_0 \leq f_1 \leq \cdots \leq f_K,$$

$$-\infty < b_0 < b_1 < \cdots < b_L < +\infty, \quad 0 = g_0 \leq g_1 \leq \cdots \leq g_L.$$
consider step functions \( f, g \) of the form

\[
\begin{align*}
  f(x) &= \sum_{i=1}^{K} f_i \chi_{(a_{\sigma(i)}-1,a_{\sigma(i)})}(x) = \sum_{j=1}^{L} f_{g^{-1}(J)} \chi_{(a_{j-1},a_{j})}(x), \\
  g(y) &= \sum_{j=1}^{L} g_j \chi_{(b_{\sigma(j)-1},b_{\sigma(j)})}(y) = \sum_{j=1}^{L} g_{\sigma^{-1}(J)} \chi_{(b_{j-1},b_{j})}(y)
\end{align*}
\]  

(5.5)

for \( x, y \in \mathbb{R} \), where \( \chi_M \) is the characteristic function of a set \( M \subset \mathbb{R} \), and \( g : \{1, \ldots, K\} \rightarrow \{1, \ldots, K\} \), \( \sigma : \{1, \ldots, L\} \rightarrow \{1, \ldots, L\} \) are some permutations of indices.

We now define

\[ F_k = f_k - f_{k-1} \quad \text{for} \quad k = 1, \ldots, K. \]  

(5.6)

Then \( f_i = \sum_{k=1}^{i} F_k \) for all \( i \), and we have

\[
\begin{align*}
  f(x) &= \sum_{i=1}^{K} \sum_{k=1}^{i} F_k \chi_{(a_{\sigma(i)}-1,a_{\sigma(i)})}(x) = \sum_{i=1}^{K} \sum_{k=1}^{K} F_k \chi_{(a_{\sigma(i)}-1,a_{\sigma(i)})}(x) \\
  &= \sum_{k=1}^{K} \sum_{i=k}^{K} F_k \chi_{(a_{\sigma(i)}-1,a_{\sigma(i)})}(x).
\end{align*}
\]  

(5.7)

We further introduce for \( k = 1, \ldots, K \) the numbers

\[
a_k^* = \frac{1}{2} \sum_{i=k}^{K} (a_{\sigma(i)} - a_{\sigma(i)-1}), \quad a_{K+1}^* = 0,
\]  

(5.8)

and for \( x \in \mathbb{R} \) put

\[
f^*(x) = \sum_{k=1}^{K} F_k \chi_{(-a_k^*,a_k^*)}(x) = \sum_{i=1}^{K} f_i \chi_{(-a_i^*,a_i^*)}(x)
\]  

(5.9)

Similarly, we put

\[ G_\ell = g_\ell - g_{\ell-1} \quad \text{for} \quad \ell = 1, \ldots, L. \]  

(5.10)

Then

\[
g(y) = \sum_{j=1}^{L} \sum_{\ell=1}^{j} G_\ell \chi_{(b_{\sigma(j)-1},b_{\sigma(j)})}(y) = \sum_{j=1}^{L} \sum_{\ell=1}^{L} G_\ell \chi_{(b_{\sigma(j)-1},b_{\sigma(j)})}(y)
\]  

(5.11)

As before, we introduce for \( \ell = 1, \ldots, L \) the numbers

\[
b_{\ell}^* = \frac{1}{2} \sum_{j=\ell}^{L} (b_{\sigma(j)} - b_{\sigma(j)-1}), \quad b_{L+1}^* = 0,
\]  

(5.12)

and for \( y \in \mathbb{R} \) put

\[
g^*(y) = \sum_{\ell=1}^{L} G_\ell \chi_{(-b_{\ell}^*,b_{\ell}^*)}(y) = \sum_{j=1}^{L} g_j \chi_{(-b_j^*,b_j^*)}(y)
\]  

(5.13)

We now prove the following crucial inequality.
Lemma 5.6 Let \( f, g \) be as in (5.5), and let \( f^*, g^* \) be given by (5.9), (5.13), respectively. Let \( h \) be as in Lemma 5.2. Then we have \(|f|_p = |f^*|_p, \ |g|_q = |g^*|_q\) for all \( p \geq 1, \ q \geq 1, \) and
\[
\iint_{\mathbb{R}^2} f(x) g(y) h(x-y) \, dy \, dx \leq \iint_{\mathbb{R}^2} f^*(x) g^*(y) h(x-y) \, dy \, dx. \tag{5.14}
\]

Proof. The fact that the \( L^p \) norms of \( f \) and \( f^* \) coincide, follows immediately from (5.5) and (5.9), taking into account the fact that for all \( i \) we have \( a_{q(i)} - a_{q(i)-1} = 2(a_i^* - a_{i+1}^*). \) The same argument works for \( g \) and \( g^* \), indeed.

We further have by (5.7), (5.11) that
\[
\iint_{\mathbb{R}^2} f(x) g(y) h(x-y) \, dy \, dx = \sum_{k=1}^{K} \sum_{\ell=1}^{L} F_k G_\ell \sum_{i=k}^{K} \sum_{j=\ell}^{L} \int_{a_{q(i)-1}}^{b_{q(i)}} \int_{b_{q(j)-1}}^{b_{q(j)}} h(x-y) \, dy \, dx. \tag{5.15}
\]

By Corollary 5.5, we have for every \( k \) and \( \ell \) that
\[
\sum_{i=k}^{K} \sum_{j=\ell}^{L} \int_{a_{q(i)-1}}^{b_{q(j)}} \int_{b_{q(j)-1}}^{b_{q(j)}} h(x-y) \, dy \, dx \leq \int_{a_k}^{b_k} \int_{a_\ell}^{b_\ell} h(x-y) \, dy \, dx,
\]
and (5.14) follows from (5.9), (5.13), and (5.15).

We are now ready to pass to the proof of Proposition 5.1.

Proof of Proposition 5.1. We restrict ourselves to the case that \( f \) and \( g \) are non-negative step functions of the form (5.5). The general case then follows from the density of step functions in \( L^p(\mathbb{R}), \ L^q(\mathbb{R}) \). By Lemma 5.6 we have
\[
\iint_{\mathbb{R}^2} f(x) g(y) \, |x-y|^{-1/r} \, dx \, dy \leq \iint_{\mathbb{R}^2} f^*(x) g^*(y) \, |x-y|^{-1/r} \, dx \, dy. \tag{5.16}
\]

For \( y \in \mathbb{R} \) set
\[
F(y) = \int_{\mathbb{R}} f^*(x) \, |x-y|^{-1/r} \, dx.
\]

The function \( f^* \) is even, nondecreasing in \((-\infty, 0)\) and nonincreasing in \((0, +\infty)\), hence
\[
|f|_p^p \geq \int_{-|x|}^{|x|} (f^*(\xi))^p \, d\xi \geq 2|x|(f^*(x))^p \quad \forall x \in \mathbb{R}. \tag{5.17}
\]

Choosing \( \alpha = p/q' \), we thus obtain for every \( y \in \mathbb{R} \) that
\[
F(y) \leq \int_{\mathbb{R}} (f^*(x))^\alpha \, |2x|^{(1-\alpha)/p} \, f_1^{1-\alpha} \, |x-y|^{-1/r} \, dx = 2^{1+1/r} \, |f|_p^{1-\alpha/q'} \int_{\mathbb{R}} (f^*(x))^{p/q'} \, |x|^{-1+1/r} \, |x-y|^{-1/r} \, dx
\]

\[
= 2^{1+1/r} \, |f|_p^{1-\alpha/q'} \int_{\mathbb{R}} (f^*(yt))^{p/q'} \, |t|^{-1+1/r} \, |t-1|^{-1/r} \, dt.
\]
We now use the Minkowski inequality (4.6) to estimate the \( L^{q'} \) norm of \( F \). We have
\[
|F|_{q'} \leq 2^{-1+1/r} |f|_{p}^{1-p/q'} \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} (f^*(yt))^{p/q'} dt \right)^{1/q'} dy \right)^{1/q'} 
\]
Minkowski
\[
\leq 2^{-1+1/r} |f|_{p}^{1-p/q'} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} (f^*(yt))^{p/q'} dt \right)^{1/q'} dy dt 
\]
\[
= 2^{-1+1/r} |f|_{p}^{1-p/q'} \int_{\mathbb{R}} \left| t \right|^{-1+1/r} \left( \int_{\mathbb{R}} (f^*(yt))^{p/q'} dt \right)^{1/q'} dy dt 
\]
\[
= 2^{-1+1/r} |f|_{p} \int_{\mathbb{R}} \left| t \right|^{-1/p} \left| t-1 \right|^{-1/r} \, dt .
\]
By Hölder’s inequality, Lemma 5.6, and inequality (5.16), the left-hand side of (5.1) is estimated from above by \( |g|_{q} |F|_{q'} \). Hence, (5.1) holds with
\[
H_{pr} = 2^{-1+1/r} \int_{\mathbb{R}} \left| t \right|^{-1/p} \left| t-1 \right|^{-1/r} \, dt .
\]
(5.18)

6 Smooth approximation of \( L^p \) functions

We fix a smooth (\( C^1 \) is enough for our purposes) function \( \varphi : \mathbb{R}^N \to [0, \infty) \) such that \( \varphi(x) = 0 \) outside the set \( B(1) := \{ x \in \mathbb{R}^N ; |x| \leq 1 \} \), and
\[
\int_{B(1)} \varphi(x) \, dx = 1 .
\]
(6.1)

For \( u \in L^p(\mathbb{R}^N) \), \( x \in \mathbb{R}^N \), and a parameter \( \sigma \in (0, 1] \) we set
\[
u^\sigma(x) = \sigma^{-N} \int_{\mathbb{R}^N} \varphi \left( \frac{x-y}{\sigma} \right) u(y) \, dy .
\]
(6.2)

For all \( \sigma \in (0, 1] \), the function \( u^\sigma \) is continuously differentiable, and we have
\[
\int_{\mathbb{R}^N} \left| u^\sigma - u \right|^p \, dx = \int_{\mathbb{R}^N} \left| \int_{B(1)} \varphi(z)(u(x-\sigma z) - u(x)) \, dz \right|^p \, dx
\]
Hölder
\[
\leq \left( \int_{B(1)} \varphi^p(x) \, dx \right)^{p/p'} \int_{B(1)} \int_{\mathbb{R}^N} |u(x-\sigma z) - u(x)|^p \, dx \, dz ,
\]
(6.3)

hence
\[
\nu^\sigma \to u \quad \text{strongly in } L^p(\mathbb{R}^N) \quad \text{as } \sigma \to 0+
\]
(6.4)
as a consequence of the Mean Continuity Theorem, see [4, Chap. 2, Sect. 1.2].

In the sequel, we will use the following relation between the \( L^q \) norm of \( u^\sigma \) and \( L^p \) norm of \( u \), which follows directly from Proposition 4.6:
\[
|u^\sigma|_q \leq \sigma^{-N(1/p-1/q)} |\varphi|_r |u|_p \quad \forall q \geq p ,
\]
(6.5)
where \( r \) is as in (4.10).
7 Sobolev embeddings

We now state and prove the main result of this text.

**Theorem 7.1** Let \( p, q \in (1, \infty) \) be such that
\[
\frac{1}{p} \geq \frac{1}{q} > \frac{1}{p} - \frac{1}{N},
\]
and set
\[
\kappa := 1 - N \left(\frac{1}{p} - \frac{1}{q}\right) \in (0, 1).
\]
Then there exists \( C_{pq} > 0 \) such that for every \( u \in W^{1,p}(\mathbb{R}^N) \) and every \( \sigma \in (0, 1) \) we have
\[
|u^\sigma - u|_q \leq C_{pq} \sigma^{\kappa} |\nabla u|_p. \tag{7.1}
\]

**Proof.** Notice first that for every \( x \in \mathbb{R}^N \) and \( \sigma \in (0, 1) \) we obtain, integrating by parts, that
\[
\frac{\partial}{\partial \sigma} u^\sigma(x) = \sigma^{-N} \int_{\mathbb{R}^N} \sum_{i=1}^N \frac{\partial}{\partial y_i} \left( \frac{x_i - y_i}{\sigma} \varphi \left( \frac{x - y}{\sigma} \right) \right) u(y) \, dy
= -\sigma^{-N} \int_{\mathbb{R}^N} \left\langle \Phi \left( \frac{x - y}{\sigma} \right), \nabla u(y) \right\rangle \, dy, \tag{7.2}
\]
where we set \( \Phi(\xi) = \xi \varphi(\xi) \). This yields in particular,
\[
|u^\beta(x) - u^\alpha(x)| \leq \int_{\alpha}^\beta \sigma^{-N} \left| \int_{\mathbb{R}^N} \left\langle \Phi \left( \frac{x - y}{\sigma} \right), \nabla u(y) \right\rangle \, dy \right| \, d\sigma \tag{7.3}
\]
for every \( 0 < \alpha < \beta \leq 1 \). To estimate the difference \( u^\beta - u^\alpha \) in (7.3) in the space \( L^q(\mathbb{R}^N) \), we make use of the Minkowski and Young II inequalities with \( r \) as in (4.10), using the notation \( \int_X, \int_Y \) for \( \int_{\mathbb{R}^N} \) as in Proposition 4.3. More specifically, we have
\[
|u^\beta - u^\alpha|_q \leq \left( \int_X \left( \int_\alpha^\beta \sigma^{-N} \left| \int_{\mathbb{R}^N} \left\langle \Phi \left( \frac{x - y}{\sigma} \right), \nabla u(y) \right\rangle \, dy \right|^q \, d\sigma \right)^{1/q} \right)^{1/q}
\leq Minkowski \int_{\alpha}^\beta \int_{\mathbb{R}^N} \left| \Phi \left( \frac{x - y}{\sigma} \right) \right|^r \, dy \, d\sigma \tag{7.4}
\leq Young II \int_{\alpha}^\beta \int_{\mathbb{R}^N} \left| \nabla u(x) \right|^p \, dx \, d\sigma
\leq |\nabla u|_p \left( \int_{B(1)} \left| \Phi (x) \right|^r \, dx \right)^{1/r} \int_{\alpha}^\beta \sigma^{N(1/r - 1)} \, d\sigma.
\]
We have \( N(1/r - 1) = \kappa - 1 \), hence
\[
|u^\beta - u^\alpha|_q \leq C_{pq} (\beta^{\kappa} - \alpha^{\kappa}) |\nabla u|_p \tag{7.5}
\]
with \( C_{pq} = |\Phi|_r / \kappa \). Hence, for every sequence \( \sigma_i \to 0^+ \), \( u^{\sigma_i} \) is a Cauchy sequence in \( L^q(\mathbb{R}^N) \).

By (6.4), \( u^{\sigma_i} \) converge to \( u \) in \( L^p(\mathbb{R}^N) \), hence \( u \in L^q(\mathbb{R}^N) \) and \( u^{\sigma_i} \) converge to \( u \) (strongly) in \( L^q(\mathbb{R}^N) \). Letting \( \alpha \) tend to 0 and replacing \( \beta \) by \( \sigma \), we thus obtain (7.1). □
Corollary 7.2 Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set with Lipschitzian boundary, and let

$$\frac{1}{q} > \frac{1}{p} - \frac{1}{N}.$$ 

Then the space $W^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$.

Proof. Assume first $q \geq p$, and set $u_* = E_p u$, where $E_p : W^{1,p}(\Omega) \to W^{1,p}_B$ is the prolongation operator from Theorem 3.1. By Theorems 3.1 and 7.1, there exist constants $C_1$ and $C_2$ such that for every $\sigma \in (0,1]$ we have

$$|u_*^\sigma - u_*|_q \leq C_1 \sigma^\kappa |\nabla u_*|_p \leq C_2 \sigma^\kappa \|u\|_{1,p,\Omega}.$$  \hspace{1cm} (7.6)

By (6.5) and Theorem 3.1 we have

$$|u_*^\sigma|_q \leq C_3 \sigma^{\kappa-1} |u_*|_p \leq C_3 c_p \sigma^{\kappa-1} |u|_{p,\Omega},$$  \hspace{1cm} (7.7)

with $C_3 = \|\varphi\|_p$. Consequently, there exists a constant $C_4 > 0$ such that

$$|u|_{q,\Omega} \leq |u_*|_q \leq C_4 \left( \sigma^{\kappa-1} |u|_{p,\Omega} + \sigma^\kappa \|u\|_{1,p,\Omega} \right)$$  \hspace{1cm} (7.8)

for all $\sigma \in (0,1]$. According to (1.1), $W^{1,p}(\Omega)$ is thus embedded in $L^q(\Omega)$. To see that the embedding is compact, consider a bounded set $M \subset W^{1,p}(\Omega)$ and an arbitrary $\varepsilon > 0$. We fix $\sigma > 0$ such that, with the notation of Theorem 7.1, we have

$$C_{pq} \sigma^\kappa |\nabla u_*|_p < \varepsilon \quad \forall u \in M.$$  \hspace{1cm} (7.9)

With this fixed $\sigma$, every element $u_*^\sigma$ of the set $M_\sigma = \{u_*^\sigma : u \in M\}$ vanishes outside of the set $(1 + \sigma)B(1) =: B(1 + \sigma)$. Moreover, $M_\sigma$ is bounded in $C^1(B(1 + \sigma))$, hence, by Proposition 2.1, there exist $u_1, \ldots, u_n \in M$ such that

$$\forall u \in M \ \exists k \in \{1, \ldots, n\} \ \forall x \in B(1 + \sigma) : |u_*^\sigma(x) - u_k^\sigma(x)| < \frac{\varepsilon}{4 \text{meas}(B(1 + \sigma))}.$$  \hspace{1cm} (7.10)

We then have, by (7.9), (7.10), and Theorem 7.1, that

$$|u_* - u_k^\sigma|_q \leq |u_*^\sigma - u_k^\sigma|_q + \frac{\varepsilon}{2} < \frac{\varepsilon}{2}.$$  \hspace{1cm} (7.11)

For $k = 1, \ldots, n$ set $M_k = \{u \in M : |u_* - u_k^\sigma|_q < \varepsilon/2\}$, and $J = \{k \in \{1, \ldots, n\} : M_k \neq \emptyset\}$. For every $k \in J$ we fix one representative $\hat{u}_k \in M_k$, so that for every $u \in M_k$ we have $|u - \hat{u}_k|_{q,\Omega} < \varepsilon$ and $M = \bigcup_{k \in J} M_k$. The proof is thus complete for $q \geq p$. Let now $q < p$. Hölder’s inequality yields

$$|u_*^\sigma - u_*|_q \leq (\text{meas}(B(1 + \sigma)))^{1/q - 1/p} |u_*^\sigma - u_*|_p,$$

hence the above argument remains valid. \hfill \blacksquare

Corollary 7.3 Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set with Lipschitzian boundary, and let $p > N$. Then the space $W^{1,p}(\Omega)$ is compactly embedded in $C(\Omega)$. 
Proof. We repeat the argument of the proof of Theorem 7.1 and Corollary 7.2, putting
\[ \kappa := 1 - \frac{N}{p} \in (0, 1). \]
A computation analogous to (7.4) yields for every \( x \in \mathbb{R}^N \) that
\[
|u^\beta(x) - u^\alpha(x)| \leq \int_\alpha^\beta \sigma^{-N} \int_{\Omega} \Phi \left( \frac{x - y}{\sigma} \right) \left| \nabla u_*(y) \right| dy d\sigma
\leq \int_\alpha^\beta \sigma^{-N} \int_{\mathbb{R}^N} \Phi \left( \frac{y}{\sigma} \right) |y|^{p'} dy \left( \int_{\mathbb{R}^N} \left| \nabla u_*(x) \right|^p dx \right)^{1/p} d\sigma
\leq |\nabla u_*|_p \left( \int_{B(1)} \Phi(x) |x|^{p'} dx \right)^{1/p} \int_\alpha^\beta \sigma^{-N/p} d\sigma,
\]
and we proceed as above. \( \blacksquare \)

8 Limit cases and counterexamples

Proposition 8.1 Let \( \Omega \subset \mathbb{R}^N, N \geq 2 \), be an open bounded connected set with Lipschitzian boundary, and let
\[ \frac{1}{q} = \frac{1}{p} - \frac{1}{N}. \]
Then the space \( W^{1,p'}(\Omega) \) is embedded in \( L^q(\Omega) \).

Proof. We proceed in principle as in the proof of Theorem 7.1. The main difference is that the number \( \kappa \) is zero here and we have to proceed more carefully. We represent \( x \in \mathbb{R}^N \) as \( x = (x', x_N), x' \in \mathbb{R}^{N-1} \), and rewrite inequality (7.3) as
\[
|u^\beta(x', x_N) - u^\alpha(x', x_N)| \leq \int_\alpha^\beta \sigma^{-N} \int_{\mathbb{R}^N} \Phi \left( \frac{x' - y', x_N - y_N}{\sigma} \right) \left| \nabla u(y', y_N) \right| dy' dy_N d\sigma.
\]
With \( r = N/(N - 1) \), we now repeat the computation from (7.4), restricted to the component \( x' \), to obtain
\[
|u^\beta(\cdot, x_N) - u^\alpha(\cdot, x_N)|_q
\leq \left( \int_{\mathbb{R}^{N-1}} \left( \int_\alpha^\beta \sigma^{-N} \int_{\mathbb{R}^{N-1}} \Phi \left( \frac{x' - y', x_N - y_N}{\sigma} \right) \left| \nabla u(y', y_N) \right| dy' dy_N d\sigma \right)^q dx' \right)^{1/q}
\leq \int_\alpha^\beta \sigma^{-N} \int_{\mathbb{R}^{N-1}} \Phi \left( \frac{y'}{\sigma} \right) |y'| \left( \int_{\mathbb{R}^{N-1}} \left| \nabla u(x', y_N) \right|^p dx' \right)^{1/p} d\sigma dy_N
\leq \int_\alpha^\beta \sigma^{-N} \int_{\mathbb{R}^{N-1}} \left| \nabla u_*(y_N) \right|_{p} \Phi \left( \frac{y_N}{\sigma} \right) d\sigma dy_N.
\]

(8.3)
The function $|\Phi(\cdot, x_N - y_N)|_q$ vanishes if $\sigma < |x_N - y_N|$. Moreover, $\Phi$ is bounded by a constant $\Phi_0 > 0$. Hence, using the fact that $-N + N - 1/r = -2 + 1/N$, we have

$$
\int_{\alpha}^{\beta} \sigma^{-N+N-1/r} |\Phi(\cdot, x_N - y_N)/\sigma)|_r \, d\sigma \leq \Phi_0 \int_{|x_N - y_N|}^{\infty} \sigma^{-2+1/N} \, d\sigma = \Phi_0 r |x_N - y_N|^{1/r}.
$$

We thus have

$$
|u^\beta(\cdot, x_N) - u^\alpha(\cdot, x_N)|_q \leq \Phi_0 r \int_{R} |\nabla u(\cdot, y_N)|_p |x_N - y_N|^{-1/r} \, dy_N.
$$

At this point, we use the Hardy-Littlewood inequality (5.1), with $q$ replaced by $q'$. Indeed,

$$
\frac{1}{q'} + \frac{1}{p} + \frac{1}{r} = 2.
$$

Hence, for every function $g \in L^q(R)$ we have by Proposition 5.1 that

$$
\int_{R} |u^\beta(\cdot, x_N) - u^\alpha(\cdot, x_N)|_q g(x_N) \, dx_N \leq C |g|_{q'} |\nabla u|_p
$$

with some constant $C > 0$, hence, by the reverse Hölder inequality (4.9), we have

$$
|u^\beta - u^\alpha|_q \leq C |\nabla u|_p. \quad (8.4)
$$

Since $u^\sigma$ converge strongly to $u$ in $L^p(R^N)$ and their $L^q$ norms are bounded, we conclude that they converge strongly in $L^q(R^N)$ as well and the embedding formula follows.

We now show a few examples to illustrate that the embedding inequalities are (at least qualitatively) optimal.

(i) To see that the embedding in Proposition 8.1 is not compact, and that $W^{1,p}(\Omega)$ is not embedded in $L^q(\Omega)$ if

$$
\frac{1}{q} < \frac{1}{p} - \frac{1}{N},
$$

it suffices to fix any open set $\Omega$, some $x_0 \in \Omega$, find $s_0 > 0$ such that $x_0 + s_0 B(1) \subset \Omega$, and consider the family of functions

$$
u_s(x) = s^{1-N/p} \varphi \left( \frac{x - x_0}{s} \right), \quad s \in (0, s_0),
$$

with $\varphi$ as in (6.2). We have

$$
|u_s|_{p, \Omega} = s |\varphi|_p, \quad \left| \frac{\partial u_s}{\partial x_i} \right|_{p, \Omega} = \left| \frac{\partial \varphi}{\partial x_i} \right|_p, \quad \left| u_s \right|_{q, \Omega} = s^\alpha |\varphi|_q \quad \forall s \in (0, s_0),
$$

where $\alpha = 1 - N(1/p - 1/q)$. In the case (8.1), we have $\alpha = 0$. Using the fact that $u_s$ converge to 0 in $L^p(\Omega)$ as $s \to 0^+$, we conclude that the family $\{u_s\}$, having constant nonzero norm in $L^q(\Omega)$, does not contain any convergent subsequence in $L^q(\Omega)$, hence the embedding is not compact. In the case (8.5), we have $\alpha < 0$, hence the family $\{u_s\}$ is unbounded in $L^q(\Omega)$ and no embedding takes place.
In another limit case
\[ p = N , \]
the space \( W^{1,p}(\Omega) \) is embedded in \( L^\infty(\Omega) \) if and only if \( p = N = 1 \), and the embedding is not compact. For \( N \geq 2 \), it suffices to consider \( \Omega = B(1) \), and
\[ u(x) = \left( -\log \left( \frac{|x|}{2} \right) \right)^\alpha, \]
for any \( 0 < \alpha < 1 - 1/N \). Then \( u \) is unbounded, but belongs to \( W^{1,N}(B(1)) \). For \( N = 1 \), the embedding of \( W^{1,1}(\Omega) \) into \( C(\bar{\Omega}) \) (hence \( L^\infty(\Omega) \)) for every bounded interval \( \Omega \) is obvious. To see that it is not compact, we may consider for \( n \in \mathbb{N} \) the sequence
\[ u_n(x) = \begin{cases} \sin \frac{1}{x} & \text{for } x \in \left[ \frac{1}{(n+1)\pi}, \frac{1}{n\pi} \right] \\ 0 & \text{otherwise.} \end{cases} \]
It is bounded in \( W^{1,1}(0, 1/\pi) \), but \( \sup |u_n(x) - u_m(x)| = 1 \) for all \( m \neq n \), hence it is not precompact in \( L^\infty(0, 1/\pi) \).

The assumption on the Lipschitzian boundary is substantial. We show that there exists an open simply connected set \( \Omega \subset \mathbb{R}^2 \) such that \( W^{1,p}(\Omega) \) is not embedded in \( L^q(\Omega) \) for any \( q > p \geq 1 \). This set can be defined as (see Fig. 4)

\[ \Omega = \{ x = (x_1, x_2) \in \mathbb{R}^2; 0 < x_1 < 1, 0 < x_2 < e^{-1/x_1} \}. \]

For any \( q > p \) we set
\[ u_{pq}(x) = e^{2/(p+q)x_1}. \]
Then \( u_{pq} \in W^{1,p}(\Omega) \), but \( u_{pq} \notin L^q(\Omega) \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{non-lipschitzian_boundary.png}
\caption{Non-Lipschitzian boundary.}
\end{figure}

9 Anisotropic embeddings

In evolution problems, one deals with functions which depend on a space variable \( x \in \Omega \) and time \( t \in \omega \), where \( \omega \subset \mathbb{R} \) is an open interval corresponding to the time of the process. For \( 1 \leq p, q < \infty \), we introduce the spaces
\[ L^p(\omega; L^q(\Omega)) = \left\{ u \in L^1(\Omega \times \omega) : |u|_{p,q;\Omega,\omega} := \left( \int_\omega |u(\cdot, t)|_q^{p,\Omega} dt \right)^{1/p} < \infty \right\}, \]
(9.1)
with obvious modifications for $p = \infty$ or $q = \infty$.

We state explicitly one possible embedding result for such spaces, without going into much detail in the proof, which is fully analogous to the above ones.

**Theorem 9.1** Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set with Lipschitzian boundary, let $\omega$ be a bounded open interval, and let $W^{p_0,q_0;p_1,q_1}(\omega;\Omega)$ be the space

$$W^{p_0,q_0;p_1,q_1}(\omega;\Omega) = \left\{ u \in L^1(\Omega \times \omega) : \frac{\partial u}{\partial t} \in L^{p_0}(\omega; L^{q_0}(\Omega)), \right.$$

$$\left. \frac{\partial u}{\partial x_i} \in L^{p_1}(\omega; L^{q_1}(\Omega)) \text{ for } i = 1, \ldots, N \right\}.$$

If

$$\frac{p'_0}{p_1 q_0} + \frac{1}{q_1} < \frac{1}{N}, \quad (9.2)$$

then the space $W^{p_0,q_0;p_1,q_1}(\omega;\Omega)$ is compactly embedded in $C(\bar{\Omega} \times \bar{\omega})$. If $q_2 \geq \max\{q_0,q_1\}$, $p_2 \geq \max\{p_0,p_1\}$, and

$$\left(1 - \frac{1}{p_0} + \frac{1}{p_2}\right) \left(1 \frac{1}{q_0} + \frac{1}{q_2}\right) > \left(\frac{1}{p_1} - \frac{1}{p_2}\right) \left(\frac{1}{q_0} - \frac{1}{q_2}\right), \quad (9.3)$$

then $W^{p_0,q_0;p_1,q_1}(\omega;\Omega)$ is compactly embedded in $L^{p_2}(\omega; L^{q_2}(\Omega))$.

**Hint for the proof.** Consider as before the extensions to the space $W^{p_0,q_0;p_1,q_1}(\mathbb{R};\mathbb{R}^N)$, where the norms $\| \cdot \|_{p_i,q_i,\Omega,\omega}$ are denoted again for simplicity as $\| \cdot \|_{p_i,q_i}, i = 0, 1$. For $\sigma \in (0,1]$ and $u \in W^{p_0,q_0;p_1,q_1}(\mathbb{R};\mathbb{R}^N)$, we define regularizations analogous to (6.2) in the form

$$u^\sigma(x,t) = \sigma^{-N-\lambda} \int_{\mathbb{R}} \int_{\mathbb{R}^N} \varphi \left( \frac{x-y}{\sigma}, \frac{t-s}{\sigma^\lambda} \right) u(y,s) \, dy \, ds, \quad (9.4)$$

where $\varphi$ is a smooth nonnegative function on $\mathbb{R}^{N+1}$, which vanishes outside $B(1) \times (-1,1)$, and

$$\int_{-1}^{1} \int_{B(1)} \varphi(x,t) \, dx \, dt = 1.$$

The number $\lambda$ is to be chosen as

$$\lambda = \frac{1 + N \left( \frac{1}{q_0} - \frac{1}{q_1} \right)}{\frac{1}{p_0} + \frac{1}{p_1}}. \quad (9.5)$$

Note that $\lambda > 0$ by (9.3). A computation similar to (7.2)–(7.4) yields

$$\frac{\partial}{\partial \sigma} u^\sigma(x,t) = -\lambda \sigma^{-N-1} \int_{\mathbb{R}^N} \Phi_0 \left( \frac{x-y}{\sigma}, \frac{t-s}{\sigma^\lambda} \right) \frac{\partial u}{\partial s}(y,s) \, dy \, ds$$

$$-\sigma^{-N-\lambda} \int_{\mathbb{R}^N} \left( \Phi_1 \left( \frac{x-y}{\sigma}, \frac{t-s}{\sigma^\lambda} \right), \nabla_y u(y,s) \right) \, dy \, ds, \quad (9.6)$$
where $\Phi_0(\xi, \tau) = \tau \varphi(\xi, \tau)$, $\Phi_1(\xi, \tau) = \xi \varphi(\xi, \tau)$, hence

$$|u^\beta(x, t) - u^\alpha(x, t)| \leq \lambda I_0(x, t) + I_1(x, t)$$

(9.7)

for $0 < \alpha < \beta \leq 1$, where

$$I_0(x, t) = \int_\alpha^\beta \sigma^{-N-1} \int_\mathbb{R} \int_{\mathbb{R}^N} |\Phi_0 \left( \frac{x - y}{\sigma}, \frac{t - s}{\sigma^\lambda} \right)| \left| \frac{\partial u}{\partial s} (y, s) \right| dy \, ds \, d\sigma,$$

$$I_1(x, t) = \int_\alpha^\beta \sigma^{-N-\lambda} \int_\mathbb{R} \int_{\mathbb{R}^N} |\Phi_1 \left( \frac{x - y}{\sigma}, \frac{t - s}{\sigma^\lambda} \right)| \left| \nabla_y u(y, s) \right| dy \, ds \, d\sigma.$$  

(9.8)

Let (9.3) hold. With the intention to use Young’s inequality for convolutions again, we introduce the numbers $r_0, s_0, r_1, s_1$ by the identities

$$\frac{1}{r_0} = 1 - \frac{1}{q_0} + \frac{1}{q_2}, \quad \frac{1}{s_0} = 1 - \frac{1}{p_0} + \frac{1}{p_2}, \quad \frac{1}{r_1} = 1 - \frac{1}{q_1} + \frac{1}{q_2}, \quad \frac{1}{s_1} = 1 - \frac{1}{p_1} + \frac{1}{p_2}.$$  

(9.9)

We use again the notation $\int_X dx$, $\int_Y dy$ for $\int_{\mathbb{R}^N} dx$, $\int_{\mathbb{R}^N} dy$, and $\int_I dt$, $\int_S ds$ for $\int_{\mathbb{R}} dt$, $\int_{\mathbb{R}} ds$. For $t \in \mathbb{R}$, we have

$$[I_0(\cdot, t)]_{q_2} \leq \int_\alpha^\beta \sigma^{-N-1} \int_X \left( \int_Y \left( \int_Y \left[ \Phi_0 \left( \frac{x - y}{\sigma}, \frac{t - s}{\sigma^\lambda} \right) \left| \frac{\partial u}{\partial s} (y, s) \right| dy \right] dx \right)^{q_2} \right)^{1/q_2} ds \, d\sigma$$

Young II

$$\leq \int_\alpha^\beta \sigma^{-N-1} \int_X \left[ \Phi_0 \left( \frac{t - s}{\sigma^\lambda} \right) \left| \frac{\partial u}{\partial s} (\cdot, s) \right|_{q_0} \right] ds \, d\sigma$$

$$= \int_\alpha^\beta \sigma^{-N-1+N/r_0} \int_X \left[ \Phi_0 \left( \frac{t - s}{\sigma^\lambda} \right) \left| \frac{\partial u}{\partial s} (\cdot, s) \right|_{q_0} \right] ds \, d\sigma,$$

(9.10)

hence

$$[I_0]_{p_2, q_2} \leq \int_\alpha^\beta \sigma^{-N-1+N/r_0} \left( \int_T \left( \int_S \left[ \Phi_0 \left( \frac{t - s}{\sigma^\lambda} \right) \left| \frac{\partial u}{\partial s} (\cdot, s) \right|_{q_0} \right] ds \right)^{p_2} \right)^{1/p_2} dt \, d\sigma$$

Young II

$$\leq \int_\alpha^\beta \sigma^{-N-1+N/r_0} \left[ \Phi_0 \left( \frac{t - s}{\sigma^\lambda} \right) \left| \frac{\partial u}{\partial s} \right|_{p_0, q_0} \right] ds \, d\sigma$$

$$= \left| \Phi_0 \right|_{s_0, r_0} \left| \frac{\partial u}{\partial s} \right|_{p_0, q_0} \int_\alpha^\beta \sigma^{-N-1+N/r_0+\lambda/s_0} \, d\sigma.$$  

(9.11)

Similarly,

$$[I_1(\cdot, t)]_{q_2} \leq \int_\alpha^\beta \sigma^{-N-\lambda} \int_X \left( \int_Y \left[ \Phi_1 \left( \frac{x - y}{\sigma}, \frac{t - s}{\sigma^\lambda} \right) \left| \nabla_y u(y, s) \right| dy \right] dx \right)^{q_2} \, ds \, d\sigma$$

Young II

$$\leq \int_\alpha^\beta \sigma^{-N-\lambda} \int_X \left[ \Phi_1 \left( \frac{t - s}{\sigma^\lambda} \right) \right] \left| \nabla_y u(\cdot, s) \right|_{q_1} ds \, d\sigma$$

$$= \int_\alpha^\beta \sigma^{-N-\lambda+N/r_1} \int_X \left[ \Phi_1 \left( \frac{t - s}{\sigma^\lambda} \right) \right] \left| \nabla_y u(\cdot, s) \right|_{q_1} ds \, d\sigma,$$

(9.12)
hence

\[ |I_1|_{p_2,q_2} \leq \int_\alpha^\beta \sigma^{-N-\lambda+N/r_1} \left( \int_T \left( \int_S |\Phi_1 \left( \frac{t-s}{\sigma \lambda} \right) \right)|\nabla_y u(\cdot, s)|_{q_1} ds \right)_{r_1}^{p_2} dt \right)^{1/p_2} d\sigma \]

\[ \leq \int_\alpha^\beta \sigma^{-N-\lambda+N/r_1} \left| \int_T \Phi_1 \left( \frac{t-s}{\sigma \lambda} \right) \right|_{s_1,r_1} |\nabla_y u|_{p_1,q_1} d\sigma \]

Young II

\[ = |\Phi_1|_{s_1,r_1} |\nabla_y u|_{p_1,q_1} \int_\sigma^\beta \sigma^{-N-\lambda+N/r_1+\lambda/s_1} d\sigma . \]  

(9.13)

Set

\[ \kappa = N \left( \frac{1}{p_0} + \frac{1}{p_1} \right)^{-1} \left( \frac{1}{p_0} - \frac{1}{p_1} + \frac{1}{p_2} \right) \left( \frac{1}{N} - \frac{1}{q_1} + \frac{1}{q_2} \right) - \left( \frac{1}{q_0} - \frac{1}{q_1} \right) \left( \frac{1}{p_1} - \frac{1}{p_2} \right) \]  

Then \( \kappa > 0 \) by (9.3), and we have

\[-N - 1 + \frac{N}{r_0} + \frac{\lambda}{s_0} = -N - \lambda + \frac{N}{r_1} + \frac{\lambda}{s_1} = \kappa - 1 . \]

Combining (9.7) with (9.8), (9.11), and (9.13) yields

\[ |u^\beta - u^\alpha|_{p_2,q_2} \leq C_{p_0,p_1,p_2,q_0,q_2} (\beta^\kappa - \alpha^\kappa) \left( \frac{\partial u}{\partial t} \right)_{p_0,q_0} + |\nabla_x u|_{p_1,q_1} , \]  

(9.14)

and we obtain the result similarly as in Theorem 7.1. \( \blacksquare \)

Note that the order of integration in (9.1) cannot be reversed. For \( p \geq q \) we have by Remark 4.5 that \( L^q(\Omega; L^p(\omega)) \) is embedded into \( L^p(\omega; L^q(\Omega)) \), but the opposite inclusion does not hold, see Example 4.4. On the other hand, denoting

\[ W^{q_0,p_0,q_1,p_1}(\Omega; \omega) = \left\{ u \in L^1(\Omega \times \omega) ; \frac{\partial u}{\partial t} \in L^{q_0}(\Omega; L^{p_0}(\omega)) , \frac{\partial u}{\partial x_i} \in L^{q_1}(\Omega; L^{p_1}(\omega)) \right\} , \]

we may repeat the computations in (9.10)–(9.13) with reversed order of integration, to check that conditions (9.2) and (9.3) remain valid for the compact embedding of \( W^{q_0,p_0,q_1,p_1}(\Omega; \omega) \) into \( C(\Omega \times \bar{\omega}) \) and \( L^{q_1}(\Omega; L^{p_1}(\omega)) \), respectively. Let us mention one important particular case which frequently occurs in applications. We omit the proof which is the same as for the other cases.

**Corollary 9.2** If \( q_2 \geq \max\{q_0,q_1\} \), and

\[ \frac{1}{p_0} \left( \frac{1}{N} - \frac{1}{q_1} + \frac{1}{q_2} \right) > \frac{1}{p_1} \left( \frac{1}{q_0} - \frac{1}{q_2} \right) , \]  

(9.15)

then the space \( W^{q_0,p_0,q_1,p_1}(\Omega; \omega) \) is compactly embedded in \( L^{q_2}(\Omega; C(\bar{\omega})) \).
Embeddings of function spaces that are “anisotropic” also in the space variables, for example
\[
\frac{\partial u}{\partial x_i} \in L^p(\Omega; L^q(\Omega)), \ i = 1, \ldots, N,
\]
can be treated in the same way. The regularizations then have to be chosen in the form
\[
u^\sigma(x, t) = \sigma^{-1} - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \varphi \left( \frac{x_1 - y_1}{\sigma^{\mu_1}}, \ldots, \frac{x_N - y_N}{\sigma^{\mu_N}}, \frac{t - s}{\sigma} \right) u(y, s) \, dy \, ds, \tag{9.16}
\]
with suitably chosen exponents \(\mu_1, \ldots, \mu_N\).

## 10 Interpolations

We first recall the following classical interpolation result in \(L^p\) spaces.

**Proposition 10.1** Let \(\Omega \subset \mathbb{R}^N\) be an open set (bounded or unbounded), and let \(1 \leq p_0 < p_1 \leq \infty\) be given. If \(u \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)\), then \(u \in L^p(\Omega)\) for all \(p \in [p_0, p_1]\), and we have

\[
|u|_{p, \Omega} \leq |u|_{p_0, \Omega}^{1 - \alpha} |u|_{p_1, \Omega}^\alpha
\]
for all \(u \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)\), where

\[
\alpha = \frac{1}{p_0} - \frac{1}{p_1}.
\]

**Proof.** Set \(q = p_1/\alpha p\). Then \(q' = p_0/(1 - \alpha) p\), and we may use Hölder’s inequality to obtain

\[
|u|_{p, \Omega} = \left( \int_{\Omega} |u(x)|^{(1-\alpha)p} |u(x)|^{\alpha p} \, dx \right)^{1/p}
\leq \left( \int_{\Omega} |u(x)|^{(1-\alpha)p'q} \, dx \right)^{1/pq'} \left( \int_{\Omega} |u(x)|^{\alpha pq} \, dx \right)^{1/pq}
= |u|_{p_0, \Omega}^{1 - \alpha} |u|_{p_1, \Omega}^\alpha.
\]

\[\blacksquare\]

We now establish an interpolation formula between \(L^p\) spaces and Sobolev spaces.

**Theorem 10.2** Let \(p, q, s \in (1, \infty)\) be such that

\[
\frac{1}{s} > \frac{1}{q} > \frac{1}{p} - \frac{1}{N},
\]
and set

\[
\kappa := 1 - N \left( \frac{1}{p} - \frac{1}{q} \right), \quad \gamma = N \left( \frac{1}{s} - \frac{1}{q} \right).
\]

Then there exists \(C_{pq} > 0\) such that for every \(u \in W^{1,p}(\mathbb{R}^N) \cap L^s(\mathbb{R}^N)\) and every \(\sigma \in (0, 1]\) we have

\[
|u|_q \leq C_{pq} (|\sigma^{\kappa} |u|_s + |\sigma^{\gamma} \nabla u|_p). \tag{10.1}
\]
Proof. The assertion follows from (6.5) and (7.1) provided $q \geq p$. In particular, for $q = p$ we have $\kappa = 1$, $\gamma = \gamma_0 := N(1/s - 1/p)$, and

$$|u|_p \leq C_{ppsp}(\sigma^{-\gamma_0}|u|_s + \sigma|\nabla u|_p).$$

(10.2)

Let now $q < p$. By Proposition 10.1 we have

$$|u|_q \leq |u|_s^{1-\alpha}|u|_p^\alpha,$$

where

$$\alpha = \frac{1}{s} - \frac{1}{q}.$$

This yields

$$|u|_p \leq C^\alpha_{ppsp}(\sigma^{-\alpha\gamma_0}|u|_s + \sigma^\alpha|u|_s^{1-\alpha}|\nabla u|_p^\alpha).$$

We now use inequality (4.4) with $p$ replaced by $1/\alpha$, and with $x = \mu\sigma\alpha|\nabla u|_p^\alpha$, $y = |u|_s^{-\alpha}/\mu$, where we set $\mu = \sigma^{(1-\alpha)\alpha\gamma_0}$, and obtain

$$\sigma^\alpha|u|_s^{1-\alpha}|\nabla u|_p^\alpha \leq \alpha\sigma^{1+(1-\alpha)\gamma_0}|\nabla u|_p + (1 - \alpha)\sigma^{-\gamma_0}|u|_s.$$

Hence,

$$|u|_p \leq 2C^\alpha_{ppsp}(\sigma^{-\alpha\gamma_0}|u|_s + \sigma^{1+(1-\alpha)\gamma_0}|\nabla u|_p),$$

which is precisely (10.1). $\blacksquare$

We conclude this text with the famous Gagliardo-Nirenberg inequality.

**Corollary 10.3 (Gagliardo-Nirenberg inequality)** Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set with Lipschitzian boundary, and let

$$\frac{1}{s} > \frac{1}{q} > \frac{1}{p} - \frac{1}{N}.$$

Set

$$\varrho = \frac{1}{s} - \frac{1}{q} + \frac{1}{p}.$$

Then there exists a constant $K_{ppsp} > 0$ such that for every $u \in W^{1,p}(\Omega)$ we have

$$|u|_{q,\Omega} \leq K_{ppsp}\left(|u|_{s,\Omega} + |u|_s^{1-\varrho}\|u\|_q^\varrho\right).$$

(10.3)

**Proof.** As in the proof of Corollary 7.2, we set $u_s = E_{p\mu}u$. By Theorem 10.2, we have

$$|u|_q \leq C_{ppsp}(\sigma^{-\gamma}|u|_s + \sigma^\alpha|\nabla u|_p).$$

(10.4)

If $|\nabla u_s|_p > |u|_s$, then we set

$$\sigma = \left(\frac{|u|_s}{|\nabla u|_p}\right)^{1/(\gamma + \kappa)},$$

otherwise we choose $\sigma = 1$. In both cases we obtain

$$|u|_q \leq 2C_{ppsp}\left(|u|_s + |u|_s^{\kappa/(\gamma + \kappa)}|\nabla u|_p^{\gamma/(\gamma + \kappa)}\right).$$

(10.5)

We have $\kappa/(\gamma + \kappa) = 1 - \varrho$, $\gamma/(\gamma + \kappa) = \varrho$, and the desired result follows from Theorem 3.1. $\blacksquare$

It is in principle possible to derive from (9.14) the corresponding interpolation inequalities also for anisotropic spaces. The general formulas then become, however, rather complicated and we omit them here.
References


