On a singular limit for the Navier-Stokes-Fourier system with an unexpected term

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Navier–Stokes–Fourier system, primitive system

Mass conservation:

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance:

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta)} = \operatorname{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) + \boxed{\frac{1}{\varepsilon} \varrho \nabla_x G}$$

Internal energy balance (heat equation):

$$\partial_t(\varrho e(\varrho, \vartheta)) + \operatorname{div}_x(\varrho e(\varrho, \vartheta) \mathbf{u}) + \operatorname{div}_x \mathbf{q}(\vartheta, \nabla_x \vartheta)$$
$$= \boxed{\varepsilon^2 \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u}} - p(\varrho, \vartheta) \operatorname{div}_x \mathbf{u}$$

Newton's law

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta) \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \operatorname{div}_x \mathbf{u} \mathbb{I} \right) + \eta(\vartheta) \operatorname{div}_x \mathbf{u} \mathbb{I}$$

Fourier's law

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta) \nabla_x \vartheta$$

Oberbeck–Boussinesq system, target system

Incompressibility:

$$\mathrm{div}_x \mathbf{U} = 0$$

Momentum balance:

$$\overline{\varrho}\Big(\partial_t \mathbf{U} + \mathbf{U}\cdot\nabla_x \mathbf{U}\Big) + \nabla_x \Pi = \mu(\overline{\vartheta})\Delta_x \mathbf{U} + r\nabla_x G$$

Heat equation:

$$\overline{\varrho} c_p(\overline{\varrho},\overline{\vartheta})\Big(\partial_t \Theta + \mathbf{U}\cdot\nabla_x \Theta\Big) - \overline{\varrho}\ \overline{\vartheta}\alpha(\overline{\varrho},\overline{\vartheta})\mathbf{U}\cdot\nabla_x G = \kappa(\overline{\vartheta})\Delta_x \Theta$$

Boussinesq relation:

$$\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \varrho}\nabla_x r + \frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}\nabla_x \Theta = \overline{\varrho}\nabla_x G.$$

$$\alpha(\overline{\varrho},\overline{\vartheta}) \equiv \frac{1}{\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}\left(\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \varrho}\right)^{-1}$$

$$c_p(\overline{\varrho},\overline{\vartheta}) \equiv \frac{\partial e(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta} + \overline{\varrho}^{-1}\overline{\vartheta}\alpha(\overline{\varrho},\overline{\vartheta})\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}$$

Spatial domain

Periodic strip

$$\Omega = \mathcal{T}^{d-1} \times [0,1]$$

$$\mathcal{T}^{d-1} = \left([-1,1]|_{\{-1;1\}} \right)^{d-1}$$

Conservative boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \mathbb{S} \cdot \mathbf{n} \times \mathbf{n}|_{\partial\Omega} = 0, \ \nabla_x \vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Dirichlet boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \ \vartheta|_{\partial\Omega} = \vartheta_B$$

Singular limit in the conservative case

From Navier–Stokes–Fourier to Oberbeck–Boussinesq

$$\mathbf{u}_\varepsilon \to \mathbf{U}$$

$$\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \to r$$

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \Theta$$

Convergence for the conservative boundary conditions, EF, A. Novotný 2009

If the initial data are ill prepared:

$$\mathbf{u}_\varepsilon(0,\cdot),\ \frac{\varrho_\varepsilon(0,\cdot) - \overline{\varrho}}{\varepsilon},\ \frac{\vartheta_\varepsilon(0,\cdot) - \overline{\vartheta}}{\varepsilon} \text{ bounded,}$$

then $\mathbf{U}$, r, and Θ solve the Oberberck–Boussinesq system with the conservative boundary conditions

$$\mathbf{U}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ \mathbb{S}(\overline{\vartheta}, \nabla_x \mathbf{U})\cdot\mathbf{n}\times\mathbf{n}|_{\partial\Omega} = 0,\ \nabla_x\Theta\cdot\mathbf{n}|_{\partial\Omega} = 0$$

Singular limit for the Dirichlet boundary conditions

Scaled boundary conditions for the temperature:

$$\vartheta_\varepsilon|_{\partial\Omega} = \overline{\vartheta} + \varepsilon\Theta_B$$

Initial data:

$$\mathbf{u}_\varepsilon(0,\cdot) \to \mathbf{U}_0 \text{ in } L^2(\Omega; R^d),\ \mathbf{U}_0|_{\partial\Omega} = 0,\ \mathrm{div}_x \mathbf{U}_0 = 0$$

$$\frac{\varrho_\varepsilon(0,\cdot) - \overline{\varrho}}{\varepsilon} \to r_0 \text{ in } L^1(\Omega) \text{ and weakly-(*) in } L^\infty(\Omega)$$

$$\frac{\vartheta_\varepsilon(0,\cdot) - \overline{\vartheta}}{\varepsilon} \to \mathfrak{T}_0 \text{ in } L^1(\Omega) \text{ and weakly-(*) in } L^\infty(\Omega)$$

Boussinesq relation

$$\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \varrho}\nabla_x r_0 + \frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}\nabla_x \mathfrak{T}_0 = \overline{\varrho}\nabla_x G.$$

Convergence for the Dirichlet boundary conditions

Convergence, P. Bella, EF, F. Oschmann 2022:

$$\lambda(\overline{\varrho}, \overline{\vartheta}) = \frac{\overline{\vartheta} \alpha(\overline{\varrho}, \overline{\vartheta})}{\overline{\varrho} c_p(\overline{\varrho}, \overline{\vartheta})} \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \in (0,1)$$

$$\mathbf{u}_\varepsilon \to \mathbf{U}$$

$$\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \to r$$

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} - \boxed{\lambda(\overline{\varrho}, \overline{\vartheta}) \frac{1}{|\Omega|} \int_\Omega \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \, \mathrm{d}x} \to \Theta,$$

where $(\mathbf{U}, r, \Theta)$ solves the Oberbeck–Boussinesq system with *nonlocal boundary conditions*

$$\Theta|_{\partial\Omega} = \Theta_B - \frac{\lambda(\overline{\varrho}, \overline{\vartheta})}{1 - \lambda(\overline{\varrho}, \overline{\vartheta})} \frac{1}{|\Omega|} \int_\Omega \Theta \, \mathrm{d}x$$

Weak solutions to the primitive system

Mass conservation:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance:

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) + \frac{1}{\varepsilon}\varrho \nabla_x G$$

Gibbs' relation

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right)$$

Entropy balance (inequality):

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta}\right)$$
$$\geq \frac{1}{\vartheta}\left(\varepsilon^2 \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta}\right)$$

Energy conservation, conservative boundary conditions

Total energy:

$$E_\varepsilon(\varrho, \vartheta, \mathbf{u}) = \frac{\varepsilon^2}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta)$$

Energy balance (inequality):

$$\partial_t E_\varepsilon(\varrho, \vartheta, \mathbf{u}) + \mathrm{div}_x (E_\varepsilon(\varrho, \vartheta, \mathbf{u})\mathbf{u}) + \mathrm{div}_x (p(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \mathbf{q}(\vartheta, \nabla_x \vartheta)$$
$$= \boxed{(\leq)} \varepsilon^2 \mathrm{div}_x \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) \cdot \mathbf{u} \right) + \varepsilon \varrho \nabla_x G \cdot \mathbf{u}$$

Energy conservation (balance), conservative boundary conditions:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega E_\varepsilon(\varrho, \vartheta, \mathbf{u}) \ \mathrm{d}x = (\leq) \varepsilon \int_\Omega \varrho \nabla_x G \cdot \mathbf{u} \ \mathrm{d}x$$

Basic properties of weak solutions

- **Global existence.** Under certain physically grounded restrictions imposed on the constitutive relations, the weak solutions exist globally in time for any finite energy initial data
- **Compatibility.** Any sufficiently smooth weak solution is a classical solution of the Navier–Stokes–Fourier system
- **Weak–strong uniqueness.** If $[\widetilde{\varrho}, \widetilde{\vartheta}, \widetilde{\mathbf{u}}]$ is a classical solution and $[\varrho, \vartheta, \mathbf{u}]$ a weak solution starting from the same initial data, then $\varrho = \widetilde{\varrho}$, $\vartheta = \widetilde{\vartheta}$, $\mathbf{u} = \widetilde{\mathbf{u}}$.

Problems with the Dirichlet boundary conditions

Lack of control of the boundary heat flux:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} E_{\varepsilon}(\varrho,\vartheta,\mathbf{u})\ \mathrm{d}x + \boxed{\int_{\partial\Omega} \mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\mathbf{n}\ \mathrm{d}\sigma_x} = (\leq)\varepsilon\int_{\Omega} \varrho\nabla_x G\cdot\mathbf{u}\ \mathrm{d}x$$

Ballistic energy:

$$E_{\varepsilon}(\varrho,\vartheta,\mathbf{u}) - \widetilde{\vartheta}\varrho s(\varrho,\vartheta)$$

$$\widetilde{\vartheta}|_{\partial\Omega} = \vartheta_B$$

Ballistic energy flux:

$$\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\mathbf{n} - \frac{\widetilde{\vartheta}}{\vartheta}\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\mathbf{n} = 0 \text{ on } \partial\Omega$$

Weak solutions revisited

Ballistic energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varepsilon^2 \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \widetilde{\vartheta}\varrho s(\varrho,\vartheta)\ \mathrm{d}x$$

$$+\int_\Omega \frac{\widetilde{\vartheta}}{\vartheta}\left(\varepsilon^2 \mathbb{S}(\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{u} - \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right)\ \mathrm{d}x$$

$$\leq \int_\Omega \left(\varepsilon\varrho\nabla_x G\cdot\mathbf{u} - \varrho s(\varrho,\vartheta)\partial_t\widetilde{\vartheta} - \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\widetilde{\vartheta} - \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\widetilde{\vartheta}}{\vartheta}\right)\ \mathrm{d}x$$

for any $\widetilde{\vartheta} > 0$, $\widetilde{\vartheta}|_{\partial\Omega} = \vartheta_B$

- **Compatibility [Chaudhuri–EF 2021].** Smooth weak solutions are classical solutions
- **Weak–strong uniqueness [Chaudhuri–EF 2021].** A weak solution coincides with the strong solution as long as the latter exists

Relative energy as Bregman distance

Thermodynamic stability:

$$\frac{\partial p(\varrho,\vartheta)}{\partial \varrho} > 0, \ \ \frac{\partial e(\varrho,\vartheta)}{\partial \vartheta} > 0 \text{ for all } \varrho, \vartheta > 0$$

Relative energy:

$$E_\varepsilon\left(\varrho,\vartheta,\mathbf{u}\middle|\widetilde{\varrho},\widetilde{\vartheta},\widetilde{\mathbf{u}}\right) = \frac{1}{2}\varrho|\mathbf{u}-\widetilde{\mathbf{u}}|^2$$

$$+\frac{1}{\varepsilon^2}\left[\varrho e - \widetilde{\vartheta}\Big(\varrho s - \widetilde{\varrho} s(\widetilde{\varrho},\widetilde{\vartheta})\Big) - \left(e(\widetilde{\varrho},\widetilde{\vartheta}) - \widetilde{\vartheta} s(\widetilde{\varrho},\widetilde{\vartheta}) + \frac{p(\widetilde{\varrho},\widetilde{\vartheta})}{\widetilde{\varrho}}\right)(\varrho - \widetilde{\varrho}) - \widetilde{\varrho} e(\widetilde{\varrho},\widetilde{\vartheta})\right]$$

Bregmann distance:

$$E_\varepsilon\left(\varrho,S,\mathbf{m}\middle|\widetilde{\varrho},\widetilde{S},\widetilde{\mathbf{m}}\right) = E_\varepsilon(\varrho,S,\mathbf{m})$$

$$-\left\langle \partial_{\varrho,S,\mathbf{m}} E_\varepsilon(\widetilde{\varrho},\widetilde{S},\widetilde{\mathbf{m}}); (\varrho-\widetilde{\varrho}, S-\widetilde{S}, \mathbf{m}-\widetilde{\mathbf{m}})\right\rangle - E_\varepsilon(\widetilde{\varrho},\widetilde{S},\widetilde{\mathbf{m}})$$

$$(\varrho,S,\mathbf{m}) \mapsto E_\varepsilon(\varrho,S,\mathbf{m}) \text{ (strictly) convex}$$

Relative energy inequality

$$\left[\int_\Omega E_\varepsilon \left(\varrho, \vartheta, \mathbf{u} \Big| \widetilde{\varrho}, \widetilde{\vartheta}, \widetilde{\mathbf{u}}\right) \ \mathrm{d}x\right]_{t=0}^{t=\tau}$$

$$+ \int_0^\tau \int_\Omega \frac{\widetilde{\vartheta}}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + \frac{1}{\varepsilon^2} \frac{\kappa(\vartheta)|\nabla_x \vartheta|^2}{\vartheta} \right) \ \mathrm{d}x\mathrm{d}t$$

$$\leq -\frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \left(\varrho(s - s(\widetilde{\varrho}, \widetilde{\vartheta}))\partial_t \widetilde{\vartheta} + \varrho(s - s(\widetilde{\varrho}, \widetilde{\vartheta}))\mathbf{u} \cdot \nabla_x \widetilde{\vartheta} - \frac{\kappa(\vartheta)\nabla_x \vartheta}{\vartheta} \cdot \nabla_x \widetilde{\vartheta} \right) \ \mathrm{d}x\mathrm{d}t$$

$$- \int_0^\tau \int_\Omega \left[\varrho(\mathbf{u} - \widetilde{\mathbf{u}}) \otimes (\mathbf{u} - \widetilde{\mathbf{u}}) + \frac{1}{\varepsilon^2} p(\varrho, \vartheta)\mathbb{I} - \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) \right] : \nabla_x \widetilde{\mathbf{u}} \ \mathrm{d}x\mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \varrho \left[\frac{1}{\varepsilon} \nabla_x G - \partial_t \widetilde{\mathbf{u}} - (\widetilde{\mathbf{u}} \cdot \nabla_x)\widetilde{\mathbf{u}} \right] \cdot (\mathbf{u} - \widetilde{\mathbf{u}}) \ \mathrm{d}x\mathrm{d}t$$

$$+ \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \left[\left(1 - \frac{\varrho}{\widetilde{\varrho}} \right) \partial_t p(\widetilde{\varrho}, \widetilde{\vartheta}) - \frac{\varrho}{\widetilde{\varrho}} \mathbf{u} \cdot \nabla_x p(\widetilde{\varrho}, \widetilde{\vartheta}) \right] \ \mathrm{d}x\mathrm{d}t$$

any trio of continuously differentiable functions $(\widetilde{\varrho}, \widetilde{\vartheta}, \widetilde{\mathbf{u}})$

$$\widetilde{\varrho} > 0, \ \widetilde{\vartheta} > 0, \ \widetilde{\vartheta}|_{\partial\Omega} = \widetilde{\vartheta}, \ \widetilde{\mathbf{u}}|_{\partial\Omega} = 0.$$

Constitutive theory, I

Gibbs' law

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right)$$

Thermodynamic stability

$$\frac{\partial p(\varrho,\vartheta)}{\partial \varrho} > 0, \ \ \frac{\partial e(\varrho,\vartheta)}{\partial \vartheta} > 0$$

Pressure EOS

$$p(\varrho,\vartheta) = p_{\rm m}(\varrho,\vartheta) + p_{\rm rad}(\vartheta)$$

$$p_{\rm m}(\varrho,\vartheta) = \frac{2}{3}\varrho e_{\rm m}(\varrho,\vartheta), \ \ p_{\rm rad}(\vartheta) = \frac{a}{3}\vartheta^4, \ \ a > 0$$

Internal energy

$$e(\varrho,\vartheta) = e_{\rm m}(\varrho,\vartheta) + e_{\rm rad}(\varrho,\vartheta), \ \ e_{\rm rad}(\varrho,\vartheta) = \frac{a}{\varrho}\vartheta^4.$$

Constitutive theory, II

- $$\text{Gibbs' law} \ \Rightarrow \ p_m(\varrho, \vartheta) = \vartheta^{\frac{5}{2}} P\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}} \right)$$
- $$\text{Thermodynamic stability} \ \Rightarrow \ \frac{P(Z)}{Z^{\frac{5}{3}}} \searrow p_\infty > 0 \text{ as } Z \to \infty$$
- **Entropy.**
$$s(\varrho, \vartheta) = \mathcal{S}\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}} \right) + \frac{4a}{3}\frac{\vartheta^3}{\varrho}, \ \mathcal{S}' < 0$$
- **Third law of thermodynamics.**
$$\mathcal{S}(Z) \searrow 0 \text{ as } Z \to \infty$$

Constitutive theory, III

Viscosity.

$$0 < \underline{\mu}(1 + \vartheta) \leq \mu(\vartheta), \ |\mu'(\vartheta)| \leq \overline{\mu}$$

$$0 \leq \eta(\vartheta) \leq \overline{\eta}(1 + \vartheta)$$

Thermal conductivity.

$$0 < \underline{\kappa}(1 + \vartheta^\beta) \leq \kappa(\vartheta) \leq \overline{\kappa}(1 + \vartheta^\beta), \ \beta > 6$$

Global existence [Chaudhuri, EF, 2021].
Under the above constitutive restrictions, the problem admits global–in–time weak solutions for any finite energy initial data and any sufficiently smooth boundary data

Target problem – smooth solutions

Incompressibility:

$$\mathrm{div}_x \mathbf{U} = 0$$

Momentum balance:

$$\overline{\varrho}\Big(\partial_t \mathbf{U} + \mathbf{U}\cdot\nabla_x \mathbf{U}\Big) + \nabla_x \Pi = \mu(\overline{\vartheta})\Delta_x \mathbf{U} + r\nabla_x G$$

Heat equation:

$$\overline{\varrho} c_p(\overline{\varrho},\overline{\vartheta})\Big(\partial_t \Theta + \mathbf{U}\cdot\nabla_x \Theta\Big) - \overline{\varrho}\ \overline{\vartheta}\alpha(\overline{\varrho},\overline{\vartheta})\mathbf{U}\cdot\nabla_x G = \kappa(\overline{\vartheta})\Delta_x \Theta$$

Boussinesq relation:

$$\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \varrho}\nabla_x r + \frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}\nabla_x \Theta = \overline{\varrho}\nabla_x G.$$

Nonlocal boundary conditions

$$\Theta|_{\partial\Omega} = \Theta_B - \frac{\lambda(\overline{\varrho},\overline{\vartheta})}{1-\lambda(\overline{\varrho},\overline{\vartheta})}\frac{1}{|\Omega|}\int_\Omega \Theta \ \mathrm{d}x,\ \ 0 < \lambda(\overline{\varrho},\overline{\vartheta}) < 1$$

Strong solutions of the target problem

A. Abbatiello, EF , 2022:
Suppose that

$$G \in W^{1,\infty}(\Omega),\ \Theta_B \in C^2(\overline{\Omega}),$$

$$\Theta_0 \in W^{2,p}(\Omega),\ \mathbf{U}_0 \in W^{2,p}(\Omega; R^d),\ \mathrm{div}_x \mathbf{U}_0 = 0,\ \text{for any } 1 \leq p < \infty,$$

together with the compatibility conditions

$$\mathbf{U}_0 = 0,\ \Theta_0 + \frac{\lambda}{1-\lambda}\frac{1}{|\Omega|}\int_\Omega \Theta_0 \ \mathrm{d}x = \Theta_B \text{ on } \partial\Omega$$

Then there exists $T_{\max} > 0$, $T_{\max} = \infty$ if $d = 2$, such that the target OB system with the initial data

$$\mathbf{U}(0,\cdot) = \mathbf{U}_0,\ \Theta(0,\cdot) = \Theta_0,$$

admits a (unique) strong solution $\mathbf{U}$, Θ in the regularity class

$$\mathbf{U} \in L^p(0,T; W^{2,p}(\Omega; R^d)),\ \partial_t \mathbf{U} \in L^p(0,T; L^p(\Omega; R^d)),\ \Pi \in L^p(0,T; W^{1,p}(\Omega)),$$
$$\Theta \in L^p(0,T; W^{2,p}(\Omega)),\ \partial_t \Theta \in L^p(0,T; L^p(\Omega; R^d))$$

for any $1 \leq p < \infty$ and any $0 < T < T_{\max}$

Convergence to the target problem, I

Well prepared initial data

$$\varrho_\varepsilon(0,\cdot) = \overline{\varrho} + \varepsilon \varrho_{0,\varepsilon},\ \overline{\varrho} > 0 \text{ constant},\ \int_\Omega \varrho_{0,\varepsilon}\ \mathrm{d}x = 0$$

$$\vartheta_\varepsilon(0,\cdot) = \overline{\vartheta} + \varepsilon \vartheta_{0,\varepsilon},\ \mathbf{u}_\varepsilon(0,\cdot) = \mathbf{u}_{0,\varepsilon}$$

$$\|\varrho_{0,\varepsilon}\|_{L^\infty(\Omega)} \lesssim 1,\ \varrho_{0,\varepsilon} \to r_0 \text{ in } L^1(\Omega),$$

$$\|\vartheta_{0,\varepsilon}\|_{L^\infty(\Omega)} \lesssim 1,\ \vartheta_{0,\varepsilon} \to \mathfrak{T}_0 \text{ in } L^1(\Omega),$$

$$\|\mathbf{u}_{0,\varepsilon}\|_{L^\infty(\Omega;R^d)} \lesssim 1,\ \mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0 \text{ in } L^1(\Omega;R^d),$$

$\mathfrak{T}_0 \in W^{2,p}(\Omega),\ \mathbf{U}_0 \in W^{2,p}(\Omega;R^d),$ for any $1 \le p < \infty,\ \mathrm{div}_x \mathbf{U}_0 = 0,$
$\mathbf{U}_0 = 0,\ \mathfrak{T}_0 = \Theta_B$ on $\partial\Omega,$

$$\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \varrho}\nabla_x r_0 + \frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}\nabla_x \mathfrak{T}_0 = \overline{\varrho}\nabla_x G$$

Energy estimates

1-st ansatz in the relative energy inequality

$$\int_\Omega E_\varepsilon \left(\varrho_\varepsilon(0,\cdot), \vartheta_\varepsilon(0,\cdot), \mathbf{u}_\varepsilon(0,\cdot) \middle| \overline{\varrho}, \overline{\vartheta} + \varepsilon \Theta_B, 0 \right) \ \mathrm{d}x$$

Boundedness of initial values:

$$\int_\Omega E_\varepsilon \left(\varrho_\varepsilon(0,\cdot), \vartheta_\varepsilon(0,\cdot), \mathbf{u}_\varepsilon(0,\cdot) \middle| \overline{\varrho}, \overline{\vartheta} + \varepsilon \Theta_B, 0 \right) \ \mathrm{d}x \lesssim 1$$

independently of $\varepsilon \to 0$

Convergence to the target problem, II

Convergence as a consequence of energy estimates

$$\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \to \mathfrak{R} \text{ weakly-(*) in } L^\infty(0,T;L^p(\Omega)),\ \ p>1$$

$$\frac{\vartheta_\varepsilon - \overline{\varrho}}{\varepsilon} \to \mathfrak{T} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)) \text{ and weakly-(*) in } L^\infty(0,T;L^2(\Omega))$$

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } L^2(0,T;W_0^{1,2}(\Omega;R^d))$$

where

$$\mathrm{div}_x \mathbf{u} = 0$$

$$\mathfrak{T}|_{\partial\Omega} = \Theta_B$$

and

$$\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \varrho}\nabla_x \mathfrak{R} + \frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}\nabla_x \mathfrak{T} = \overline{\varrho}\nabla_x G$$

Target problem revisited

New variables:

$$\mathcal{T},\ \mathcal{T} - \lambda(\overline{\varrho}, \overline{\vartheta}) \frac{1}{|\Omega|} \int_\Omega \mathcal{T}\ \mathrm{d}x = \Theta.$$

Transformed system:

$$\mathrm{div}_x \mathbf{U} = 0$$

$$\overline{\varrho}\Big(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U} \Big) + \nabla_x \Pi = \mathrm{div}_x \mathbb{S}(\overline{\vartheta}, \nabla_x \mathbf{U}) + r \nabla_x G$$

$$\overline{\varrho} c_p(\overline{\varrho}, \overline{\vartheta}) \left(\partial_t \mathcal{T} + \mathbf{U} \cdot \nabla_x \mathcal{T} \right) - \overline{\varrho}\ \overline{\vartheta} \alpha(\overline{\varrho}, \overline{\vartheta}) \mathbf{U} \cdot \nabla_x G$$

$$= \kappa(\overline{\vartheta}) \Delta_x \mathcal{T} + \overline{\vartheta} \alpha(\overline{\varrho}, \overline{\vartheta}) \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \partial_t \frac{1}{|\Omega|} \int_\Omega \mathcal{T}\ \mathrm{d}x$$

Boussinesq relation:

$$\frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \nabla_x r + \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \nabla_x \mathcal{T} = \overline{\varrho} \nabla_x G,\ \int_\Omega r\ \mathrm{d}x = 0$$

Boundary conditions:

$$\mathbf{U}|_{\partial\Omega} = 0,\ \mathcal{T}|_{\partial\Omega} = \Theta_B,$$

Convergence to the target problem, III

2-nd ansatz in the relative energy inequality

$$E_\varepsilon \left(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon \ \Big| \overline{\varrho} + \varepsilon r, \overline{\vartheta} + \varepsilon \mathcal{T}, \mathbf{U} \right)$$

Convergence of the initial values:

$$\int_\Omega E_\varepsilon \left(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon \ \Big| \overline{\varrho} + \varepsilon r, \overline{\vartheta} + \varepsilon \mathcal{T}, \mathbf{U} \right) (0, \cdot) \ \mathrm{d}x \to 0 \text{ as } \varepsilon \to 0$$