Hereditarily bounded sets

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Essentially undecidable theories

Incompleteness/undecidability theorems:

- any consistent r.e. theory extending XXXX is incomplete
- any consistent theory extending XXXX is undecidable

What can serve as XXXX?

T essentially undecidable [TMR'53]
$\iff$ all consistent extensions of T are undecidable
$\iff$ no r.e. extension of T is complete and consistent

Convenient weak essentially undecidable theories:

- Robinson's arithmetic Q
- Robinson's theory R
- adjunctive set theory AS
- Vaught's set theory VS

Vaught's set theory

Weak set theory *VS* introduced in [Vau'67]

Language: $\in$

Axioms:

$$(\mathrm{V}_n) \qquad \forall x_0, \dots, x_{n-1}\, \exists y\, \forall t \left(t \in y \leftrightarrow \bigvee_{i<n} t = x_i \right)$$

for each standard $n \in \omega$

NB: (V_n) implies (V_m) for $n \geq m > 0$

- *VS* is essentially undecidable
- finite fragments $VS_n = (\mathrm{V}_0) + (\mathrm{V}_n)$ not ess. und.
  - VS_n interpretable in any theory with pairing

Theories with pairing

Assume $T \vdash \exists x \, \exists y \; x \neq y$

Pairing function in T: definable function $p(x,y)$ s.t. T proves

$$p(x,y) = p(x',y') \to x = x' \wedge y = y'$$

Non-functional pairing: a formula $\pi(x,y,p)$ s.t. T proves

$$\forall x \, \forall y \, \exists p \, \pi(x,y,p)$$
$$\pi(x,y,p) \wedge \pi(x',y',p) \to x = x' \wedge y = y'$$

Example: VS_2 has non-functional pairing $\{\{x\},\{x,y\}\}$

See [Vis'08] for more background

Decidable theories with pairing

Theories with variable-length sequence encoding (sequential theories [Pud'85]) interpret $Q \implies$ ess. und.

In contrast: there are decidable theories with pairing

- [Mal'61,'62] theories of locally free algebras ($\approx$ term algebras, also with "commutativity" constraints) incl. acyclic pairing functions: $\langle \mathbb{N}, 2^x 3^y \rangle$
- [Ten'72] p.f. acyclic up to a few exceptions e.g.: $2^x(2y+1)-1$, $\max\{x^2, y^2+x\}+y$, $\binom{x+y+1}{2}+x$

Even with more arithmetical structure:

- [Sem'83] $\langle \mathbb{N}, +, 2^x \rangle$ (has p.f. $2^x + 2^{x+y}$ [CR'99])
- [CR'01] $\langle \mathbb{N}, S, \binom{x+y+1}{2}+x \rangle$

Decidable extensions of VS_k

Encode k-sets by pairs:

Corollary

- Any theory with pairing interprets VS_k for each k
- For any k, VS_k has a decidable completion

These extensions of VS_k are quite unnatural as theories of sets

- e.g., extensionality fails: consider $\langle x, y\rangle$ and $\langle y, x\rangle$

Problem (informal)

Find a natural decidable extension of VS_k
with a transparent meaning

Hereditarily finite/bounded sets

The set H_ω of hereditarily finite sets:

- The smallest set s.t. $\forall x\,(x \subseteq H_\omega \land x \text{ finite} \implies x \in H_\omega)$
- $x \in H_\omega \iff \forall y \in \mathrm{tc}(\{x\})\ y \text{ finite}$
- $H_\omega = V_\omega = \bigcup_{n\in\omega} V_n$, where $V_0 = \varnothing$, $V_{n+1} = \mathcal{P}(V_n)$

$\mathbf{H}_\omega = \langle H_\omega, \in\rangle$ is bi-interpretable with $\langle \mathbb{N}, +, \cdot\rangle$

The set H_k of sets hereditarily of size $\le k$:

- The smallest set s.t. $\forall x\,(x \subseteq H_k \land |x| \le k \implies x \in H_k)$
- $x \in H_k \iff \forall y \in \mathrm{tc}(\{x\})\ |y| \le k$
- $H_k = \bigcup_n V_{n,\le k}$, where $V_{0,\le k} = \varnothing$, $V_{n+1,\le k} = \mathcal{P}_{\le k}(V_{n,\le k})$

NB: $H_\omega = \bigcup_{k\in\omega} H_k$

$\mathbf{H}_k = \langle H_k, \in\rangle$ is a natural (minimal!) model of VS_k

The theory of $\mathbf{H}_k$

The goal of this talk:

- an explicit axiomatization S_k for $\mathrm{Th}(\mathbf{H}_k)$
- characterization of elementary equivalence of tuples
- S_k is decidable, with ~~insane~~ lowest possible complexity

NB: some cases are easy/already understood

- $\mathbf{H}_0$ is a one-element structure
- $\mathbf{H}_1 \simeq \langle \mathbb{N}, S(x) = y \rangle$
- $\mathbf{H}_2$ is definitionally equivalent to $\langle H_2, \varnothing, \{x, y\} \rangle$
  $\{x, y\}$ free commutative operation [Mal'62]

The theory S_k

S_k is axiomatized by:

- the axioms (V_0) and (V_k) of VS_k
- extensionality

$$(\mathrm{E}) \qquad \forall x, y\, \big(\forall t\, (t \in x \leftrightarrow t \in y) \to x = y\big)$$

- boundedness (all sets have $\le k$ elements)

$$(\mathrm{B}_k) \qquad \forall x, u_0, \dots, u_k \Big(\bigwedge_{i \le k} u_i \in x \to \bigvee_{i < j \le k} u_i = u_j\Big)$$

- acyclicity: for each $n \in \omega$,

$$(\mathrm{C}_n) \qquad \forall x_0, \dots, x_n\, \neg\Big(\bigwedge_{i<n} x_i \in x_{i+1} \wedge x_n \in x_0\Big)$$

Transitive closures

$\mathbf{A} \vDash S_k$, $\overline{a} \in A$, $\ell = \mathrm{lh}(\overline{a})$: define $\mathrm{tc}_n^{\mathbf{A}}(\overline{a}) \subseteq A$

$$\mathrm{tc}_0^{\mathbf{A}}(\overline{a}) = \{a_i : i < \ell\}$$

$$\mathrm{tc}_{n+1}^{\mathbf{A}}(\overline{a}) = \mathrm{tc}_n^{\mathbf{A}}(\overline{a}) \cup \bigcup_{u \in \mathrm{tc}_n^{\mathbf{A}}(\overline{a})} \{v \in A : v \in^{\mathbf{A}} u\}$$

$$\mathrm{tc}^{\mathbf{A}}(\overline{a}) = \bigcup_{n \in \omega} \mathrm{tc}_n^{\mathbf{A}}(\overline{a})$$

As structures: $\mathbf{tc}_n^{\mathbf{A}}(\overline{a}) = \langle \mathrm{tc}_n^{\mathbf{A}}(\overline{a}), \in^{\mathbf{A}}, \overline{a} \rangle$

NB: $|\mathrm{tc}_n^{\mathbf{A}}(\overline{a})| \le \ell \cdot k^{\le n}$ where $k^{\le n} = \sum_{i=0}^{n} k^i = \frac{k^{n+1}-1}{k-1}$

$$\mathbf{A}, \overline{a} \sim \mathbf{B}, \overline{b} \iff \mathbf{tc}^{\mathbf{A}}(\overline{a}) \simeq \mathbf{tc}^{\mathbf{B}}(\overline{b})$$

$$\mathbf{A}, \overline{a} \sim_n \mathbf{B}, \overline{b} \iff \mathbf{tc}_n^{\mathbf{A}}(\overline{a}) \simeq \mathbf{tc}_n^{\mathbf{B}}(\overline{b})$$

Characterization of elementary equivalence

Ehrenfeucht–Fraïssé argument:

Theorem

Let $\mathbf{A}, \mathbf{B} \vDash S_k$, $\overline{a} \in A$, $\overline{b} \in B$, $\ell = \mathrm{lh}(\overline{a}) = \mathrm{lh}(\overline{b})$, $n < \omega$. Then

$$\mathbf{A}, \overline{a} \equiv \mathbf{B}, \overline{b} \iff \mathbf{A}, \overline{a} \sim \mathbf{B}, \overline{b}.$$

More precisely, for all $n \in \omega$,

$$\mathbf{A}, \overline{a} \equiv_{\ell(k^{\le n}-1)} \mathbf{B}, \overline{b} \implies \mathbf{A}, \overline{a} \sim_n \mathbf{B}, \overline{b},$$
$$\mathbf{A}, \overline{a} \sim_{t_k(n)} \mathbf{B}, \overline{b} \implies \mathbf{A}, \overline{a} \equiv_n \mathbf{B}, \overline{b},$$

where $t_k(0) = 0$, $t_k(n+1) = k^{\le t_k(n)+1} + t_k(n) + 1$.

Completeness, decidability, q. elimination

Corollary

- S_k is a complete theory, thus $S_k = \mathrm{Th}(\mathbf{H}_k)$
- $S_k = \mathrm{Th}(\mathbf{H}_k)$ is decidable
- in S_k, any formula is equivalent to a Boolean combination of bounded existential formulas

In particular: $\mathrm{Th}(\mathbf{H}_k)$ is a decidable extension of VS_k

Complexity

Superexponential function: $2_0^x = x$, $2_{n+1}^x = 2^{2_n^x}$

Theorem [FR'79]

T consistent theory with pairing $\implies \exists\gamma > 0$ s.t. any decision procedure for T has complexity $\geq 2_{\gamma n}^0$

$t_k(n) \leq 2_n^{c_k}$ for some constant c_k

Turning the Ehrenfeucht–Fraïssé argument into an algorithm:

Theorem

S_k is decidable in time $2_{n/4}^{c_k}$

Fine-tuned algorithm

Handle blocks of quantifiers in one go:

Theorem

Given a sentence φ with

- $\varphi \in \exists_r$
- n: number of symbols
- q: max length of quantifier blocks

we can decide whether $S_k \vdash \varphi$ in

$$\begin{cases} \mathrm{NTIME}(n^{O(1)}) & r = 1 \\ \mathrm{NTIME}\big((kq)^{O(kq)} n^{O(1)}\big) & r = 2 \\ \mathrm{NTIME}\big(2_{r-1}^{O(qk \log k)} n^{O(1)}\big) & r \geq 3 \end{cases}$$

Summary

We identified $\mathrm{Th}(\mathbf{H}_k)$ as a natural extension of VS_k:

- decidable, of iterated exponential complexity
- transparent explicit axiomatization
- combinatorial characterization of elementary equivalence
- quantifier elimination

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