One-dimensional Fredholm boundary-value problems with a parameter in Sobolev spaces

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THE INTERNATIONAL ONLINE CONFERENCE
"CURRENT TRENDS IN ABSTRACT AND APPLIED ANALYSIS"

We study

continuity in a parameter of solutions of the most general (generic) classes of one-dimensional inhomogeneous boundary-value problems for systems of linear ordinary differential equations of an arbitrary order in Sobolev spaces on a finite interval.

Let a finite interval $(a,b) \subset \mathbb{R}$ and parameters $\{m,n,r\} \subset \mathbb{N}$, $1 \le p \le \infty$, be given.

Linear boundary-value problem

$$(Ly)(t) := y^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) y^{(r-j)}(t) = f(t), \quad t \in (a,b), \tag{1}$$

$$By = c. \tag{2}$$

Here matrix-valued functions $A_{r-j}(\cdot) \in (W_p^n)^{m \times m}$, vector-valued function $f(\cdot) \in (W_p^n)^m$, vector $c \in \mathbb{C}^{rm}$, linear continuous operator

$$B\colon (W_p^{n+r})^m \to \mathbb{C}^{rm} \tag{3}$$

are arbitrarily chosen; vector-valued function $y(\cdot) \in (W_p^{n+r})^m$ is unknown.

The solutions of equation (1) fill the space $(W_p^{n+r})^m$ if its right-hand side $f(\cdot)$ runs through the space $(W_p^n)^m$. Hence, the condition (2) with operator (3) is generic condition for this equation.

It includes all known types of classical boundary conditions and numerous nonclassical conditions containing the derivatives (in general fractional) $y^{(k)}(\cdot)$ with $0 < k \le n+r$.

Complex Sobolev space $W_p^{n+r} := W_p^{n+r}\big([a,b];\mathbb{C}\big)$

$$W_p^{n+r}\big([a,b];\mathbb{C}\big) := \big\{y \in C^{n+r-1}[a,b] : y^{(n+r-1)} \in AC[a,b],\, y^{(n+r)} \in L_p[a,b]\big\}$$

This space is Banach relative to the norm

$$\|y\|_{n+r,p} = \sum_{k=0}^{n+r-1} \big\|y^{(k)}\big\|_p + \big\|y^{(n+r)}\big\|_p,$$

where $\|\cdot\|_p$ is the norm in $L_p\big([a,b];\mathbb{C}\big)$.
By $\|\cdot\|_{n+r,p}$, we also denote the norms in Banach spaces

$$(W_p^{n+r})^m := W_p^{n+r}\big([a,b];\mathbb{C}^m\big) \quad \text{and} \quad (W_p^{n+r})^{m\times m} := W_p^{n+r}\big([a,b];\mathbb{C}^{m\times m}\big).$$

They consist of the vector-valued functions and matrix-valued functions, respectively, all components of which belong to W_p^{n+r}.

With problem (1), (2), we associate the linear operator

$$(L,B)\colon (W_p^{n+r})^m \to (W_p^n)^m \times \mathbb{C}^{rm}. \tag{4}$$

A linear continuous operator $T\colon X \to Y$, where X and Y are Banach spaces, is called a Fredholm operator if its kernel $\ker T$ and cokernel $Y/T(X)$ are finite-dimensional. If this operator is Fredholm, then its range $T(X)$ is closed in Y and the index $\operatorname{ind} T := \dim\ker T - \dim(Y/T(X)) \in \mathbb{Z}$. By $[BY_k]$, we denote the numerical $m \times m$ matrix, in which j-th column is result of the action of B on j-th column of $Y_k(\cdot)$.

Definition 1.

A block numerical matrix

$$\mathrm{M(L,B)} := ([BY_0], \dots, [BY_{r-1}]) \in \mathbb{C}^{rm\times rm} \tag{5}$$

is characteristic matrix to problem (1), (2). It consists of r rectangular block columns $[BY_k(\cdot)] \in \mathbb{C}^{m\times m}$.

Theorem 1.

The operator (4) is invertible if and only if the matrix $M(L,B)$ is nondegenerate.

Boundary-value problem depending on a parameter $\varepsilon \in [0, \varepsilon_0)$, $\varepsilon_0 > 0$

$$L(\varepsilon)y(t,\varepsilon) := y^{(r)}(t,\varepsilon) + \sum_{j=1}^{r} A_{r-j}(t,\varepsilon)y^{(r-j)}(t,\varepsilon) = f(t,\varepsilon),\ t \in (a,b), \quad (6)$$

$$B(\varepsilon)y(\cdot;\varepsilon) = c(\varepsilon). \quad (7)$$

Here $A_{r-j}(\cdot,\varepsilon) \in (W_p^n)^{m\times m}$, $f(\cdot,\varepsilon) \in (W_p^n)^m$, $c(\varepsilon) \in \mathbb{C}^{rm}$, linear continuous operator $B(\varepsilon)\colon (W_p^{n+r})^m \to \mathbb{C}^{rm}$ are arbitrarily chosen; vector-valued function $y(\cdot,\varepsilon) \in (W_p^{n+r})^m$ is unknown.

Problem (6), (7) is a Fredholm one with zero index for every $\varepsilon \in [0, \varepsilon_0)$.

Definition 2.

The solution to the problem (6), (7) depends continuously on a parameter ε at $\varepsilon = 0$ if the conditions are satisfied:

($*$) there exists a positive number $\varepsilon_1 < \varepsilon_0$ such that, for any $\varepsilon \in [0, \varepsilon_1)$ and arbitrary chosen $f(\cdot;\varepsilon) \in (W_p^n)^m$, $c(\varepsilon) \in \mathbb{C}^{rm}$, this problem has a unique solution $y(\cdot;\varepsilon) \in (W_p^{n+r})^m$;

($**$) the convergence of right-hand sides $f(\cdot;\varepsilon) \to f(\cdot;0)$ and $c(\varepsilon) \to c(0)$ implies the convergence of solutions

$$y(\cdot;\varepsilon) \to y(\cdot;0) \quad \text{in} \quad (W_p^{n+r})^m \quad \text{as} \quad \varepsilon \to 0+.$$

Consider the following conditions:

(0) the homogeneous boundary-value problem

$$L(0)y(t,0)=0, \quad t\in(a,b), \quad B(0)y(\cdot,0)=0$$

has only the trivial solution;

(I) $A_{r-j}(\cdot;\varepsilon)\to A_{r-j}(\cdot;0)$ in $(W_p^n)^{m\times m}$ for every $j\in\{1,\dots,r\}$;

(II) $B(\varepsilon)y\to B(0)y$ in $\mathbb{C}^{rm}$ for every $y\in(W_p^{n+r})^m$.

Theorem 2.

The solution to the problem (6), (7) depends continuously on the parameter ε at $\varepsilon=0$ if and only if this problem satisfies Conditions (0), (I), and (II).

Gnyp, Mikhailets, and Murach (2016) gave a constructive criterion of continuous dependence on a parameter in Sobolev spaces W_p^{n+r}, where $1 \leq p < \infty$. The proof of criterion is based on the fact that the linear continuous operator $B\colon (W_p^{n+r})^m \to \mathbb{C}^{rm}$ admits the unique analytic representation

$$By = \sum_{k=0}^{n+r-1} \alpha_k y^{(k)}(a) + \int_a^b \Phi(t) y^{(n+r)}(t)\mathrm{d}t, \quad y(\cdot) \in (W_p^{n+r})^m. \qquad (8)$$

Here, the matrices $\alpha_k \in \mathbb{C}^{rm\times m}$, and the matrix-valued function $\Phi(\cdot) \in L_{p'}\left([a,b];\mathbb{C}^{rm\times m}\right)$, $1/p + 1/p' = 1$.

Our method of proof allows to investigate such problems in Sobolev spaces W_p^{n+r}, where $1 \leq p \leq \infty$, and some others function spaces.

We supplement our result with a two-sided estimate of the error $\left\|y(\cdot;0)-y(\cdot;\varepsilon)\right\|_{n+r,p}$ of solution $y(\cdot;\varepsilon)$ via its discrepancy

$$\widetilde{d}_{n,p}(\varepsilon) := \left\|L(\varepsilon)y(\cdot;0)-f(\cdot;\varepsilon)\right\|_{n,p} + \left\|B(\varepsilon)y(\cdot;0)-c(\varepsilon)\right\|_{\mathbb{C}^{rm}}.$$

Here, we interpret $y(\cdot;0)$ as an approximate solution to problem (6), (7).

Theorem 3.

Let the problem (6), (7) satisfies Conditions (0), (I), and (II). Then there exist positive numbers $\varepsilon_2 < \varepsilon_1$, γ_1, and γ_2, such that

$$\gamma_1\,\widetilde{d}_{n,p}(\varepsilon) \leq \left\|y(\cdot;0)-y(\cdot;\varepsilon)\right\|_{n+r,p} \leq \gamma_2\,\widetilde{d}_{n,p}(\varepsilon)$$

for any $\varepsilon \in (0,\varepsilon_2)$. Here, the numbers ε_2, γ_1, and γ_2 do not depend on $y(\cdot;0)$, and $y(\cdot;\varepsilon)$.

Thus, the error and discrepancy of the solution to problem (6), (7) are of the same degree of smallness [2, 5].

For any $\varepsilon \in [0, \varepsilon_0)$, $\varepsilon_0 > 0$, we associate with the system (6)

multi-point Fredholm boundary condition

$$B(\varepsilon)y(\cdot,\varepsilon) = \sum_{j=0}^{N} \sum_{k=1}^{\omega_j(\varepsilon)} \sum_{l=0}^{n+r-1} \beta_{j,k}^{(l)}(\varepsilon) y^{(l)}(t_{j,k}(\varepsilon), \varepsilon) = q(\varepsilon), \qquad (9)$$

where the numbers $\{N, \omega_j(\varepsilon)\} \subset \mathbb{N}$, vectors $q(\varepsilon) \in \mathbb{C}^{rm}$, matrices $\beta_{j,k}^{(l)}(\varepsilon) \in \mathbb{C}^{m\times m}$, and points $\{t_j, t_{j,k}(\varepsilon)\} \subset [a,b]$ are arbitrarily given.

It is not assumed that the coefficients $A_{r-j}(\cdot,\varepsilon)$, $\beta_{j,k}^{(l)}(\varepsilon)$ or points $t_{j,k}(\varepsilon)$ have a certain regularity on the parameter ε as $\varepsilon > 0$. It will be required that for each fixed $j \in \{1,\dots,N\}$ all the points $t_{j,k}(\varepsilon)$ have a common limit as $\varepsilon \to 0+$, but for the zero-point series $t_{0,k}(\varepsilon)$ this requirement will not be necessary. We consider the case where the points of the interval $[a,b]$ appearing in boundary conditions are not fixed and depend on a numerical parameter and the number of these points may change.

The solution $y(\cdot,\varepsilon)$ to problem (6), (9) is continuous on the parameter ε if it exists, is unique, and satisfies the limit relation

$$\left\| y(\cdot,\varepsilon) - y(\cdot,0) \right\|_{n+r,p} \to 0 \quad \text{as} \quad \varepsilon \to 0+. \qquad (10)$$

Assumptions as $\varepsilon \to 0+$:

(α) $t_{j,k}(\varepsilon) \to t_j$ for all $j \in \{1,\dots,N\}$, and $k \in \{1,\dots,\omega_j(\varepsilon)\}$;

(β) $\sum_{k=1}^{\omega_j(\varepsilon)} \beta_{j,k}^{(l)}(\varepsilon) \to \beta_j^{(l)}$ for all $j \in \{1,\dots,N\}$, and $l \in \{0,\dots,n+r-1\}$;

(γ) $\sum_{k=1}^{\omega_j(\varepsilon)} \left\|\beta_{j,k}^{(l)}(\varepsilon)\right\| \left|t_{j,k}(\varepsilon) - t_j\right| \to 0$ for all $j \in \{1,\dots,N\}$, $k \in \{1,\dots,\omega_j(\varepsilon)\}$, and $l \in \{0,\dots,n+r-1\}$;

(δ) $\sum_{k=1}^{\omega_0(\varepsilon)} \left\|\beta_{0,k}^{(l)}(\varepsilon)\right\| \to 0$ for all $k \in \{1,\dots,\omega_0(\varepsilon)\}$, and $l \in \{0,\dots,n+r-1\}$.

Assumptions (β) and (γ) imply that the norms of the coefficients $\beta_{j,k}^{(l)}(\varepsilon)$ can increase as $\varepsilon \to 0+$, but not too fast.

Theorem 4.

Let the boundary-value problem (6), (9) for $p = \infty$ satisfies the assumptions (α), (β), (γ), (δ). Then it satisfies the limit condition (II). If, moreover, the conditions (0) and (I) are fulfilled, then for a sufficiently small ε its solution exists, is unique and satisfies the limit relation (10).

Assumptions as $\varepsilon \to 0+$:

(γ_p) $\sum_{k=1}^{\omega_j(\varepsilon)} \|\beta_{j,k}^{(n+r-1)}(\varepsilon)\| \left|t_{j,k}(\varepsilon) - t_j\right|^{1/p'} = O(1)$ for all $j \in \{1,\dots,N\}$, and $k \in \{1,\dots,\omega_j(\varepsilon)\}$;

(γ') $\sum_{k=1}^{\omega_j(\varepsilon)} \|\beta_{j,k}^{(l)}(\varepsilon)\| \left|t_{j,k}(\varepsilon) - t_j\right| \to 0$ for all $j \in \{1,\dots,N\}$, $k \in \{1,\dots,\omega_j(\varepsilon)\}$, and $l \in \{0,\dots,n+r-2\}$.

Theorem 7.

Let the boundary-value problem (6), (9) for $1 \le p < \infty$ satisfies the assumptions (α), (β), (γ_p), (γ'), (δ). Then it satisfies the limit condition (II). If, moreover, the conditions (0) and (I) are fulfilled, then for a sufficiently small ε its solution exists, is unique and satisfies the limit relation (10) [4, 6].

Remark 5.

The systems of conditions (α), (β), (γ), (δ) and (α), (β), (γ_p), (γ'), (δ) do not guarantee uniform convergence of continuous operators $B(\varepsilon)$ to $B(0)$ as $\varepsilon \to 0+$.

Linear boundary-value problem

$$(Ly)(t) := y^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) y^{(r-j)}(t) = f(t), \quad t \in (a,b), \tag{11}$$

$$By = c, \tag{12}$$

where $1 \le p < \infty$, $A_{r-j}(\cdot)$, $f(\cdot)$, c, and linear continuous operator B satisfy the above conditions to problem (1), (2).

A sequence of multipoint boundary-value problems

$$(L_k y_k)(t) := y_k^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) y_k^{(r-j)}(t) = f(t), \quad t \in (a,b), \tag{13}$$

$$B_k y_k := \sum_{j=0}^{N} \sum_{l=0}^{n+r-1} \beta_k^{(l,j)} y^{(l)}(t_{k,j}) = c. \tag{14}$$

Theorem 6.

For the boundary-value problem (11), (12) there is a sequence of multipoint boundary-value problems of the form (13), (14) such that they are well-posedness for sufficiently large k and the asymptotic property is fulfilled

$$y_k \to y \quad \text{in} \quad (W_p^{n+r})^m \quad \text{for} \quad k \to \infty.$$

The sequence can be chosen independently of f and c, and constructed explicitly.

Analysis of differential operators with distributions in coefficients

For some classes of ordinary differential operators with distributions in coefficients, they can be specifically defined as quasi-differential operators. Therefore, each of these operators is a limit in the sense of uniform resolvent convergence of differential operators with smooth coefficients. Due to this, some properties of differential operators with distributions in coefficients can be obtained based on known results of differential operators with smooth coefficients by the limit transition.

Publication of results

1. *Atlasiuk, O. M.; Mikhailets, V. A.* Fredholm one-dimensional boundary-value problems in Sobolev spaces. Ukrainian Math. J. 70 (2019), no. 10, 1526–1537. (Scopus, WoS, SJR Q2) *DOI: 10.1007/s11253-019-01588-w*
2. *Atlasiuk, O. M.; Mikhailets, V. A.* Fredholm one-dimensional boundary-value problems with parameter in Sobolev spaces. Ukrainian Math. J. 70 (2019), no. 11, 1677–1687. (Scopus, WoS, SJR Q2) *DOI: 10.1007/s11253-019-01599-7*
3. *Atlasiuk, O. M.; Mikhailets, V. A.* On solvability of inhomogeneous boundary-value problems in Sobolev spaces. Dopov. Nac. akad. nauk Ukr. (2019), no. 11, 3–7. *DOI: 10.15407/dopovidi2019.11.003*

4. *Atlasiuk, O. M.* Limit theorems for solutions of multipoint boundary-value problems in Sobolev spaces. J. Math. Sci. (Scopus, SJR Q3) 247 (2020), no. 2, 238–247. *DOI: 10.1007/s10958-020-04799-w*

5. *Atlasiuk, O. M.; Mikhailets, V. A.* On Fredholm parameter-dependent boundary-value problems in Sobolev spaces. Dopov. Nac. akad. nauk Ukr. (2020), no. 6, 3–6. *DOI: 10.15407/dopovidi2020.06.003*

6. *Atlasiuk, O. M.* Limit theorems for solutions of multipoint boundary-value problems with a parameter in Sobolev spaces. Ukrainian Math. J. 72 (2021), no. 8, 1175–1184. (Scopus, WoS, SJR Q3) *DOI: 10.1007/s11253-020-01859-x*

Thank you for your attention!