Statistical solutions to compressible Navier–Stokes system: Analysis and Numerics

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Navier–Stokes system

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{u})$$

Periodic boundary conditions

$$\mathbb{T}^d = \left([-1,1]|_{\{-1,1\}} \right)^d, \ d = 2,3$$

Initial data

$$\varrho(0,\cdot) = \varrho_0, \ \inf \varrho_0 > 0, \ (\varrho \mathbf{u})(0,\cdot) = \mathbf{m}_0 = \varrho_0 \mathbf{u}_0$$

$$p(\varrho) = a\varrho^\gamma, \ a > 0, \ \gamma > 1$$

Concepts of solutions

strong (classical) solutions $\subset$ **weak** solutions $\subset$ **dissipative** solutions

Strong solutions
Local in time existence for smooth data., global in time existence for the data close to equilibrium, uniqueness and continuous dependence on the data

Weak solutions
Global in time existence for $\gamma > \frac{d}{2}$, uniqueness – open problem, possibility to select a solution semigroup, measurable dependence of solutions on the data

Dissipative solutions
Limits of consistent approximations – numerical schemes.

Dissipative solutions

$$\int_0^\infty \int_{\mathbb{T}^d} \left[\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \right] \, \mathrm{d}x \mathrm{d}t = - \int_{\mathbb{T}^d} \varrho_0 \varphi(0, \cdot) \, \mathrm{d}x$$

for any $\varphi \in C^1_c([0,\infty) \times \mathbb{T}^d)$

$$\int_0^\infty \int_{\mathbb{T}^d} \left[\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \, \mathrm{d}x \mathrm{d}t$$

$$= \int_0^\infty \int_{\mathbb{T}^d} \mathbb{S}(\mathbb{D}_x \mathbf{u}) : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}x \mathrm{d}t - \int_{\mathbb{T}^d} \varrho_0 \mathbf{u}_0 \cdot \boldsymbol{\varphi}(0, \cdot) \, \mathrm{d}x - \int_0^\infty \int_{\mathbb{T}^d} \mathfrak{R} : \nabla_x \boldsymbol{\varphi} \mathrm{d}t$$

for any $\boldsymbol{\varphi} \in C^1_c([0,\infty) \times \mathbb{T}^d; R^d)$

$$\int_{\mathbb{T}^d} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right) (\tau, \cdot) \, \mathrm{d}x + \int_{\mathbb{T}^d} \mathfrak{E}(\tau, \cdot) + \int_0^\tau \int_{\mathbb{T}^d} \mathbb{S}(\mathbb{D}_x \mathbf{u}) : \mathbb{D}_x \mathbf{u} \, \mathrm{d}x \mathrm{d}t$$

$$\leq \int_{\mathbb{T}^d} \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + P(\varrho_0) \right) \, \mathrm{d}x$$

Compatibility

$$0 \leq \mathfrak{R}, \ 0 \leq \mathrm{trace}[\mathfrak{R}] \leq c \mathfrak{E}$$

Solvability of the Navier–Stokes system

- **Local existence of smooth solutions** [Valli, Zajaczkowski [1986]]

$$\varrho_0 \in W^{k,2}(Q),\ \inf \varrho_0 > 0,\ \mathbf{u}_0 \in W^{k,2}(Q; R^d),\ k \geq 3$$

$$+$$

compatibility conditions

$$\Rightarrow$$

There exists a regular (classical) solution

$$\varrho \in C([0, T_{\max}); W^{k,2}(Q)),\ \mathbf{u} \in C([0, T_{\max}); W^{k,2}(Q; R^d)),\ T_{\max} > 0$$

- **Global existence of weak solutions** [Lions [1998], EF [2000]]

$$\varrho_0 \geq 0,\ \int_Q \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + P(\varrho_0) \right] \ \mathrm{d}x < \infty,\ \gamma > \frac{d}{2}$$

$$\Rightarrow$$

There exists global in time weak solution

$$\varrho \in C([0, T]; L^1(Q)) \cap C_{\rm weak}([0, T]; L^\gamma(Q)),$$

$$\varrho \mathbf{u} \in C_{\rm weak}([0, T]; L^{\frac{2\gamma}{\gamma+1}}(Q; R^d)),\ \mathbf{u} \in L^2(0, T; W_0^{1,2}(Q; R^d)) \text{ for any } T > 0$$

Conditional regularity, weak–strong uniqueness

A priori **bounds** [Sun, Wang, and Zhang [2011]]

$$\|\varrho(t,\cdot)\|_{W^{k,2}(Q)} + \|\mathbf{u}(t,\cdot)\|_{W^{k,2}(Q)}$$

$$\leq \Lambda\left(T, \|\varrho_0\|_{W^{k,2}(Q)},\ \inf \varrho_0, \|\mathbf{u}_0\|_{W^{k,2}(Q)},\ \boxed{\|\varrho\|_{L^\infty(0,T)\times Q)}, \|\mathbf{u}\|_{L^\infty(0,T)\times Q)}}\right)$$

$$t \in [0, T],\ \ k \geq 3$$

Weak (dissipative) –strong uniqueness [EF, Jin, Novotný [2012], Abbatiello, EF [2020]]
Any dissipative solutions emanating from sufficiently regular initial data coincides with the unique strong solutions as long as the latter exists

Corollary
Any dissipative solution emanating from sufficiently regular initial data that remain uniformly bounded is a classical solution

Statistical solutions – framework

Data (phase) space

$$\mathcal{D} = \left\{ [\varrho_0, \mathbf{m}_0] \ \middle| \varrho_0 \in L^\gamma(Q),\ \mathbf{m}_0 \in L^{\frac{2\gamma}{\gamma+1}}(Q; R^d)) \int_Q E(\varrho_0, \mathbf{m}_0)\ \mathrm{d}x < \infty \right\}$$

$$\subset X_{\mathcal{D}} = W^{-k,2}(Q) \times W^{-k,2}(Q; R^d) \ \text{ – Polish space}$$

Probability measures

$$\mathfrak{P}[\mathcal{D}] \text{ – the set of probability measures on } X_{\mathcal{D}} \text{ supported by } \mathcal{D}$$

Statistical solution

- Family of Markov operators

$$M_t : \mathfrak{P}[\mathcal{D}] \to \mathfrak{P}[\mathcal{D}]$$

- $$M_0(\nu) = \nu \text{ for any } \nu \in \mathfrak{P}[\mathcal{D}]$$

- $$M_t\left(\sum_{i=1}^N \alpha_i \nu_i\right) = \sum_{i=1}^N \alpha_i M_t(\nu_i),\ \alpha_i \geq 0,\ \sum_{i=1}^N \alpha_i = 1$$

- $$M_{t+s} = M_t \circ M_s \text{ for any } t \geq 0 \text{ and a.a. } s \geq 0$$

- $$t \mapsto M_t \text{ continuous with respect to the weak topology on } \mathfrak{P}[\mathcal{D}]$$

- $$M_t(\delta_{[\varrho_0, \mathbf{m}_0]}) = \delta_{(\varrho(t,\cdot), \mathbf{m}(t,\cdot))}$$

$$[\varrho(t,\cdot), \mathbf{m}(t,\cdot)]$$

solution of the Navier–Stokes system with the data $[\varrho_0, \mathbf{m}_0]$

Statistical solution – pushforward measure

Semiflow selection

$$\mathbf{U} : [0, \infty) \times \mathcal{D} \to \mathcal{D}$$

Pushforward measure

$$\nu_0 \in \mathfrak{P}[\mathcal{D}] \text{ given}$$

$$M_t(\nu_0)[B] = \nu_0[\mathbf{U}^{-1}(t, B)]$$

$$\int_{X_\mathcal{D}} F(\varrho, \mathbf{m}) \, \mathrm{d}M_t(\nu_0) = \int_\mathcal{D} F(\mathbf{U}(t; \varrho_0, \mathbf{m}_0)) \, \mathrm{d}\nu_0(\varrho_0, \mathbf{m}_0)$$

for any

$$F \in BC(X_\mathcal{D})$$

[Fanelli and EF [2020]]

Tools from probability theory I

Skorokhod (representation) theorem
Let $(\mathbf{U}^M)_{M=1}^{\infty}$ be a sequence of random variables ranging in a Polish space X. Suppose that their laws are tight in X, meaning for any $\varepsilon > 0$, there exists a compact set $K(\varepsilon) \subset X$ such that

$$\mathbb{P}[\mathbf{U}^M \in X \setminus K(\varepsilon)] \leq \varepsilon \text{ for all } M = 1, 2, \dots.$$

Then there is a subsequence $M_n \to \infty$ and a sequence of random variables $(\widetilde{\mathbf{U}}^{M_n})_{n=1}^{\infty}$ defined on the standard probability space

$$\left(\widetilde{\Omega} = [0,1], \mathfrak{B}[0,1], \mathrm{d}y\right)$$

satisfying:

- $$\widetilde{\mathbf{U}}^{M_n} \approx_X \mathbf{U}^{M_n} \text{ (they are equally distributed random variables)},$$
- $$\widetilde{\mathbf{U}}^{M_n} \to \widetilde{\mathbf{U}} \text{ in } X \text{ for every } y \in [0,1].$$

Tools from probability theory II

Gyöngy–Krylov theorem
Let X be a Polish space and $(\mathbf{U}^M)_{M\geq 1}$ a sequence of $X-$valued random variables.
Then $(\mathbf{U}^M)_{M=1}^{\infty}$ converges in probability if and only if for any sequence of joint laws of

$$(\mathbf{U}^{M_k}, \mathbf{U}^{N_k})_{k=1}^{\infty}$$

there exists further subsequence that converge weakly to a probability measure μ on $X \times X$ such that

$$\mu\left[(x, y) \in X \times X,\ x = y\right] = 1.$$

Numerical approximation

(Initial) data

$$\varrho_0,\ \mathbf{m}_0 = \varrho_0 \mathbf{u}_0 \in \mathcal{D} \subset X_{\mathcal{D}}$$

Numerical approximation

$$\varrho^h,\ \mathbf{u}^h,\ h = h(\ell) \to 0 \text{ as } \ell \to \infty$$

Numerical scheme
$(\varrho^h, \mathbf{u}^h) \in V_h$, where $V_h \subset L^\infty((0,T) \times \mathbb{T}^d); R^{d+1})$ is a finite dimensional space,

$$\inf \varrho^h > 0 \text{ for any } h,$$

$$\mathcal{A}\left(h, [\varrho_0, \mathbf{u}_0,], \varrho^h, \mathbf{u}^h\right) = 0,$$

where

$$\mathcal{A} : (0,\infty) \times \mathcal{D} \times V_h \to R^m,\ m = m(h)$$

is a Borel measurable (typically continuous) mapping representing a finite system of algebraic equations called *numerical scheme*

Convergent numerical approximation

We say that a numerical approximation is *convergent* if for any sequence of data

$$[\varrho_0^N, \mathbf{u}_0^N] \in \mathcal{D} \to [\varrho_0, \mathbf{u}_0] \text{ in } X_{\mathcal{D}} \text{ as } N \to \infty,$$

the numerical approximation $(\varrho^{h,N}, \mathbf{u}^{h,N})$ satisfies:

- $$\varrho^{h,N} > 0;$$

- $$\varrho^{h,N} \to \varrho \text{ in } L^1((0,T) \times \mathbb{T}^d),$$
  $$\mathbf{u}^{h,N} \to \mathbf{u} \text{ in } L^1((0,T) \times \mathbb{T}^d; R^d) \text{ as } N \to \infty,\ h \to 0,$$

  for any $0 < T < T_{\max}$, where $(\varrho, \mathbf{u})$ is the unique classical solution of the problem with the data $[\varrho_0, \mathbf{u}_0]$ defined on the maximal time interval $[0, T_{\max})$.

Bounded graph property

If $N = N(\ell) \nearrow \infty$, $h = h(\ell) \searrow 0$,

$$\left[\varrho_0^N, \mathbf{u}_0^N\right] \in \mathcal{D} \to [\varrho_0, \mathbf{u}_0] \text{ in } X_{\mathcal{D}} \text{ as } N \to \infty,$$

and the associated numerical approximation satisfies

$$\sup_{h,N} \left\| (\varrho^{h,N}, \mathbf{u}^{h,N}) \right\|_{L^\infty((0,T)\times\mathbb{T}^d;R^{d+1})} < \infty,$$

then

$$\varrho^{h,N} \to \varrho \text{ in } L^1((0,T) \times \mathbb{T}^d),$$

$$\mathbf{u}^{h,N} \to \mathbf{u} \text{ in } L^1((0,T) \times \mathbb{T}^d; \mathbb{T}^d) \text{ as } h \to 0, \ N \to \infty,$$

where $(\varrho, \mathbf{u})$ is the unique classical solution of the Navier–Stokes system with the initial the data $[\varrho_0, \mathbf{u}_0]$.

Corollary
Any convergent numerical scheme possesses the bounded graph property

Random data, weak approach

$$\varrho_0, \mathbf{u}_0 \in \mathcal{D} \subset X_{\mathcal{D}}$$

weak approach $\Leftrightarrow$ determining distribution (law) of solutions

Generating sequences of random data

$$[\varrho_0^n, \mathbf{u}_0^n] \in \mathcal{D}$$

$$\frac{1}{N} \sum_{n=1}^{N} F\left[\varrho_0^n, \mathbf{u}_0^n\right] \to \mathbb{E}\left[F[\varrho_0, \mathbf{u}_0]\right] \text{ as } N \to \infty$$

for any $F \in BC(X_{\mathcal{D}})$

Expected value

$$\mathbb{E}\left[F[\varrho_0, \mathbf{u}_0]\right] = \int_{X_D} F\left(\hat{\varrho}, \hat{\mathbf{u}}\right) \, \mathrm{d}\mathcal{L}[\varrho_0, \mathbf{u}_0]$$

Distribution of the initial data

$$\mathcal{L}[\varrho_0, \mathbf{u}_0] \in \mathfrak{P}[\mathcal{D}] - \text{probability measure on the space of data}$$

Weak approach, main goal I

$[\varrho_0^n, \mathbf{u}_0^n] \in \mathcal{D} \ \to \ [\varrho^{h,n}, \mathbf{u}^{h,n}]$ numerical approximation

Sequence of empirical measures

$$\frac{1}{N} \sum_{n=1}^{N} \delta_{\varrho^{h,n}, \mathbf{u}^{h,n}}$$

Convergence in law

$$\frac{1}{N} \sum_{n=1}^{N} F[\varrho^{h,n}, \mathbf{u}^{h,n}] \to \mathbb{E}\left[F[\varrho, \mathbf{u}]\right] \text{ as } h \to 0, \ N \to \infty$$

for any $F \in BC\Big(W^{-m,2}((0,T) \times \mathbb{T}^d) \times W^{-m,2}((0,T) \times \mathbb{T}^d; R^d)\Big)$

Limit solution

$$\mathbb{E}\left[F[\varrho, \mathbf{u}]\right] = \int_{X_D} F\left[(\varrho, \mathbf{u})[\hat{\varrho}, \hat{\mathbf{u}}]\right] \mathrm{d}\mathcal{L}[\varrho_0, \mathbf{u}_0]$$

$(\varrho, \mathbf{u})$ - smooth (whence unique) statistical solution of the Navier-Stokes system

Weak approach, main goal II

Convergence of empirical means

$$\frac{1}{N}\sum_{n=1}^{N}(\varrho^{h,n},\mathbf{u}^{h,n}) \to \mathbb{E}\left[\varrho,\mathbf{u}\right] \ \text{as } N\to\infty,\ \ h\to 0$$

in $L^q((0,T)\times\mathbb{T}^d;R^{d+1})$, $q\geq 1$

Expected value

$$\mathbb{E}\left[\varrho,\mathbf{u}\right] = \int_{X_D}(\varrho,\mathbf{u})[\hat{\varrho},\hat{\mathbf{u}}]\ \mathrm{d}\mathcal{L}[\varrho_0,\mathbf{u}_0]$$

Bochner integral in a suitable Banach space

Neither the approximate sequence $[\varrho_0^n,\mathbf{u}_0^n]$ nor the associated numerical solutions $(\varrho^{h,n},\mathbf{u}^{h,n})$ are uniquely determined by the data $[\varrho_0,\mathbf{u}_0]$. Practical implementations deal with a large number of *samples* – sequences $[\varrho_0^n,\mathbf{u}_0^n]$ – generated independently mimicking the Strong law of large numbers

[Mishra, Schwab et al.]

Random data, strong approach

Data as random variable

$$[\varrho_0, \mathbf{u}_0] : \{\Omega, \mathcal{B}, \mathcal{P}\} \to X_{\mathcal{D}}.$$

Main goal

Identify the exact solution $(\varrho, \mathbf{u})$ as a random variable on the same probability space

Stochastic collocation method

$\Omega = \cup_{n=1}^{N} \Omega_n^N,\ \Omega_n^N\ \mathcal{P} - \text{measurable},\ \Omega_i^N \cap \Omega_j^N = \emptyset \text{ for } i \neq j,\ \cup_{n=1}^{N} \Omega_N^n = \Omega$

Approximate random data

$$[\varrho_{0,N}, \mathbf{u}_{0,N}] = \sum_{n=1}^{N} \mathbb{1}_{\Omega_N^n}(\omega)[\varrho_0, \mathbf{u}_0](\omega_n),\ \omega_n \in \Omega_N^n.$$

$$\sum_{n=1}^{N} \mathbb{1}_{\Omega_N^n}(\omega)[\varrho_0, \mathbf{u}_0](\omega_n) \to [\varrho_0, \mathbf{u}_0] \text{ in } X_{\mathcal{D}}\ \mathcal{P} - \text{a.s.}$$

Collocation method - convergence of data approximation

Probability space, class $\mathcal{R}$

$$\Omega - \text{compact metric space}$$

$$\mathcal{R}(\Omega, \mathbb{P}) = \left\{ f : \Omega \to R \ \middle|\ f \text{ bounded}, \ \mathbb{P}\{\omega \in \Omega \ \middle|\ f \text{ is not continuous at } \omega\} = 0 \right\}$$

Unconditional convergence of data approximation

Suppose the (initial data) belong to the class $\mathcal{R}$ (in a weak sense - Fourier modes).

Then

$$\sum_{n=1}^{N} \mathbb{1}_{\Omega_N^n}(\omega)[\varrho_0, \mathbf{u}_0](\omega_n) \to [\varrho_0, \mathbf{u}_0] \text{ in } X_{\mathcal{D}} \ \mathcal{P} - \text{a.s.}$$

independently of the choice of the collocation points provided diameters of the partition tend to zero

[EF, Lukáčová-Medvid'ová [2021]]

Boundedness in probability, weak approach

Approximate solutions

$$h = h(\ell),\ N = N(\ell),\ h(\ell) \searrow 0,\ N(\ell) \nearrow \infty \text{ as } \ell \to \infty.$$

$$\frac{1}{N} \sum_{n=1}^{N} \delta_{[\varrho^{h,n}, \mathbf{u}^{h,n}]}$$

Boundedness in probability (weak)
For any $\varepsilon > 0$, there is $M = M(\varepsilon)$ such that

$$\frac{\# \left\{ \| \varrho^{h,n}, \mathbf{u}^{h,n} \|_{L^\infty((0,T)\times \mathbb{T}^d; R^{d+1})} > M,\ n \leq N \right\}}{N} < \varepsilon \text{ for any } \ell = 1, 2, \dots$$

Boundedness in probability, strong approach

Approximate solutions

$$h = h(\ell),\ \ N = N(\ell),\ \ h(\ell) \searrow 0,\ \ N(\ell) \nearrow \infty \text{ as } \ell \to \infty.$$

$$\sum_{n=1}^{N} \mathbb{1}_{\Omega_N^n}(\omega)[\varrho^{h,n}, \mathbf{u}^{h,n}]$$

Boundedness in probability (strong)
For any $\varepsilon > 0$, there is $M = M(\varepsilon)$ such that

$$\sum_{n \leq N, \left\{ \| \varrho^{n,h}, \mathbf{u}^{n,h} \|_{L^\infty((0,T)\times\mathbb{T}^d;R^{d+1})} > M \right\}} |\Omega_n^N| < \varepsilon \text{ for } \ell = 1, 2, \dots$$

Weak to strong

Weak (statistical data)

$$\frac{1}{N}\sum_{n=1}^{N}\delta_{[\varrho_0^n,\mathbf{u}_0^n]}$$

Application of Skorokhod representation theorem

$$\mathcal{L}[\varrho_{0,N},\mathbf{u}_{0,N}] = \mathcal{L}\left[\frac{1}{N}\sum_{n=1}^{N}\delta_{[\varrho_0^n,\mathbf{u}_0^n]}\right]$$

$$[\varrho_{0,N},\mathbf{u}_{0,N}] \to [\widetilde{\varrho}_0,\widetilde{\mathbf{u}}_0] \text{ in } X_{\mathcal{D}} \ \mathrm{d}\mathcal{P} - \text{a.s.}$$

on a probability basis $\{\Omega,\mathcal{B},\mathcal{P}\}$

$$[\widetilde{\varrho}_0,\widetilde{\mathbf{u}}_0] \sim [\varrho_0,\mathbf{u}_0]$$

$\sim$ - equivalence in law

Convergence of approximate solutions, I

Approximate (numerical) solutions

$$(\varrho^{h,N}, \mathbf{u}^{h,N}),\ N = 1, 2, \dots,\ \mathcal{P}\left\{ \left\| \varrho^{h,N}, \mathbf{u}^{h,N} \right\|_{L^\infty((0,T)\times\mathbb{T}^d;R^{d+1})} \geq M \right\} \leq \varepsilon.$$

Application of Skorokhod theorem

$$Y_{h,N} = \left\{ [\varrho_{0,N}, \mathbf{u}_{0,N}]; (\varrho^{h,N}, \mathbf{u}^{h,N}); \Lambda_{h,N} \right\},\quad \text{with } \Lambda_{h,N} = \| \varrho^{h,N}, \mathbf{u}^{h,N} \|_{L^\infty},$$

a sequence of random variables ranging in the Polish space

$$X = X_{\mathcal{D}} \times W^{-m,2}((0,T) \times \mathbb{T}^d; R^{d+1}) \times R,\ m > d + 1.$$

Convergence of approximate solutions, II

$$\mathcal{L}[Y_{h,N}] \text{ tight in } X$$

$$\Rightarrow$$

$$\left\{[\widetilde{\varrho}_{0,N_k}, \widetilde{\mathbf{u}}_{0,N_k}]; \left(\widetilde{\varrho}^{h_k,N_k}, \widetilde{\mathbf{u}}^{h_k,N_k}\right); \widetilde{\Lambda}_{h_k,N_k}\right\}$$

$$\sim \left\{[\varrho_{0,N_k}, \mathbf{u}_{0,N_k}]; \left(\varrho^{h_k,N_k}, \mathbf{u}^{h_k,N_k}\right), \Lambda_{h_k,N_k}\right\},$$

$$[\widetilde{\varrho}_{0,N_k}, \widetilde{\mathbf{u}}_{0,N_k}] \to [\widetilde{\varrho}_0, \widetilde{\mathbf{u}}_0] \text{ in } X_{\mathcal{D}}\ \widetilde{\mathcal{P}} - \text{a.s.},$$

$$\text{where } [\widetilde{\varrho}_0, \widetilde{\mathbf{u}}_0] \sim [\varrho_0, \mathbf{u}_0]$$

$$\left(\widetilde{\varrho}^{h_k,N_k}, \widetilde{\mathbf{u}}^{h_k,N_k}\right) \to (\widetilde{\varrho}, \widetilde{\mathbf{u}}) \text{ in } W^{-m,2}((0,T) \times \mathbb{T}^d; R^{d+1})\ \widetilde{\mathcal{P}} - \text{a.s.},$$

and

$$\widetilde{\Lambda}_{h_k,N_k} = \|(\widetilde{\varrho}^{h_k,N_k}, \widetilde{\mathbf{u}}^{h_k,N_k})\|_{L^\infty} \to \widetilde{\Lambda}\ \widetilde{\mathcal{P}} - \text{a.s.}$$

on a probability space $\{\widetilde{\Omega}; \widetilde{\mathcal{B}}; \widetilde{\mathcal{P}}\}$

Convergence of approximate solutions, conclusion

Bounded graph property

$$\left(\widetilde{\varrho}^{h_k, N_k}, \widetilde{\mathbf{u}}^{h_k, N_k}\right) \to (\widetilde{\varrho}, \widetilde{\mathbf{u}}) \text{ strongly in } L^q((0,T) \times \mathbb{T}^d; R^{d+1}) \ \ \widetilde{\mathcal{P}} - \text{a.s.}$$

$$\text{for any } 1 \leq q < \infty$$

where $(\widetilde{\varrho}, \widetilde{\mathbf{u}})$ is the unique (statistical) solution of the Navier–Stokes system

Gyöngy–Krylov criterion

$$\left(\varrho^{h,N}, \mathbf{u}^{h,N}\right) \to (\varrho, \mathbf{u}) \text{ in } L^q((0,T) \times \mathbb{T}^d; R^{d+1}) \text{ in } \mathcal{P} - \text{probability}$$

on the original probability basis

Convergence in expectations

Strong convergence in expectations [EF [2022]]
Suppose that the energy of the numerical solutions is bounded in expectations, meaning

$$\sum_{n=1}^{N} |\Omega_n^M| \int_{\mathbb{T}^d} \left[\frac{1}{2} \varrho^{h,n} |\mathbf{u}^{h,n}|^2 + P(\varrho^{h,n}) \right] (\tau, \cdot) \, \mathrm{d}x \stackrel{<}{\sim} 1 \text{ for } \tau \in (0,T), \ \ell = 1, 2, \dots$$

Then

$$\mathbb{E} \left[\left\| \sum_{n=1}^{N} \mathbb{1}_{\Omega_n^N} \varrho^{h,n} - \varrho \right\|^r_{L^\gamma((0,T)\times\mathbb{T}^d)} \right] \to 0 \text{ as } \ell \to \infty \text{ for any } 1 \le r < \gamma,$$

$$\mathbb{E} \left[\left\| \sum_{n=1}^{N} \mathbb{1}_{\Omega_n^N} \varrho^{h,n} \mathbf{u}^{h,n} - \varrho \mathbf{u} \right\|^s_{L^{\frac{2\gamma}{\gamma+1}}((0,T)\times\mathbb{T}^d;R^d)} \right] \to 0 \text{ as } \ell \to \infty$$

$$\text{for any } 1 \le s < \frac{2\gamma}{\gamma+1}$$

r-barycenter

r-barycenter
$\mathbb{E}_r[Y]$ of a random variable Y defined on a Polish space $(X; d_X)$:

$$\mathbb{E}_r[Y] \in X, \ \mathbb{E}\left[d_X\left(Y; \mathbb{E}_r[Y]\right)^r\right] = \min_{Z \in X} \mathbb{E}\left[d_X\left(Y; Z\right)^r\right], \ r \geq 1,$$

meaning

$$E_r(Y) = \arg\min_{Z \in X} \mathbb{E}\left[d_X\left(Y; Z\right)^r\right].$$

If $X = L^q((0, T) \times \mathbb{T}^d; R^d)$ and $1 < r < \infty$, then

- there exists a unique $r-$barycenter for any Y, $\mathbb{E}\left[\|Y\|_{L^q}^r\right] < \infty$,
- $\mathbb{E}_r[Y]$ depends only on the distribution (law) of Y

Convergence of barycenters

Strong convergence of barycenters [EF [2022]]
Suppose that the energy of the numerical solutions is bounded in expectations.
Then

- $$\frac{1}{N}\sum_{n=1}^{N} \varrho^{h,n} \to \mathbb{E}\left[\varrho\right] \text{ in } L^{\gamma}((0,T)\times\mathbb{T}^d),$$
  $$\frac{1}{N}\sum_{n=1}^{N} \varrho^{h,n}\mathbf{u}^{h,n} \to \mathbb{E}\left[\varrho\mathbf{u}\right] \text{ in } L^{\frac{2\gamma}{\gamma+1}}((0,T)\times\mathbb{T}^d;R^d)$$
  as $\ell \to \infty$
- $$\mathbb{E}_r\left[\frac{1}{N}\sum_{n=1}^{N} \delta_{\varrho^{h,n}}\right] \to \mathbb{E}_r[\varrho] \text{ in } L^{\gamma}(\mathbb{T}^d),\ 1 < r < \gamma,$$
  $$\mathbb{E}_s\left[\frac{1}{N}\sum_{n=1}^{N} \delta_{\varrho^{h,n}\mathbf{u}^{h,n}}\right] \to \mathbb{E}_s[\varrho\mathbf{u}] \text{ in } L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d;R^d),\ 1 < s < \frac{2\gamma}{\gamma+1}$$
  as $\ell \to \infty$.

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