Ergodic theory for energetically open compressible fluid flows

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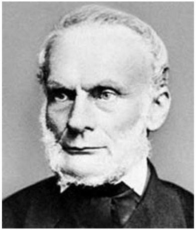
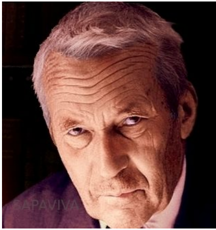
Motivation



Rudolf Clasius
1822–1888

Basic principles of thermodynamics of closed systems
Die Energie der Welt ist constant. Die Entropie der Welt strebt einem Maximum zu.



Andrey Nikolaevich Kolmogorov
1903–1987

Turbulence - ergodic hypothesis
Time averages along trajectories of the flow converge, for large enough times, to an ensemble average given by a certain probability measure

Dynamical systems

Dynamical system

$$\mathbf{U}(t,\cdot) : [0,\infty) \times X \to X$$

- **Closed system:** $\mathbf{U}(t, X_0) \to \mathbf{U}_\infty$ equilibrium solution as $t \to \infty$
- **Open system:** $\frac{1}{T}\int_0^T F(\mathbf{U}(t,X_0))\mathrm{d}t \to \int_X F(X)\ \mathrm{d}\mu,\ T \to \infty$

μ a.s. in X_0

Principal mathematical problems:

- **Low regularity of global in time solutions**
  Global in time solutions necessary. For many problems in fluid dynamics – Navier–Stokes or Euler system – only weak solutions available
- **Lack of uniqueness**
  Solutions do not, or at least are not known to, depend uniquely on the initial data. Spaces of trajectories: Sell, Nečas, Temam and others
- **Propagation of oscillations**
  Realistic systems are partly hyperbolic: propagation of oscillations "from the past", singularities

Abstract setting

George Roger Sell 1937–2015

Space of entire trajectories

$$\mathcal{T} = C_{\mathrm{loc}}(R; X), \ \ t \in (-\infty, \infty)$$

ω**–limit set**

$$\omega[\mathbf{U}(\cdot, X_0)] \subset \mathcal{T}$$

$$\omega[\mathbf{U}(\cdot, X_0)] = \left\{ \mathbf{V} \in \mathcal{T} \ \middle| \ \mathbf{U}(\cdot + t_n, X_0) \to \mathbf{V} \text{ in } \mathcal{T} \text{ as } t_n \to \infty \right\}$$

Necessary ingredients

- **Dissipativity** – ultimate boundedness of trajectories
- **Compactness** – in appropriate spaces

Strong and weak ergodic hypothesis

Krylov – Bogolyubov construction

$$T \mapsto \frac{1}{T}\int_0^T \delta_{\mathbf{U}(\cdot+t,X_0)}\mathrm{d}t \text{ – a family of probability measures on } \mathcal{T}$$

$$\text{tightness in } \mathcal{T} \ \Rightarrow \ T_n \mapsto \frac{1}{T_n}\int_0^{T_n} \delta_{\mathbf{U}(\cdot+t,X_0)}\mathrm{d}t \to \mu \in \mathcal{P}[\mathcal{T}]$$

$[\mathcal{T}, \mu]$ stationary statistical solution

$$\textbf{Ergodic hypothesis} \ \Leftrightarrow \ \mu \text{ is unique } \Rightarrow \ T \mapsto \frac{1}{T}\int_0^T \delta_{\mathbf{U}(\cdot+t,X_0)}\mathrm{d}t \to \mu$$

$$\text{unique } \approx \text{ unique on } \omega[\mathbf{U}(\cdot, X_0)]$$

Weak ergodic hypothesis

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \delta_{\mathbf{U}(\cdot+t,X_0)}\mathrm{d}t = \mu \text{ exists in the narrow sense in } \mathcal{P}[\mathcal{T}]$$

$[\mathcal{T}, \mu]$ stationary statistical solution

Barotropic Navier–Stokes system

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \varrho \mathbf{g}$$

Constitutive equations

- barotropic (isentropic) pressure–density EOS $p = p(\varrho)$ $(p = a\varrho^\gamma)$
- Newton's rheological law

$$\mathbb{S} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I},\ \mu > 0,\ \eta \geq 0$$

- Gravitational external force

$$\mathbf{g} = \nabla_x F,\ F = F(x)$$

Energy

$$E(\varrho, \mathbf{m}) \equiv \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - \varrho F,\ P'(\varrho)\varrho - P(\varrho) = p(\varrho),\ \mathbf{m} = \varrho \mathbf{u}$$

Energetically insulated system

Conservative boundary conditions

$$\Omega \subset R^d \text{ bounded (sufficiently regular) domain}$$

- **impermeability** $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$
- **no–slip** $[\mathbf{u}]_{\rm tan}|_{\partial\Omega} = 0$

Long–time behavior – Clausius scenario

- Total mass conserved

$$\int_\Omega \varrho(t, \cdot) \ {\rm d}x = M_0$$

- Total energy – Lyapunov function

$$\frac{\rm d}{{\rm d}t} \int_\Omega E(\varrho, \mathbf{m}) \ {\rm d}x + \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ {\rm d}x = (\leq) 0, \ \int_\Omega E(\varrho, \mathbf{m}) \ {\rm d}x \searrow \mathcal{E}_\infty$$

- Stationary solution

$$\mathbf{m}_\infty = 0, \ \nabla_x p(\varrho_\infty) = \varrho_\infty \nabla_x F, \ \int_\Omega \varrho_\infty \ {\rm d}x = M_0, \int_\Omega E(\varrho_\infty, 0) \ {\rm d}x = \mathcal{E}_\infty$$

Energetically open system

In/out flow boundary conditions

$$\mathbf{u} = \mathbf{u}_b \text{ on } \partial\Omega$$

$$\Gamma_{\rm in} = \left\{ x \in \partial\Omega \ \Big| \ \mathbf{u}_b(x) \cdot \mathbf{n}(x) < 0 \right\}, \ \Gamma_{\rm out} = \left\{ x \in \partial\Omega \ \Big| \ \mathbf{u}_b(x) \cdot \mathbf{n}(x) \geq 0 \right\}$$

Density (pressure) on the inflow boundary

$$\varrho = \varrho_b \text{ on } \Gamma_{\rm in}$$

Energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2} \varrho |\mathbf{u} - \mathbf{u}_b|^2 + P(\varrho) \ \mathrm{d}x + \int_\Omega \mathbb{S} : \nabla_x \mathbf{u} \ \mathrm{d}x\mathrm{d}t$$

$$+ \int_{\Gamma_{\rm in}} P(\varrho_b) \mathbf{u}_b \cdot \mathbf{n} \ \mathrm{d}S_x + \int_{\Gamma_{\rm out}} P(\varrho) \mathbf{u}_b \cdot \mathbf{n} \ \mathrm{d}S_x$$

$$= (\leq) - \int_\Omega [\varrho \mathbf{u} \otimes \mathbf{u} + p(\varrho)\mathbb{I}] : \nabla_x \mathbf{u}_b \ \mathrm{d}x + \frac{1}{2} \int_\Omega \varrho \mathbf{u} \cdot \nabla_x |\mathbf{u}_b|^2 \ \mathrm{d}x\mathrm{d}t$$

Global bounded trajectories

Global in time weak solutions
$\mathbf{U} = [\varrho, \mathbf{m} = \varrho \mathbf{u}]$ – weak solution of the Navier–Stokes system satisfying energy inequality and defined for $t > T_0$

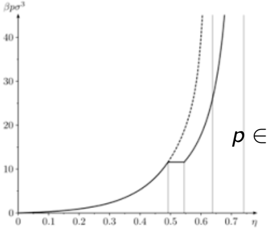
Bounded energy

$$\limsup_{t\to\infty} \int_{\Omega} E(\varrho, \mathbf{m})\, \mathrm{d}x \leq \mathcal{E}_{\infty}$$

Available results

- **Existence:** T. Chang, B. J. Jin, and A. Novotný, *SIAM J. Math. Anal.*, **51**(2):1238–1278, 2019
  H. J. Choe, A. Novotný, and M. Yang *J. Differential Equations*, **266**(6):3066–3099, 2019
- **Globally bounded solutions:** F. Fanelli, E. F., and M. Hofmanová **arxiv preprint No. 2006.02278**, 2020
  J. Březina, E. F., and A. Novotný, Communications in PDE's 2020

Hard sphere pressure EOS



$$p \in C[0,\overline{\varrho}) \cap C^1(0,\overline{\varrho}),\ \ p'(\varrho) > 0 \text{ for } \varrho > 0,\ \ \lim_{\varrho \to \overline{\varrho}-} p(\varrho) = \infty$$

Ultimate boundedness of trajectories – bounded absorbing set

$$\limsup_{t\to\infty} \int_\Omega E(\varrho, \mathbf{m})\ \mathrm{d}x \le \mathcal{E}_\infty$$

$\mathcal{E}_\infty$ – universal constant

ω – limit sets

$$p \approx a\varrho^{\gamma},\ \gamma > \frac{d}{2} \text{ or hard sphere EOS}$$

Trajectory space

$$X = \left\{ \varrho, \mathbf{m} \ \middle| \ \varrho(t,\cdot) \in L^{\gamma}(\Omega),\ \mathbf{m}(t,\cdot) \in L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d) \hookrightarrow W^{-k,2} \right\}$$

$$\mathcal{T} = C_{\rm loc}(R; L^1 \times W^{-k,2})$$

Fundamental result on compactness [Fanelli, EF, Hofmanová, 2020]

The ω–limit set $\omega[\varrho, \mathbf{m}]$ of each global in time trajectory with globally bounded energy is:

- $\boxed{\textit{non} - \textit{empty}}$
- $\boxed{\textit{compact}}$ in $\mathcal{T}$
- time shift invariant
- consists of entire (defined for all $t \in R$) weak solutions of the Navier–Stokes system

Propagation of oscillations

Equation of continuity

$$\partial_t \varrho + \mathbf{u} \cdot \nabla_x \varrho = -\varrho \mathrm{div}_x \mathbf{u}$$

Renormalized equation of continuity

$$\partial_t b(\varrho) + \mathrm{div}_x (b(\varrho)\mathbf{u}) + \Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u} = 0$$

Weak convergence

$$b(\varrho_n) \to \overline{b(\varrho)} \text{ weakly in } L^1$$

$$\partial_t \Big[\overline{b(\varrho)} - b(\varrho) \Big] + \mathrm{div}_x \left(\overline{b(\varrho)\mathbf{u}} - b(\varrho)\mathbf{u} \right)$$

$$= \Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u} - \overline{\Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u}}$$

$$\Big[\overline{b(\varrho)} - b(\varrho) \Big] (0, \cdot) = 0 \text{ is needed!}$$

Vanishing oscillation defect, I

Compactness of densities:

$$\varrho_n \equiv \varrho(\cdot + T_n) \to \varrho \text{ in } C_{\rm weak,loc}(R; L^\gamma(\Omega))$$

$$\varrho_n \log(\varrho_n) \to \overline{\varrho \log(\varrho)} \geq \varrho \log(\varrho)$$

$$\textbf{oscillation defect: } D(t) \equiv \int_\Omega \overline{\varrho \log(\varrho)} - \varrho \log(\varrho) \ {\rm d}x \geq 0$$

Renormalized equation:

$$\frac{\rm d}{{\rm d}t} D + \int_\Omega \left[\overline{\varrho {\rm div}_x \mathbf{u}} - \varrho {\rm div}_x \mathbf{u} \right] \ {\rm d}x = 0, \ 0 \leq D \leq \overline{D}, \ t \in R$$

Lions' identity

$$\overline{\varrho {\rm div}_x \mathbf{u}} - \varrho {\rm div}_x \mathbf{u} = \overline{p(\varrho)\varrho} - \overline{p(\varrho)} \ \varrho \geq 0$$

Vanishing oscillation defect, II

Crucial differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} D + \Psi(D) \leq 0, \; 0 \leq D \leq \overline{D}, \; t \in R$$

$$\Psi \in C(R), \; \Psi(0) = 0, \; \Psi(Z)Z > 0 \text{ for } Z \neq 0$$

$$\Rightarrow$$

$$D \equiv 0$$

Statistical stationary solutions

Application of Krylov – Bogolyubov method

$$\frac{1}{T_n}\int_0^{T_n} \delta_{\varrho(\cdot+t,\cdot),\mathbf{m}(\cdot+t,\cdot)}\ \mathrm{d}t \to \mu \in \mathcal{P}[\mathcal{T}] \text{ narrowly}$$

$[\mathcal{T},\mu]$ (canonical representation) – statististical stationary solution

$\mu(t)|_X$ (marginal) independent of $t \in R$

Application of Birkhoff – Khinchin ergodic theorem

$$\frac{1}{T}\int_0^{T} F(\varrho(t,\cdot),\mathbf{m}(t,\cdot))\mathrm{d}t \to \overline{F} \text{ as } T \to \infty$$

F bounded Borel measurable on X for $\mu -$ a.a. $(\varrho,\mathbf{m}) \in \omega$

Related results for incompressible Navier–Stokes system with conservative boundary conditions

F.Flandoli and D. Gatarek, F.Flandoli and M.Romito (stochastic forcing),

Back to ergodic hypothesis – conclusion

Ergodicity

$$\mu \text{ ergodic } \Leftrightarrow \ \mathcal{B} \subset \omega[\varrho, \mathbf{m}] \text{ shift invariant } \Rightarrow \ \mu[\mathcal{B}] = 1 \text{ or } \mu[\mathcal{B}] = 0$$

$$\mu \in \text{conv}\Big\{ \text{ergodic measures on } \omega[\varrho, \mathbf{m}] \Big\}$$

State of the art for compressible Navier–Stokes system

- Each bounded energy global trajectory generates a stationary statistical solution – a shift invariant measure μ – sitting on its $\omega-$limit set $\omega[\varrho, \mathbf{m}]$
- The weak ergodic hypothesis (the existence of limits of ergodic averages for any Borel measurable F) holds on $\omega[\varrho, \mathbf{m}]$ $\mu-$a.s.
- The (strong) ergodic hypothesis definitely holds for enegetically isolated systems and a class of potential forces F, where all solutions tend to equilibrium

Complete Navier–Stokes–Fourier system

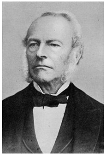
Claude Louis Marie Henri Navier [1785-1836]

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \varrho \mathbf{f}$$



George Gabriel Stokes

Entropy production

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma$$

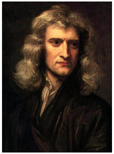
$$\sigma = (\geq) \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Constitutive relations

Joseph Fourier [1768-1830]

Fourier's law

$$\mathbf{q} = -\kappa(\vartheta)\nabla_x\vartheta$$



Isaac Newton [1643-1727]

Newton's rheological law

$$\mathbb{S} = \mu(\vartheta)\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\right) + \eta(\vartheta)\mathrm{div}_x\mathbf{u}\mathbb{I}$$

Boundary conditions

Impermeability

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

No-slip

$$\mathbf{u}_{\tan}|_{\partial\Omega} = 0$$

No-stick

$$[\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

Thermal insulation

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Long–time behavior

Dichotomy for the closed/open system

$$\mathbf{f} = \mathbf{f}(x)$$

Either

$$\mathbf{f} = \nabla_x F \ \Rightarrow \ \text{all solutions tend to a single equilibrium}$$

or

$$\mathbf{f} \neq \nabla_x F \ \Rightarrow \ \int_\Omega E(t,\cdot)\ \mathrm{d}x \to \infty \text{ as } t \to \infty$$