Ring-theoretic (in)finiteness in reduced products of Banach algebras

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Easy to see: $\sim$ is an equivalence relation on the set of idempotents of $\mathcal{A}$.

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- a commutative unital algebra is DF;
- $\mathcal{A}, \mathcal{B}$ are unital algebras, $\mathcal{A}$ is PI and $\varphi : \mathcal{A} \to \mathcal{B}$ is a unital algebra hom $\implies$ $\mathcal{B}$ is PI.

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Lemma (Rieffel, PLMS, '83 ?)

For a unital Banach algebra $\mathcal{A}$:

$$\mathcal{A} \text{ has stable rank one} \implies \mathcal{A} \text{ is DF.}$$

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- "$\ell^1(Cu_2 \setminus \{\Diamond\})$", where Cu_2 is the second Cuntz semigroup with a zero element $\Diamond$ [folk].

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- If $p \in \mathcal{A}$ is an idempotent, there is a $q \in \mathcal{A}$ projection with $p \sim q$ and ($pq = q$, $qp = p$ or $pq = p$, $qp = q$).

... But for unital C^*-algebras both versions make sense, which one to use??? It absolutely does not matter.

Proposition (folk, scattered through H.G. Dales' book "Banach Algebras and Automatic Continuity")

Let $\mathcal{A}$ be a unital C^*-algebra. Then

1. *$\mathcal{A}$ is DF as an algebra $\Leftrightarrow$ $\mathcal{A}$ is DF as a C^*-algebra;*
2. *$\mathcal{A}$ is PI as an algebra $\Leftrightarrow$ $\mathcal{A}$ is PI as a C^*-algebra.*

Proof.

(Sketch.) The main ideas used:

- If $p \in \mathcal{A}$ is an idempotent, there is a $q \in \mathcal{A}$ projection with $p \sim q$ and $(pq = q,\ qp = p$ or $pq = p,\ qp = q)$.
- Let $p, q \in \mathcal{A}$ be projections. Then $p \sim q \iff p \approx q$.

□

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$\ell^\infty(\mathcal{A}_n)$ is a unital Banach algebra endowed with pointwise operations and the sup norm

$$\|A\| = \sup_{n\in\mathbb{N}} \|a_n\| \quad (A = (a_n) \in \ell^\infty(\mathcal{A}_n)).$$

In fact, $c_0(\mathcal{A}_n) \trianglelefteq \ell^\infty(\mathcal{A}_n)$ and $c_\mathcal{U}(\mathcal{A}_n) \trianglelefteq \ell^\infty(\mathcal{A}_n)$ with $c_0(\mathcal{A}_n) \subsetneq c_\mathcal{U}(\mathcal{A}_n)$.

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The *asymptotic sequence algebra* and the *ultraproduct* of a sequence of unital Banach algebras $(\mathcal{A}_n)_{n\in\mathbb{N}}$ are defined as

$$\mathrm{Asy}(\mathcal{A}_n) := \ell^\infty(\mathcal{A}_n)/c_0(\mathcal{A}_n), \text{ and} \tag{1}$$

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$$\|\pi(A)\| = \limsup_{n\to\infty} \|a_n\|, \text{ and}$$
$$\|\pi_\mathcal{U}(A)\| = \lim_{n\to\mathcal{U}} \|a_n\| \quad (A = (a_n) \in \ell^\infty(\mathcal{A}_n)).$$

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Both $\mathrm{Asy}(\mathcal{A}_n)$ and $(\mathcal{A}_n)_{\mathcal{U}}$ are special cases of $(\mathcal{A}_n)_{\mathcal{F}}$.

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More on the proofs to follow soon. (Wishful thinking.)

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Corollary (Daws–H.)

Let $(\mathcal{A}_n)_{n\in\mathbb{N}}$ be a sequence of unital Banach algebras such that $\mathrm{Asy}(\mathcal{A}_n)$ is DF. Moreover, suppose that one of the following two conditions hold:

1. $\mathcal{A}_n = \mathcal{A}_m$ *for every* $n, m \in \mathbb{N}$;
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The C^*-case is very well known.

The situation regarding proper infiniteness is "reversed".

Recall that a unital algebra $\mathcal{A}$ is PI if there exist idempotents $p, q \in \mathcal{A}$ such that $p \sim 1 \sim q$ and $p \perp q$.

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Both of these results are somewhat harder to prove than their respective DF-counterparts.

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- If $\mathcal{A}$ is a C^*-algebra, then $\mathcal{A}$ has stable rank one $\Leftrightarrow$ $\mathrm{Asy}(\mathcal{A})$ has stable rank one. [follows from work of *e.g.* Farah–Rørdam]

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The Approximate Idempotent Lemma:

Proposition (folk)

Let $a \in \mathcal{A}$ be such that $\nu := \|a^2 - a\| < 1/4$. Then there is an idempotent $p \in \mathcal{A}$ such that $\|p - a\| \leq f_{\|a\|}(\nu)$ holds. Moreover, if $y \in \mathcal{A}$ is such that $ay = ya$ then $yp = py$.

In the Proposition above, $f_M : [0, 1/4) \to \mathbb{R}$ is some monotone increasing, non-negative continuous function for each $M > 0$. Also, $f_M \leq f_N$ when $N > M > 0$.

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By continuity of $f_{\|X\|}$, it follows that $\lim_{n\geq N} f_{\|X\|}(\nu_n) = 0$; consequently $\lim_{n\geq N} \|x_n - p'_n\| = 0$.

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$$\lim_{n \to \infty} \|1_n - a_n b_n\| = 0 \text{ and } \lim_{n \to \infty} \|p_n - b_n a_n\| = 0.$$

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Let $K \geq M$ be such that for every $n \geq K$

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This concludes the proof. □

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Clearly $p_n \in \mathcal{A}$ is an idempotent for each $n \in \mathbb{N}$,

...But why is the converse not true? What goes wrong?

Idempotents in Banach algebras can have arbitrarily big norm!

Example

Consider $\mathcal{A} := \ell^1(\mathbb{N})$ with the pointwise product. Define

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Clearly $p_n \in \mathcal{A}$ is an idempotent for each $n \in \mathbb{N}$, but $\|p_n\| = n$ and hence $(p_n) \notin \ell^\infty(\mathcal{A})$.

About the counter-example

Theorem (Daws–H.)

There is a sequence of DI (⇔ not DF) Banach algebras $(\mathcal{A}_n)_{n\in\mathbb{N}}$ *such that* $\mathrm{Asy}(\mathcal{A}_n)$ *is DF.*

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- $\ell^1(I,\nu) = \overline{\mathrm{span}\{\delta_s : s\in I\}}^{\|\cdot\|_\nu}$, hence

$$f = \sum_{s\in I} f(s)\delta_s \quad (f\in\ell^1(I,\nu))$$

where the sum converges in the norm $\|\cdot\|_\nu$.

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$(\ell^1(S, \omega), *)$ is a unital Banach algebra.

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For our counter-example, we need the *bicyclic monoid BC*. That is, the free monoid generated by elements p, q subject to the single relation that $pq = e$:

$$BC = \langle p, q : pq = e \rangle.$$

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1. $\mathcal{A}_n$ is DI for each $n \in \mathbb{N}$; and
2. $\mathrm{Asy}(\mathcal{A}_n)$ is DF. (Follows from Prop. and some actual work.)

OK, the very last one, really

Thank you for your attention :)

[Insert funny picture here, I am too technologically illiterate.]

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Sources

- I. Farah, "Combinatorial Set Theory of C^*-algebras", available on Farah's website, and forthcoming from Springer;
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