On well/ill posedness of some systems of PDE's in fluid dynamics

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Euler system for a barotropic inviscid fluid

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

Momentum equation

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = 0$$

Impermeable boundary

$$\mathbf{m} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Initial conditions

$$\varrho(0, \cdot) = \varrho_0, \ \ \mathbf{m}(0, \cdot) = \mathbf{m}_0$$

First and Second law – energy

Energy

$$\mathcal{E} = \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho),\ \ P'(\varrho)\varrho - P(\varrho) = p(\varrho)$$

$$p' \geq 0 \Rightarrow [\varrho, \mathbf{m}] \mapsto \left\{ \begin{array}{l} \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \text{ if } \varrho > 0 \\ P(\varrho) \text{ if } |\mathbf{m}| = 0 \\ \infty \text{ if } \varrho = 0, |\mathbf{m}| \neq 0 \end{array} \right. \qquad \text{is convex l.s.c}$$

Energy balance (conservation)

$$\partial_t \mathcal{E} + \mathrm{div}_x \left(\mathcal{E} \frac{\mathbf{m}}{\varrho} \right) + \mathrm{div}_x \left(p \frac{\mathbf{m}}{\varrho} \right) = 0$$

Energy dissipation

$$\partial_t \mathcal{E} + \mathrm{div}_x (\mathcal{E} \mathbf{u}) + \mathrm{div}_x (p\mathbf{u}) \boxed{\leq} 0$$

$$E = \int_Q \mathcal{E}\ \mathrm{d}x,\ \ \partial_t E \leq 0,\ \ E(0+) = \int_Q \left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right]\ \mathrm{d}x$$

Weak solutions

Field equations

$$\int_0^\infty \int_Q [\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi] \ \mathrm{d}x\mathrm{d}t = - \int_Q \varrho_0 \varphi(0,\cdot) \ \mathrm{d}x, \ \ \varphi \in C^1_c([0,\infty) \times \overline{\Omega})$$

$$\int_0^\infty \int_Q \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x\mathrm{d}t$$

$$= - \int_Q \mathbf{m}_0 \cdot \boldsymbol{\varphi}(0,\cdot) \ \mathrm{d}x, \ \ \boldsymbol{\varphi} \in C^1_c([0,T) \times \overline{\Omega}; R^N), \ \ \boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Dissipative weak solutions

$$\int_0^\infty \int_Q \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] \ \mathrm{d}x \ \partial_t \psi \ \mathrm{d}t \geq \psi(0) \int_Q \frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \ \mathrm{d}x$$

$$\psi \in C^1_c[0,\infty), \ \ \psi \geq 0$$

Well posedness

Classical solutions [Matsumura–Nishida], [Tani]

$$\varrho_0 \in W^{3,2}(Q),\ \varrho_0 > 0,\ \mathbf{m}_0 \in W^{3,2}(Q; R^N) + \text{compatibility conditions}$$

$$\Rightarrow$$

classical solution

$$\varrho \in C([0, T_{\max}); W^{3,2}(Q)),\ \mathbf{m} \in C([0, T_{\max}); W^{3,2}(Q; R^N)),\ N = 2, 3$$

$T_{\max} < \infty$ for a "generic" class of initial data

Weak–Strong uniqueness [Dafermos]

A *dissipative* weak solution coincides with the strong solution emanating from the same initial data on the time interval $[0, T_{\max})$

Well/ill posedness

Global existence well/ill posedness [Chiodaroli, E.F.]

$$\varrho_0 \in C^3(\overline{Q}),\ \varrho_0 > 0,\ \mathbf{m}_0 \in C^3(\overline{Q}; R^N),\ \mathbf{m}_0 \cdot \mathbf{n}|_{\partial Q} = 0$$

$$\Rightarrow$$

infinitely many weak solutions

$$\varrho \in L^\infty_{\rm loc}([0,\infty) \times Q),\ \mathbf{m} \in L^\infty_{\rm loc}([0,\infty) \times Q; R^N)$$

$$\varrho > 0,\ {\rm div}_x \mathbf{m} \in L^\infty_{\rm loc}([0,\infty) \times Q),\ \mathbf{m} \cdot \mathbf{n}|_{\partial Q} = 0$$

Well/ill posedness of dissipative solutions [Chiodaroli, E.F.]

$$\varrho_0 \in C^3(\overline{Q}),\ \varrho_0 > 0,\ \nabla_x \Phi_0 \in C^3(\overline{Q}),\ \nabla_x \Phi_0 \cdot \mathbf{n}|_{\partial Q} = 0$$

$$\Rightarrow$$

there exist (infinitely many) $\mathbf{v}_0 \in L^\infty(Q; R^N),\ {\rm div}_x \mathbf{v}_0 = 0$

and *infinitely many* dissipative weak solutions

$$\varrho \in L^\infty_{\rm loc}([0,\infty) \times Q),\ \mathbf{m} \in L^\infty_{\rm loc}([0,\infty) \times Q; R^N)$$

$$\varrho(0,\cdot) = \varrho_0,\ \mathbf{m}(0,\cdot) = \mathbf{v}_0 + \nabla_x \Phi_0$$

Admissible weak solutions

Global existence well/ill posedness [Chiodaroli, E.F., Luo, Xie and Xin]

$$\varrho_0 \text{ piecewise Lipschitz}, \ \varrho_0 > 0$$

$$\Rightarrow$$

$$\text{there exist (infinitely many) } \mathbf{m}_0 \in L^\infty(Q; R^N)$$

and *infinitely many* admissible weak solutions

$$\varrho \in L^\infty_{\rm loc}([0,\infty) \times Q), \ \mathbf{m} \in L^\infty_{\rm loc}([0,\infty) \times Q; R^N)$$

$$\varrho(0, \cdot) = \varrho_0, \ \mathbf{m}(0, \cdot) = \mathbf{m}_0$$

Energy conserving solutions [Luo, Xie and Xin]

If ϱ_0 is piecewise constant, one can find $\mathbf{m}_0$ as above such that the solutions satisfy the energy equation (energy conserving solutions).

Lipschitz initial data

Ill posedness for regular data [Chiodaroli, DeLellis, Kreml]

Let $T > 0$ be given.
Then there exist (infinitely many) *Lipschitz* initial data ϱ_0, $\mathbf{m}_0$ such that the barotropic Euler system admits infinitely many admissible weak solutions on the time interval $[0, T]$.

Isentropic Euler system revisited

Phase variables

mass density .... $\varrho = \varrho(t, x)$
momentum .... $\mathbf{m} = \mathbf{m}(t, x) \in R^N$
(total) energy .... $E = E(t) \in R$

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

Balance of momentum

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + a \nabla_x \varrho^\gamma = 0$$

Energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) \leq 0, \ E = \int_Q \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + \frac{a}{\gamma - 1} \varrho^\gamma \right] \ \mathrm{d}x$$

Semiflow solution

Semiflow

$$U[t, \varrho_0, \mathbf{m}_0, E_0] \mapsto [\varrho(t), \mathbf{m}(t), E(t-)], \ \ t > 0$$

Semigroup property

$$U[t+T, \varrho_0, \mathbf{m}_0, E_0] = U\left[t, U[T, \varrho_0, \mathbf{m}_0, E_0]\right] \ \text{for any } 0 < T \leq t,$$

Dissipative solution

$$\varrho \in C_{\rm weak, loc}([0, \infty); L^\gamma(Q))$$

$$\mathbf{m} \in C_{\rm weak, loc}([0, \infty); L^{\frac{2\gamma}{\gamma+1}}(Q; R^N))$$

$$E \in BV_{\rm loc}([0, \infty); R), \ \ \text{(non–increasing)}$$

Initial data

$$\varrho(0, \cdot) = \varrho_0, \ \ \mathbf{m}(0, \cdot) = \mathbf{m}_0, \ \ E(0+) \leq E_0$$

Dissipative solution, I

Stability of strong solutions

$$\widehat{\varrho},\ \widehat{\mathbf{m}},\ \widehat{E} = \int_Q \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + \frac{a}{\gamma - 1} \varrho_0^\gamma \right] \mathrm{d}x - \text{ a strong solution on } [0, T_{\max})$$

$$\Rightarrow\ \varrho(t) = \widehat{\varrho}(t),\ \mathbf{m}(t) = \widehat{\mathbf{m}}(t),\ E(t) = \widehat{E}(t) \text{ in } [0, T_{\max})$$

Maximal dissipation

$$\tilde{E}(t) \leq E(t)\ \Rightarrow E(t) = \tilde{E}(t)$$

$$\text{whenever, } \varrho_0 = \widetilde{\varrho}_0,\ \mathbf{m}_0 = \widetilde{\mathbf{m}}_0,\ E_0 = \widetilde{E}_0$$

Stability of stationary states

$$\overline{\varrho} > 0,\ \mathbf{m} \equiv 0 \text{ a stationary solution}$$

$$\varrho(T, \cdot) = \overline{\varrho},\ \mathbf{m}(T, \cdot) = 0\ \Rightarrow\ \varrho(t, \cdot) = \overline{\varrho},\ \mathbf{m}(t, \cdot) = 0 \text{ for } t \geq T$$

Dissipative solutions, II

Relative energy

$$\mathcal{E}\left(\varrho,\mathbf{m}\ \middle|\ r,\mathbf{U}\right) \equiv \frac{1}{2}\varrho\left|\frac{\mathbf{m}}{\varrho}-\mathbf{U}\right|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r),\ P(\varrho) = \frac{a}{\gamma-1}\varrho^{\gamma}$$

Relative energy inequality

$$\begin{aligned}
&\int_Q \mathcal{E}\left(\varrho,\mathbf{m}\ \middle|\ r,\mathbf{U}\right)(\tau,\cdot)\ \mathrm{d}x \\
&\leq \left[\left(E_0 - \int_Q \left[\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0)\right]\ \mathrm{d}x\right) + \int_Q \mathcal{E}\left(\varrho_0,\mathbf{m}_0\ \middle|\ r(0,\cdot),\mathbf{U}(0,\cdot)\right)\ \mathrm{d}x\right. \\
&+ \int_0^\tau \int_Q \frac{1}{r}\Big(r\left(\partial_t\mathbf{U} + \mathbf{U}\cdot\nabla_x\mathbf{U}\right) + \nabla_x p(r)\Big)\Big(\varrho\mathbf{U} - \mathbf{m}\Big)\ \mathrm{d}x\mathrm{d}t \\
&\left. + \int_0^\tau \int_Q P''(r)(r-\varrho)\Big(\partial_t r + \mathrm{div}_x(r\mathbf{U})\Big)\ \mathrm{d}x\mathrm{d}t\right] \\
&\times \exp\left(\Lambda(\gamma)\int_0^\tau \|\nabla_x\mathbf{U}\|_{L^\infty(\Omega)}\ \mathrm{d}t\right)
\end{aligned}$$

for any Lipschitz r, $r>0$, $\mathbf{U}$, $\mathbf{U}\cdot\mathbf{n}|_{\partial Q} = 0$

Dissipative measure–valued solutions, I

Basic quantities

- the Young measure:

$$(t,x) \mapsto \nu_x(t) \in L^\infty_{\mathrm{weak}-(*)}((0,\infty) \times Q; \mathcal{P}(\mathcal{S}));$$

- the kinetic and internal energy concentration defect measures:

$$t \mapsto \mathfrak{C}_{\mathrm{kin}}(t) \in L^\infty_{\mathrm{weak}-(*)}(0,\infty; \mathcal{M}^+(Q)),$$
$$t \mapsto \mathfrak{C}_{\mathrm{int}}(t) \in L^\infty_{\mathrm{weak}-(*)}(0,\infty; \mathcal{M}^+(Q)),$$

- the convective and pressure concentration defect measures:

$$t \mapsto \mathfrak{C}_{\mathrm{conv}}(t) \in L^\infty_{\mathrm{weak}-(*)}\Big(0,\infty; \mathcal{M}^+\left(Q \times S^{N-1}\right)\Big),$$
$$t \mapsto \mathfrak{C}_{\mathrm{press}}(t) \in L^\infty_{\mathrm{weak}-(*)}\left(0,\infty; \mathcal{M}^+(Q)\right).$$

Compatibility conditions

$$\mathfrak{C}_{\mathrm{conv}}(t, \mathrm{d}x, \mathrm{d}\xi) = 2 r_x(t, \mathrm{d}\xi) \otimes \mathfrak{C}_{\mathrm{kin}}(t, \mathrm{d}x), \quad \mathfrak{C}_{\mathrm{press}} = (\gamma - 1)\mathfrak{C}_{\mathrm{int}},$$

Dissipative measure–valued solutions, II

Young measure

$$\varrho(\tau, x) = \langle \nu_x(\tau); \tilde{\varrho} \rangle \geq 0, \ \mathbf{m}(\tau, x) = \langle \nu_x(\tau); \tilde{\mathbf{m}} \rangle \ \text{for a.a } x \in Q,$$

$$E(\tau) = \int_Q \left\langle \nu_x(\tau); \frac{1}{2} \frac{|\tilde{\mathbf{m}}|^2}{\tilde{\varrho}} + \frac{a}{\gamma - 1} \tilde{\varrho}^\gamma \right\rangle \ \mathrm{d}x + \int_Q \ \mathrm{d}\mathfrak{C}_{\rm kin}(\tau) + \int_Q \ \mathrm{d}\mathfrak{C}_{\rm int}(\tau)$$

Field equations

$$\left[\int_Q \varrho \varphi(\tau, \cdot) \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_Q \left[\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi \right] \ \mathrm{d}x \mathrm{d}t, \ \varrho(0, \cdot) = \varrho_0$$

$$\left[\int_Q \mathbf{m} \cdot \boldsymbol{\varphi}(\tau, \cdot) \mathrm{d}x \right]_{t=0}^{t=\tau}$$

$$= \int_0^\tau \int_Q \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \left\langle \nu_x(t); \frac{\tilde{\mathbf{m}} \otimes \tilde{\mathbf{m}}}{\tilde{\varrho}} \right\rangle : \nabla_x \boldsymbol{\varphi} + \langle \nu_x(t); a \tilde{\varrho}^\gamma \rangle \, \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x \mathrm{d}t$$

$$+ 2 \int_0^\tau \int_Q \langle r_x(t); \xi \otimes \xi \rangle : \nabla_x \boldsymbol{\varphi} \ \ \mathrm{d}\mathfrak{C}_{\rm kin} \, \mathrm{d}t + (\gamma - 1) \int_0^\tau \int_Q \mathrm{div}_x \boldsymbol{\varphi} \ \mathrm{d}\mathfrak{C}_{\rm int} \, \mathrm{d}t$$

$$\mathbf{m}(0, \cdot) = \mathbf{m}_0$$

Dissipative measure–valued solutions, III

Energy balance

$$\left[E\psi \right]_{t=\tau_1-}^{t=\tau_2+} - \int_{\tau_1}^{\tau_2} E \partial_t \psi \ \mathrm{d}t \leq 0, \ E(0-) = E_0$$

Abstract setting

Phase space

$$X = W^{-\ell,2}(Q) \times W^{-\ell,2}(Q;R^N) \times R$$

Data space

$$D = \left\{ [\varrho_0, \mathbf{m}_0, E_0] \in X \ \middle|\ \varrho_0 \geq 0,\ \int_Q \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + \frac{a}{\gamma - 1} \varrho_0^\gamma \right] \, \mathrm{d}x \leq E_0 \right\}.$$

Trajectory space

$$\Omega = C_{\rm loc}([0,\infty); W^{-\ell,2}(Q)) \times C_{\rm loc}([0,\infty); W^{-\ell,2}(Q;R^N)) \times L^1_{\rm loc}(0,\infty)$$

Method by Krylov adapted by Cardona and Kapitanski

Multi–valued solution mapping

$$\mathcal{U} : [\varrho_0, \mathbf{m}_0, E_0] \mapsto [\varrho, \mathbf{m}, E] \in 2^{\Omega}$$

Time shift

$$S_T \circ \xi, \ S_T \circ \xi(t) = \xi(T + t), \ t \geq 0.$$

Continuation

$$\xi_1 \cup_T \xi_2(\tau) = \begin{cases} \xi_1(\tau) \text{ for } 0 \leq \tau \leq T, \\ \xi_2(\tau - T) \text{ for } \tau > T. \end{cases}$$

Basic ansatz

- **(A1) Compactness:** For any $[\varrho_0, \mathbf{m}_0, E_0] \in D$, the set $\mathcal{U}[\varrho_0, \mathbf{m}_0, E_0]$ is a non–empty compact subset of Ω
- **(A2)** The mapping

$$D \ni [\varrho_0, \mathbf{m}_0, E_0] \mapsto \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] \in 2^{\Omega}$$

is **Borel measurable**, where the range of $\mathcal{U}$ is endowed with the Hausdorff metric on the subspace of compact sets in 2^{Ω}

- **(A3) Shift invariance:** For any

$$[\varrho, \mathbf{m}, E] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0],$$

we have

$$S_T \circ [\varrho, \mathbf{m}, E] \in \mathcal{U}[\varrho(T), \mathbf{m}(t), E(T-)] \text{ for any } T > 0.$$

- **(A4) Continuation:** If $T > 0$, and

$$[\varrho^1, \mathbf{m}^1, E^1] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0],\ [\varrho^2, \mathbf{m}^2, E^2] \in \mathcal{U}[\varrho^1(T), \mathbf{m}^1(T), E^1(T-)],$$

then

$$[\varrho^1, \mathbf{m}^1, E^1] \cup_T [\varrho^2, \mathbf{m}^2, E^2] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0].$$

Induction argument

System of functionals

$$I_{\lambda,F}[\varrho, \mathbf{m}, E] = \int_0^\infty \exp(-\lambda t) F(\varrho, \mathbf{m}, E)\ \mathrm{d}t,\ \ \lambda > 0$$

where

$$F : X = W^{-\ell,2}(Q) \times W^{-\ell,2}(Q; R^N) \times R \to R$$

is a bounded and continuous functional

Semiflow reduction

$$\begin{aligned} &I_{\lambda,F} \circ \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] \\ &= \Big\{ [\varrho, \mathbf{m}, E] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] \ \Big| \\ &I_{\lambda,F}[\varrho, \mathbf{m}, E] \leq I_{\lambda,F}[\tilde{\varrho}, \tilde{\mathbf{m}}, \widetilde{E}] \text{ for all } [\tilde{\varrho}, \tilde{\mathbf{m}}, \widetilde{E}] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] \Big\} \end{aligned}$$

Induction argument

$$\mathcal{U} \text{ satisfies (A1) - (A4) } \Rightarrow I_{\lambda,F} \circ \mathcal{U} \text{ satisfies (A1) - (A4)}$$